

Lecture 32: Section 5.1

Fundamental Trigonometric Identities

Reciprocal identities

Quotient identities

Pythagorean identities

Even/odd identities

Cofunction identities

Simplify trigonometric expressions

Factor trigonometric expressions

Add trigonometric expressions

Trigonometric substitution

Simplify logarithmic expressions

1. Reciprocal Identities:

$$\csc \theta = \frac{1}{\sin \theta} \quad \sec \theta = \frac{1}{\cos \theta} \quad \cot \theta = \frac{1}{\tan \theta}$$

2. Quotient Identities:

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

3. Pythagorean Identities:

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

4. Even/Odd Identities:

EVEN { $\begin{array}{ll} \cos(-\theta) = \cos \theta & \sin(-\theta) = -\sin \theta \\ \sec(-\theta) = \sec \theta & \csc(-\theta) = -\csc \theta \end{array}$ } **ODD** { $\begin{array}{l} \tan(-\theta) = -\tan \theta \\ \cot(-\theta) = -\cot \theta \end{array}$ }

$$\sin(90^\circ - 30^\circ) = \cos(30^\circ)$$

5. Cofunction Identities:

$$\sin(60^\circ) = \cos(30^\circ)$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta \quad \cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta \quad \cot\left(\frac{\pi}{2} - \theta\right) = \tan \theta$$

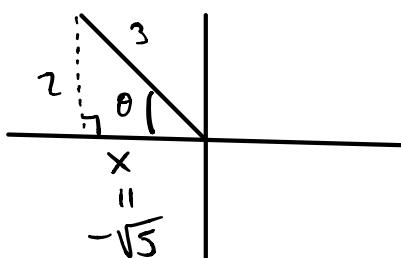
$$\sec\left(\frac{\pi}{2} - \theta\right) = \csc \theta \quad \csc\left(\frac{\pi}{2} - \theta\right) = \sec \theta$$

Checkpoint: Lecture 32, problem 1

ex. If $\sin(-\theta) = -\frac{2}{3}$ and $\tan \theta < 0$ evaluate all six trigonometric functions of θ .

$$\sin(-\theta) = -\sin(\theta) = -\frac{2}{3} \Rightarrow \sin(\theta) = \frac{2}{3}$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{+}{\cos(\theta)} < 0 \Rightarrow \cos(\theta) < 0 \\ \Rightarrow QII$$



$$\sin(\theta) = \frac{z}{3} = \frac{\text{opp}}{\text{hyp}} \quad \csc(\theta) = \frac{3}{z}$$

$$\cos(\theta) = -\frac{\sqrt{5}}{3}$$

$$\sec(\theta) = -\frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$$

$$= -\frac{3\sqrt{5}}{5}$$

$$z^2 + x^2 = 3^2$$

$$x^2 = 5$$

$$x = \pm \sqrt{5}$$

$$x = -\sqrt{5}$$

$$\tan(\theta) = -\frac{z}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$$

$$= -2\frac{\sqrt{5}}{5}$$

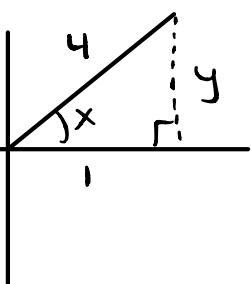
L32 - 3

$$\cot(\theta) = -\frac{\sqrt{5}}{2}$$

ex. If $\csc\left(\frac{\pi}{2} - x\right) = 4$ and $\sin x > 0$, evaluate all six trigonometric functions of x . $\nearrow y \text{ IS POSITIVE}$

$$\csc\left(\frac{\pi}{2} - x\right) = \sec(x) \Rightarrow \sec(x) = 4 = \frac{4}{1} = \frac{\text{HYP}}{\text{ADJ}}$$

$$\csc\left(\frac{\pi}{2} - x\right) = 4 \quad \nearrow x \text{ IS POSITIVE}$$



$$y^2 + 1^2 = 4^2$$

$$y^2 = 15$$

$$y = \pm \sqrt{15} \Rightarrow y = \sqrt{15}$$

$$\sin(x) = \frac{\text{OPP}}{\text{HYP}} = \frac{\sqrt{15}}{4}$$

$$\csc(x) = \frac{\text{HYP}}{\text{OPP}} = \frac{4}{\sqrt{15}} \cdot \frac{\sqrt{15}}{\sqrt{15}} = \frac{4\sqrt{15}}{15}$$

$$\cos(x) = \frac{\text{ADJ}}{\text{HYP}} = \frac{1}{4}$$

$$\sec(x) = 4$$

$$\tan(x) = \frac{\text{OPP}}{\text{ADJ}} = \frac{\sqrt{15}}{1} = \sqrt{15} \quad \cot(x) = \frac{\text{ADJ}}{\text{OPP}} = \frac{1}{\sqrt{15}} \cdot \frac{\sqrt{15}}{\sqrt{15}} = \frac{1}{15}$$

NOTE: YOU CAN ALSO USE IDENTITIES TO SOLVE

Checkpoint: Lecture 32, problem 2

NOTE: From Pythagorean identities, we have

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\sin^2 \theta = 1 - \cos^2 \theta = (1 + \cos \theta)(1 - \cos \theta)$$

$$\cos^2 \theta = 1 - \sin^2 \theta = (1 + \sin \theta)(1 - \sin \theta)$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\tan^2 \theta = \sec^2 \theta - 1 = (\sec \theta + 1)(\sec \theta - 1)$$

$$1 = \sec^2 \theta - \tan^2 \theta$$

$$1 = (\sec \theta + \tan \theta)(\sec \theta - \tan \theta)$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

$$\cot^2 \theta = \csc^2 \theta - 1 = (\csc \theta + 1)(\csc \theta - 1)$$

$$1 = \csc^2 \theta - \cot^2 \theta$$

$$1 = (\csc \theta + \cot \theta)(\csc \theta - \cot \theta)$$

ex. Simplify:

$$\begin{aligned} 1) \cot^2 \theta (1 + \tan^2 \theta) &= (\cot^2 \theta)(\sec^2 \theta) \\ &= \left(\frac{\cos^2 \theta}{\sin^2 \theta}\right)\left(\frac{1}{\cos^2 \theta}\right) \\ &= \frac{1}{\sin^2 \theta} = \csc^2 \theta \end{aligned}$$

$$\begin{aligned} 2) \frac{\tan(-\theta)}{\sec \theta} &= -\frac{\tan \theta}{\sec \theta} \\ &= \frac{\left(-\frac{\sin \theta}{\cos \theta}\right)}{\left(\frac{1}{\cos \theta}\right)} = -\frac{\sin \theta}{\cos \theta} \cdot \frac{\cos \theta}{1} = -\sin \theta \end{aligned}$$

$$\begin{aligned} 3) \frac{\sec^2 \theta - 1}{\sin^2 \theta} &= \frac{\tan^2 \theta}{\sin^2 \theta} = \frac{\left(\frac{\sin^2 \theta}{\cos^2 \theta}\right)}{\sin^2 \theta} \\ &= \frac{\sin^2 \theta}{\cos^2 \theta} \cdot \frac{1}{\sin^2 \theta} \\ &= \frac{1}{\cos^2 \theta} = \sec^2 \theta \end{aligned}$$

Checkpoint: Lecture 32, problem 3

ex. Perform the operations and simplify:

$$\begin{aligned}1) \quad & (1 + \tan x)^2 - 2 \tan x \\& = 1 + 2 \tan x + \tan^2 x - 2 \tan x \\& = 1 + \tan^2 x \\& = \sec^2 x\end{aligned}$$

$$\begin{aligned}2) \quad & \sin t + \cot t \cos t \\& = \sin t + \left(\frac{\cos t}{\sin t} \right) \cos t\end{aligned}$$

COMMON DENOMINATOR ✓

$$\begin{aligned}& \sin t + \frac{\cos^2 t}{\sin t} \\& = \frac{\sin^2 t}{\sin t} + \frac{\cos^2 t}{\sin t} = \frac{\sin^2 t + \cos^2 t}{\sin t} = \frac{1}{\sin t} = \csc t \\3) \quad & \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{1 + \sin \theta} \\& = \left(\frac{1 + \sin \theta}{1 + \sin \theta} \right) \left(\frac{\sin \theta}{\cos \theta} \right) + \left(\frac{\cos \theta}{\cos \theta} \right) \left(\frac{\cos \theta}{1 + \sin \theta} \right) \\& = \frac{\sin \theta + \sin^2 \theta}{(1 + \sin \theta) \cos \theta} + \frac{\cos^2 \theta}{\cos \theta (1 + \sin \theta)} \\& = \frac{\sin \theta + \sin^2 \theta + \cos^2 \theta}{(1 + \sin \theta) \cos \theta} \\& = \frac{\sin \theta + 1}{(1 + \sin \theta) \cos \theta} = \frac{1}{\cos \theta} = \sec \theta\end{aligned}$$

*MULTIPLY BY CONJUGATE

ex. Rewrite $\frac{1}{1 - \sin x}$ so that it is not in fractional form.

$$\begin{aligned}\frac{1}{1 - \sin x} \left(\frac{1 + \sin x}{1 + \sin x} \right) &= \frac{1 + \sin x}{1 - \sin^2 x} \\&= \frac{1 + \sin x}{\cos^2 x} \\&= \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} \\&= \sec^2 x + \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} \\&= \boxed{\sec^2 x + \tan x \sec x}\end{aligned}$$

ex. Factor: $u^3 - v^3 = (u - v)(u^2 + uv + v^2)$

1) $\sin^3 \theta - \cos^3 \theta$

$$\begin{aligned} &= (\sin \theta - \cos \theta) (\underbrace{\sin^2 \theta}_{=} + \sin \theta \cos \theta + \underbrace{\cos^2 \theta}_{=1}) \\ &= (\sin \theta - \cos \theta) (1 + \sin \theta \cos \theta) \end{aligned}$$

2) $2\underbrace{\sec^2 x}_{1+\tan^2 x} + 3 \tan x - 1$

$$= 2(1 + \tan^2 x) + 3 \tan x - 1$$

$$= \underbrace{2}_{\sec^2 x} + \underbrace{2 \tan^2 x}_{1+\tan^2 x} + \underbrace{3 \tan x - 1}_{=}$$

$$= 2 \tan^2 x + 3 \tan x + 1$$

$$= (2 \tan x + 1)(\tan x + 1)$$

Trigonometric Substitution

ex. Use the substitution $x = \underbrace{2\sin\theta}_{0 < \theta < \frac{\pi}{2}}$ to write $\frac{x}{\sqrt{4-x^2}}$ as a trigonometric function of θ .

$$\begin{aligned} \frac{2\sin\theta}{\sqrt{4-(2\sin\theta)^2}} &= \frac{2\sin\theta}{\sqrt{4-4\sin^2\theta}} = \frac{2\sin\theta}{\sqrt{4(1-\sin^2\theta)}} = \frac{2\sin\theta}{\sqrt{4\cos^2\theta}} \\ &= \frac{2\sin\theta}{2\cos\theta} \\ &= \frac{\sin\theta}{\cos\theta} = \tan\theta \end{aligned}$$

ex. Use the substitution $x = \frac{5}{2}\tan\theta, 0 < \theta < \frac{\pi}{2}$ to IN QI rewrite the equation $\sqrt{4x^2 + 25} = 10$ and then find $\sec\theta$ and $\tan\theta$.

$$\sqrt{4\left(\frac{5}{2}\tan\theta\right)^2 + 25} = 10$$

$$\sqrt{4\left(\frac{25}{4}\tan^2\theta\right) + 25} = 10$$

$$\sqrt{25\tan^2\theta + 25} = 10$$

$$\sqrt{25(\tan^2\theta + 1)} = 10$$

$$\sqrt{25\sec^2\theta} = 10$$

$$5\sec\theta = 10$$

$$\sec\theta = 2$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\tan^2\theta + 1 = 2^2$$

$$\tan^2\theta + 1 = 4$$

$$\tan^2\theta = 3$$

$$\tan\theta = \pm\sqrt{3}$$

$$\tan\theta = \sqrt{3}$$

Simplify Logarithmic Expressions

ex. Rewrite the logarithmic expression

$$\ln |\sec \theta| + \ln |\cot \theta| = \ln |\sec \theta \cot \theta|$$

as a single logarithm and simplify the result.

$$= \ln \left| \frac{1}{\cos \theta} \cdot \frac{\cos \theta}{\sin \theta} \right| = \ln \left| \frac{1}{\sin \theta} \right| = \ln |\csc \theta|$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

ex. Rewrite the logarithmic expression

$$\ln(1 - \sin^2 x) - \ln(\csc^2 x - 1)$$

as a single logarithm and simplify the result.

$$= \ln \left(\frac{1 - \sin^2 x}{\csc^2 x - 1} \right) = \ln \left(\frac{\cos^2 x}{\cot^2 x} \right) = \ln \left(\frac{\cos^2 x}{\frac{\cos^2 x}{\sin^2 x}} \right)$$

$$= \ln \left(\frac{\cos^2 x}{1} \cdot \frac{\sin^2 x}{\cos^2 x} \right) = \ln (\sin^2 x) = \ln(\sin x)^2$$

$$= 2 \ln(\sin x)$$

$= 2 \ln |\sin x|$,
AND $x \neq 0, \pi$
(OR $x \neq \pi n$)

Checkpoint: Lecture 32, problem 5

