

Lecture 30: Section 4.7

Inverse Trigonometric Functions

Inverse sine function

Inverse cosine function

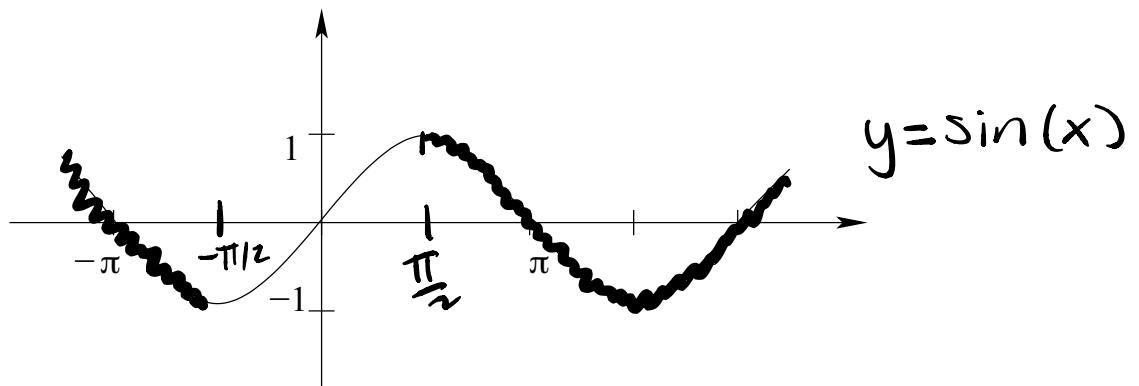
Inverse tangent function

Inverse properties

★ Review: Function - each x has only one y , passes vertical line test.

★ Review: One-to-one - each y has only one x , passes horizontal line test.

The Inverse Sine Function



NOT 1-1, SO WE NEED TO RESTRICT THE DOMAIN

Let's restrict the domain to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Then $y = \sin x$ is one-to-one.

ANGLE **Def.** The **inverse sine function** is defined by
 $y = \sin^{-1} x$ if and only if $x = \sin(y)$

with domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

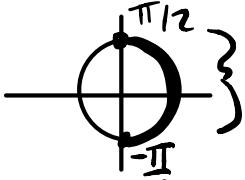
NOTE: The inverse sine function is also called **arcsine**, denoted by $y = \arcsin x$.

$$y = \arcsin(x) = \sin^{-1}(x)$$

NOTE: $\sin^{-1}(x) \neq \frac{1}{\sin(x)}$

"WHAT ANGLE BETWEEN $\frac{\pi}{2}$ AND $-\frac{\pi}{2}$ HAS A SINE VALUE OF $\frac{1}{2}$?"

ex. Find the exact value, if possible.

$$1) \sin^{-1}\left(\frac{1}{2}\right) = y \quad \sin(y) = \frac{1}{2} \text{ AND } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$


THUS $y = \frac{\pi}{3}$ or $y = 30^\circ$

$$2) \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = y \quad \sin(y) = -\frac{\sqrt{3}}{2} \text{ AND } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

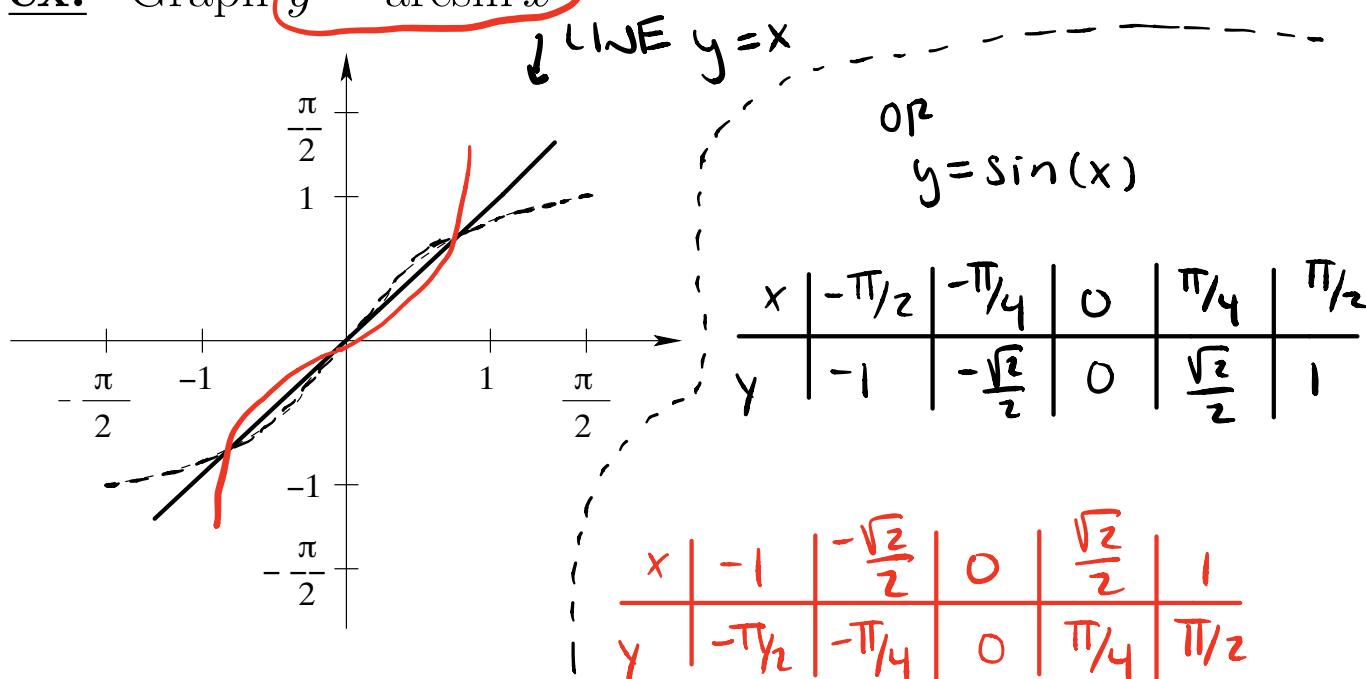
HINT: SIN VALUE IS NEGATIVE IN Q III $\Rightarrow y = -\frac{\pi}{3}$

$$3) \sin^{-1}\left(\frac{3}{2}\right)$$

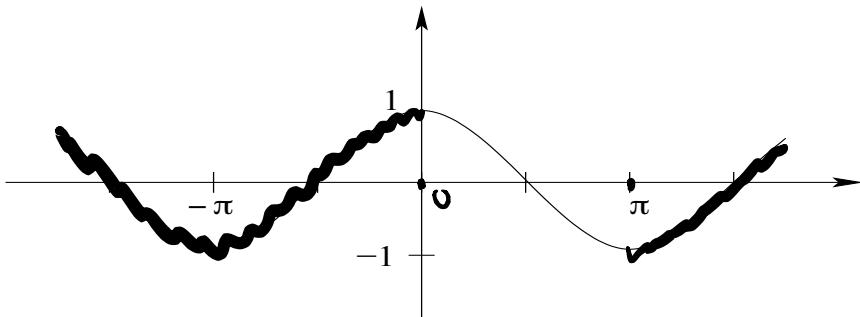
$\frac{3}{2}$ IS NOT IN THE DOMAIN OF $[-1, 1]$,
THUS NO SOLUTION

Checkpoint: Lecture 30, problem 1

ex. Graph $y = \arcsin x$



The Inverse Cosine Function



Restrict the domain to the interval $[0, \pi]$:

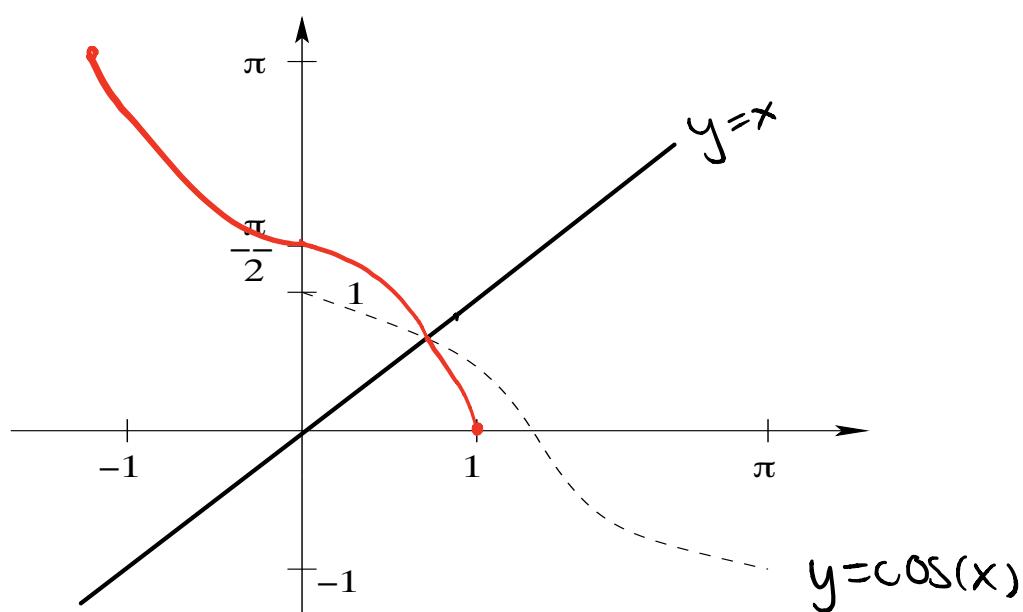
~~TO MAKE IT ONE-TO-ONE~~

Def. The **inverse cosine function** (or **arccosine function**, denoted by $y = \arccos x$) is defined by

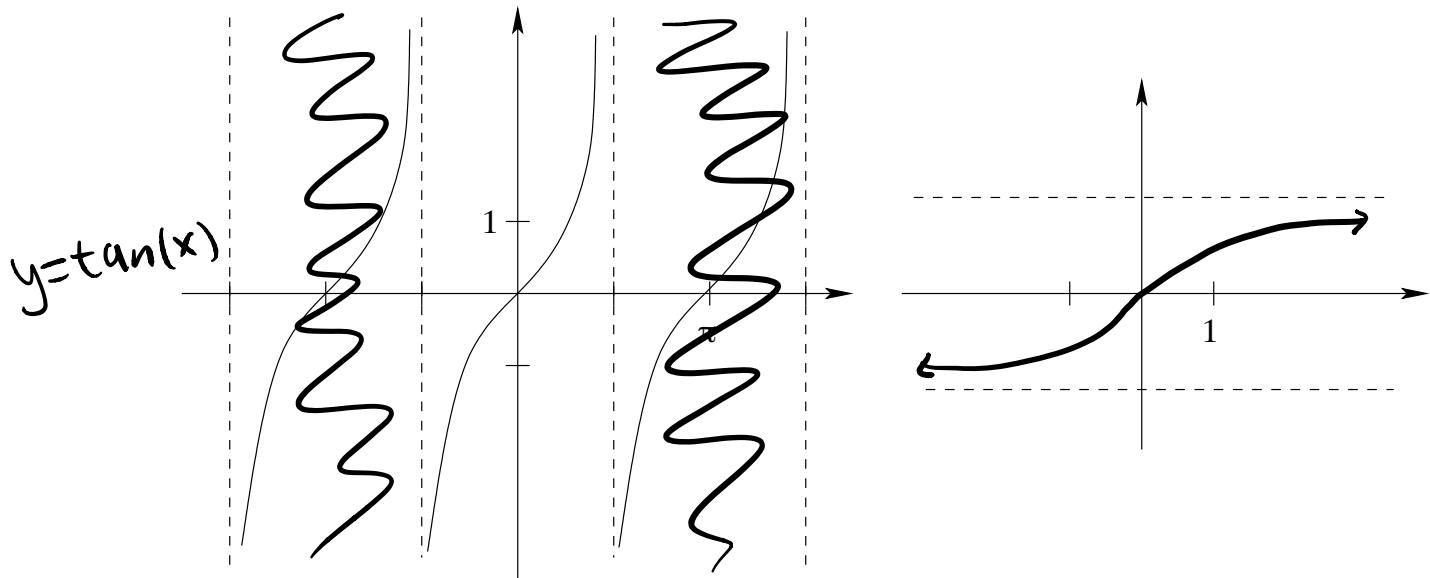
$$y = \cos^{-1} x \text{ if and only if } x = \cos(y)$$

with domain $[-1, 1]$ and range $[0, \pi]$

Graph $y = \arccos x$



The Inverse Tangent Function



Restrict the domain to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$:

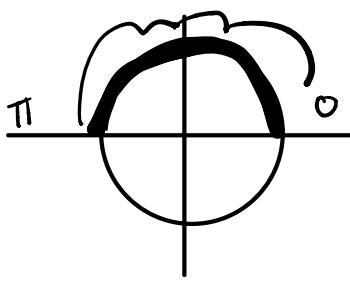
TO MAKE IT ONE-TO-ONE

Def. The **inverse tangent function** (or **arc-tangent function**, denoted by $y = \arctan x$) is defined by

$$y = \tan^{-1} x \text{ if and only if } x = \tan(y)$$

with domain $(-\infty, \infty)$ and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

NOTE: The graph of $y = \tan^{-1} x$ has two horizontal asymptotes: $y = -\frac{\pi}{2}$, $y = \frac{\pi}{2}$



ex. Find

$$1) \cos^{-1} \left(\frac{\sqrt{3}}{2} \right) = y \quad \cos(y) = \frac{\sqrt{3}}{2} \text{ AND } 0 \leq y \leq \pi$$

\nearrow
QI

$$\Rightarrow y = \frac{\pi}{6}$$

$$2) \cos^{-1} \left(-\frac{1}{2} \right) = y \quad \cos(y) = -\frac{1}{2} \text{ AND } 0 \leq y \leq \pi$$

\nearrow
QII

$$\Rightarrow y = \frac{2\pi}{3}$$

$$3) \arccos(0) = y \quad \cos(y) = 0 \text{ AND } 0 \leq y \leq \pi$$

$$\Rightarrow y = \frac{\pi}{2}$$

$$4) \arctan(0) = y \quad \tan(y) = 0 \text{ AND } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\Rightarrow y = 0$$

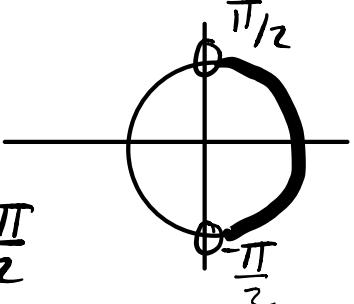
$$5) \arctan(1) = y \quad \tan(y) = 1$$

$$\Rightarrow y = \frac{\pi}{4} \text{ AND } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$6) \tan^{-1}(\sqrt{3}) = y$$

$$\tan(y) = \sqrt{3} \text{ AND } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

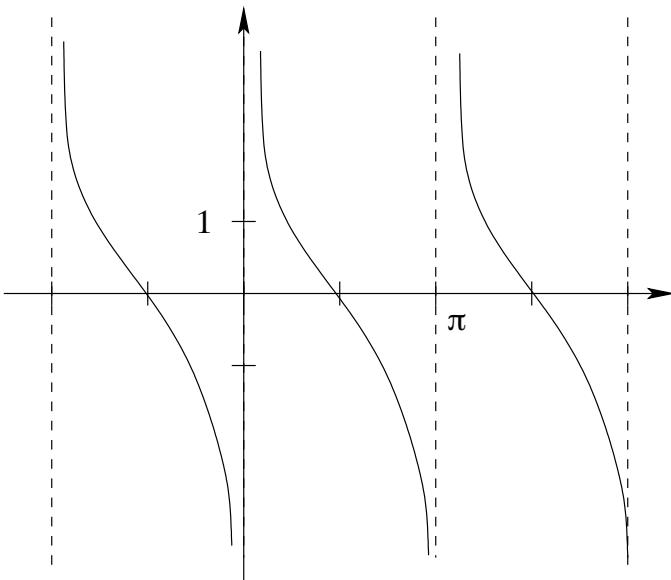
$$\Rightarrow y = \frac{\pi}{3}$$



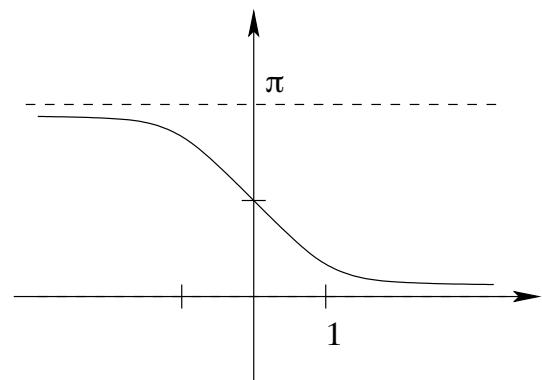
Checkpoint: Lecture 30, problem 2

(NOT RESPONSIBLE FOR THIS)

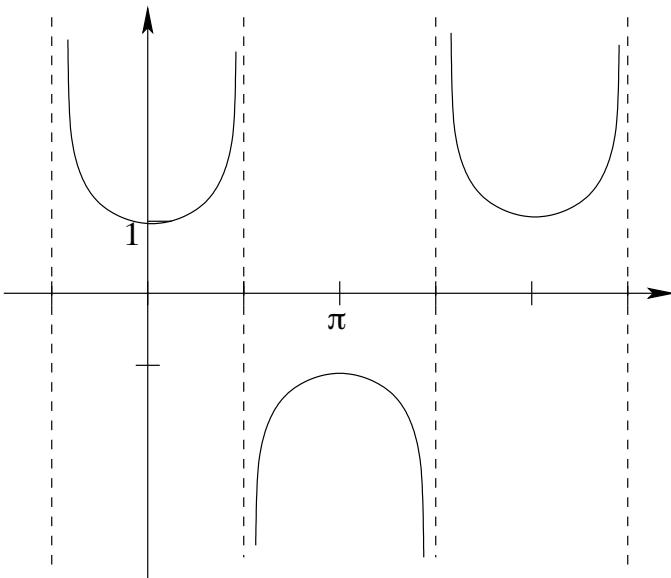
Other Inverse Trigonometric Functions



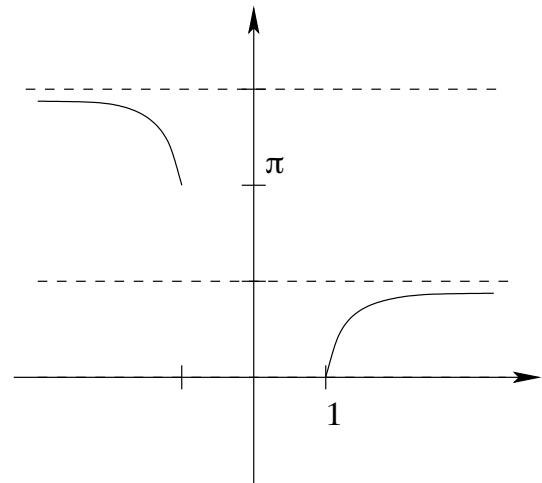
$$y = \cot x, \quad 0 < x < \pi$$



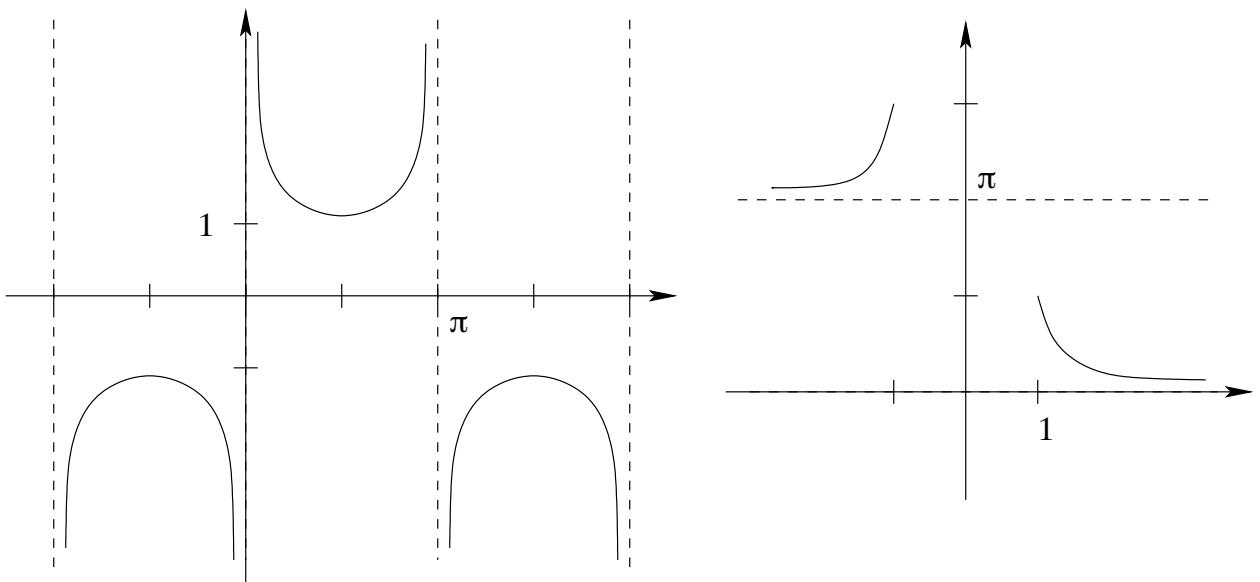
$$y = \cot^{-1} x$$



$$y = \sec x, \quad 0 \leq x < \frac{\pi}{2}, \quad \pi \leq x < \frac{3\pi}{2}$$



$$y = \sec^{-1} x$$



$$y = \csc x, \quad 0 < x \leq \frac{\pi}{2}, \quad \pi < x \leq \frac{3\pi}{2} \quad y = \csc^{-1} x$$

Inverse Properties SHORTCUTS, BUT CAN BE TRICKY!

$$1. \sin(\sin^{-1} x) = x \quad \text{for } -1 \leq x \leq 1$$

$$\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$2. \cos(\cos^{-1} x) = x \quad \text{for } -1 \leq x \leq 1$$

$$\cos^{-1}(\cos x) = x \quad \text{for } 0 \leq x \leq \pi$$

$$3. \tan(\tan^{-1} x) = x \quad \text{for all } x$$

$$\tan^{-1}(\tan x) = x \quad \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

ex. Find the exact value, if it is possible.

$$1) \arctan\left(\tan\frac{\pi}{6}\right) = \frac{\pi}{6}$$

OR: $\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$
 $\Rightarrow \arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$

\nearrow
 $-\frac{\pi}{2} < \frac{\pi}{6} < \frac{\pi}{2} \checkmark$

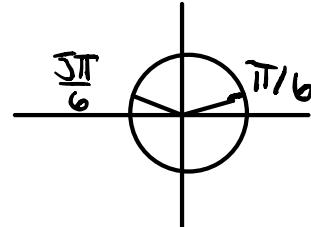
$$2) \sin(\arcsin \pi)$$

\nearrow
 MUST HAVE $-1 \leq x \leq 1$, BUT π IS NOT!
 * π IS NOT IN THE DOMAIN OF $[-1, 1]$
 \Rightarrow NO SOLUTION

$$3) \arcsin\left(\sin\frac{5\pi}{6}\right)$$

MUST HAVE $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, BUT $\frac{5\pi}{6}$ IS NOT, SO REWRITE:

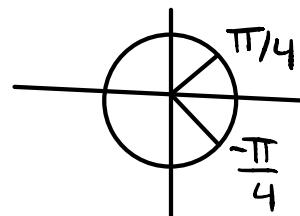
$$\arcsin(\sin(\frac{5\pi}{6})) = \frac{\pi}{6}$$



$$4) \cos^{-1}\left(\cos -\frac{\pi}{4}\right)$$

MUST HAVE $0 \leq x \leq \pi$, BUT $-\frac{\pi}{4}$ IS NOT:

$$\cos^{-1}(\cos(-\frac{\pi}{4})) = \frac{\pi}{4}$$



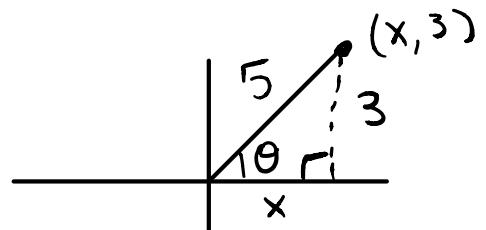
Checkpoint: Lecture 30, problem 3

ex. Find the exact value:

$$1) \cos \left(\arcsin \frac{3}{5} \right)$$

$$\text{LET } \theta = \arcsin\left(\frac{3}{5}\right) \Rightarrow \sin(\theta) = \frac{3}{5} = \frac{y}{r}$$

$$\cos(\theta) = \frac{x}{r} = \frac{4}{5}$$



$$\begin{aligned} r &= \sqrt{3^2 + x^2} \\ 5^2 &= 9 + x^2 \\ 25 &= 9 + x^2 \\ 16 &= x^2 \\ 4 &= x \end{aligned}$$

$$2) \cos(\tan^{-1}(-2)) = \cos(\theta)$$

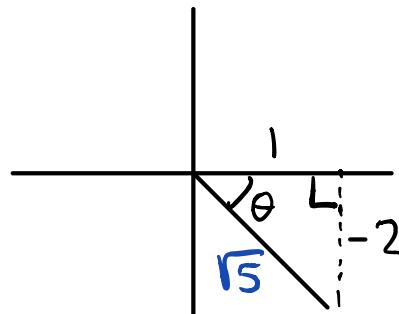
$$\text{LET } \theta = \tan^{-1}(-2)$$

$$\Rightarrow \tan(\theta) = -2 = \frac{\text{OPP}}{\text{ADJ}}$$

* INVERSE tan IS EITHER IN QI

OR QIII, HERE, tan IS NEGATIVE,
SO QIII

$$\cos(\theta) = \frac{\text{ADJ}}{\text{HYP}} = \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$



$$1^2 + (-2)^2 = h^2$$

Checkpoint: Lecture 30, problem 4

$$1+4=h^2$$

$$5=h^2$$

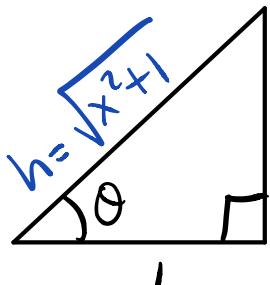
$$\sqrt{5}=h$$

ex. Rewrite the expression as an algebraic expression:

$$1) \cos(\tan^{-1} x) = \cos(\theta)$$

$$\text{LET } \theta = \tan^{-1}(x)$$

$$\tan(\theta) = x = \frac{x}{1} = \frac{\text{OPP}}{\text{ADJ}}$$



$$x \quad h^2 = x^2 + 1^2 \\ h = \sqrt{x^2 + 1}$$

$$\cos(\theta) = \frac{\text{ADJ}}{\text{HYP}}$$

$$= \frac{1}{\sqrt{x^2 + 1}} \cdot \frac{\sqrt{x^2 + 1}}{\sqrt{x^2 + 1}}$$

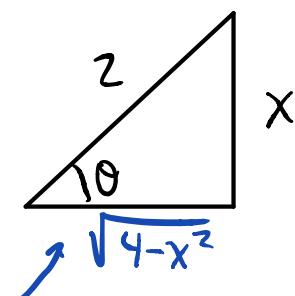
$$2) \sec\left(\arcsin \frac{x}{2}\right) = \sec(\theta)$$

$$\text{LET } \theta = \arcsin\left(\frac{x}{2}\right)$$

$$\Rightarrow \sin(\theta) = \frac{x}{2} = \frac{\text{OPP}}{\text{HYP}}$$

$$= \boxed{\frac{\sqrt{x^2 + 1}}{x^2 + 1}}$$

$$\sec(\theta) = \frac{\text{HYP}}{\text{ADJ}} = \frac{2}{\sqrt{4-x^2}} \cdot \frac{\sqrt{4-x^2}}{\sqrt{4-x^2}} = \frac{2\sqrt{4-x^2}}{4-x^2}$$

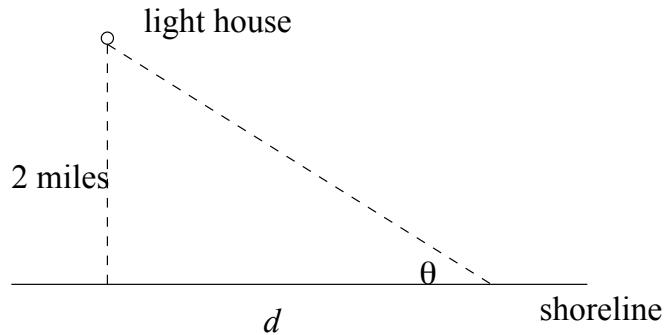


$$z^2 = x^2 + q^2 \Rightarrow q^2 = 4 - x^2 \Rightarrow q = \sqrt{4 - x^2}$$

Checkpoint: Lecture 30, problem 5

Applications

ex. A light house is located on an island that is 2 miles off a straight shoreline.



- 1) Express the angle θ formed by the beam of light and the shoreline in terms of the distance d .

$$\tan(\theta) = \frac{\text{OPP}}{\text{ADJ}} \Rightarrow \tan(\theta) = \frac{2}{d}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{2}{d}\right)$$

OR: $\theta = \arctan\left(\frac{2}{d}\right)$

- 2) Find θ when $d = 4$ miles.

$$\theta = \tan^{-1}\left(\frac{2}{4}\right)$$

$$\boxed{\theta = \tan^{-1}\left(\frac{1}{2}\right) \approx 27^\circ}$$