

## **Lecture 34: Section 5.3**

### **Solving Trigonometric Equations**

Solving trigonometric equations

Trigonometric equations of quadratic types

Trigonometric functions of multiple angles

Using inverse trigonometric funtions

WANT TO GET OUR TRIG. FUNCTION EQUAL TO SOME CONSTANT

1. Reduce the trigonometric equation to one of the form:

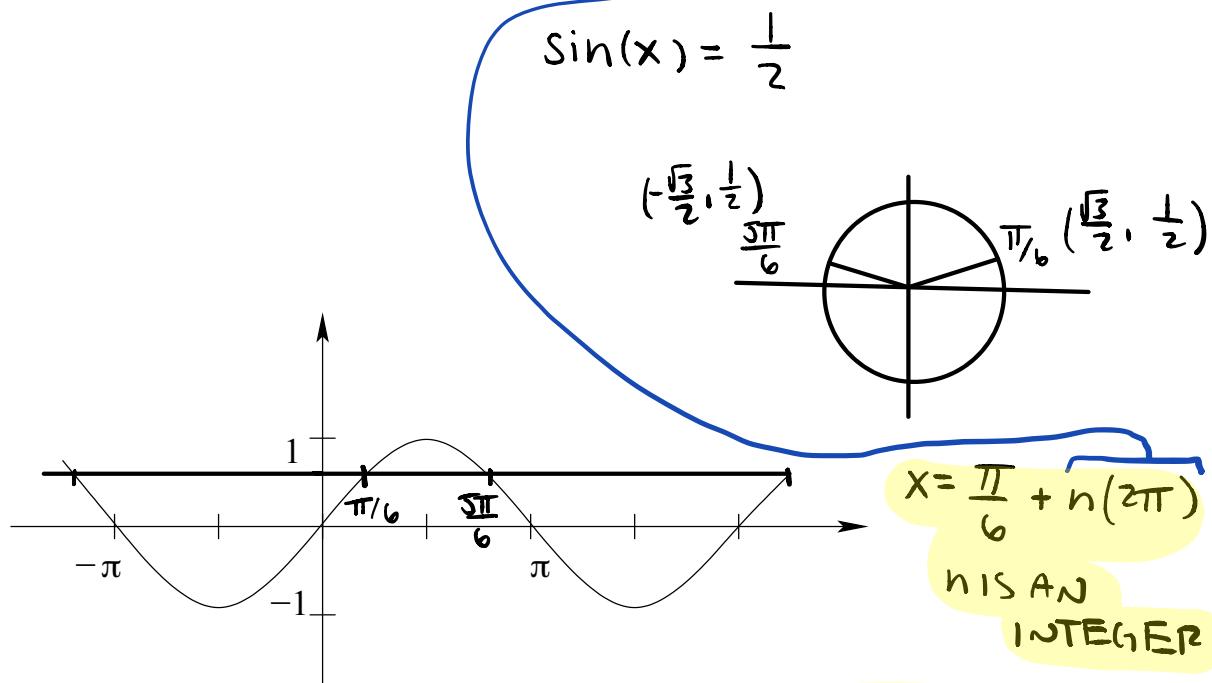
$$\sin \theta = c \quad (-1 \leq c \leq 1)$$

$$\cos \theta = c \quad (-1 \leq c \leq 1)$$

$$\tan \theta = c$$

2. Find the solution  $\theta$  in one period and then find the general solutions by adding an integer number of periods.

**ex.** Solve the equation  $2 \sin x - 1 = 0$ .



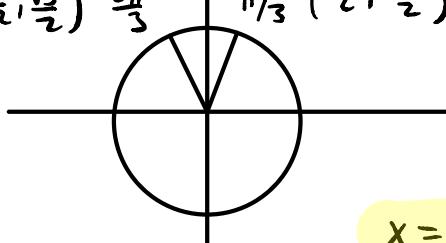
$$x = \frac{5\pi}{6} + n(2\pi)$$

WHERE n IS AN INTEGER

ex. Solve the equation  $\tan^2 x - 3 = 0$ .

$$\tan^2 x = 3$$

$$(-\frac{1}{2}, \frac{\sqrt{3}}{2}) \quad \frac{2\pi}{3} \quad \pi/3 \quad (\frac{1}{2}, \frac{\sqrt{3}}{2})$$



n TIMES THE PERIOD



$$x = \frac{\pi}{3} + n\pi, n \text{ INTEGER}$$

$$x = \frac{2\pi}{3} + n\pi, n \text{ INTEGER}$$

NOTE: FOR  $\tan$  AND  $\cot$ , ONLY NEED TO LOOK FOR SOLUTIONS IN THE INTERVAL  $[0, \pi]$  B/C PERIOD IS  $\pi$

Checkpoint: Lecture 34, problem 1

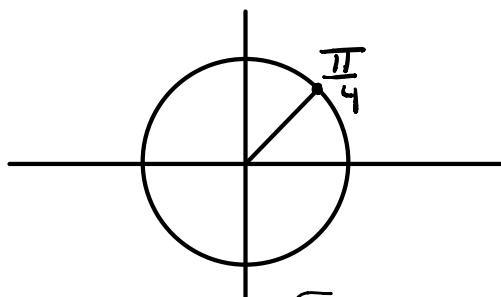
ex. Find the values of  $x$  for which the graphs of  $f(x) = \sin x$  and  $g(x) = \cos x$  intersect.

$$\sin(x) = \cos(x)$$

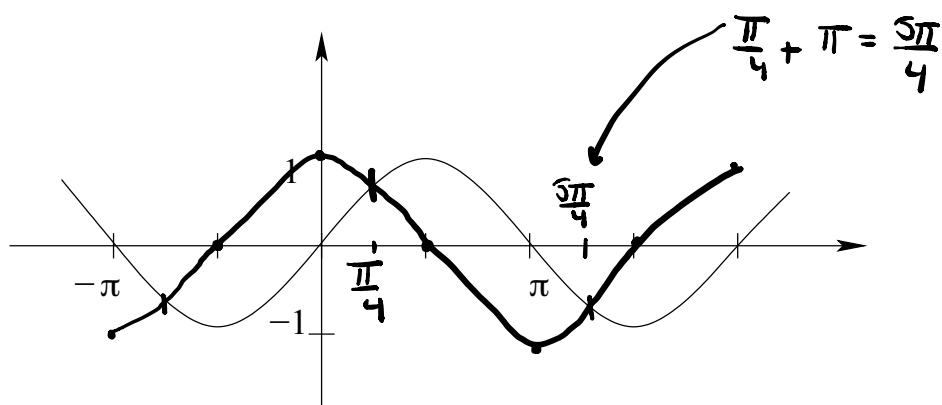
$$(\sin(x) = \cos(x)) \frac{1}{\cos(x)}$$

$$\frac{\sin(x)}{\cos(x)} = 1$$

$$\tan(x) = 1$$



$$x = \frac{\pi}{4} + n\pi$$



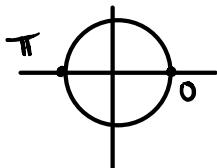
ex. Solve the equation  $\sin^3 \theta = 2 \sin \theta$ .

$$\sin^3 \theta - 2 \sin \theta = 0$$

$$\sin \theta (\sin^2 \theta - 2) = 0$$

OR:  $\theta = n\pi$

$$\sin \theta = 0$$



$$\theta = 0 + n(2\pi) = 2n\pi$$

$$\theta = \pi + n(2\pi) = \pi + 2n\pi$$

$$\sin^2 \theta - 2 = 0$$

$$\sin^2 \theta = 2$$

$$\sin \theta = \pm \sqrt{2}$$



NO SOLUTION B/C  
 $\pm \sqrt{2}$  IS NOT IN  
 THE RANGE OF  
 $[-1, 1]$  SINCE  
 $\sqrt{2} > 1$  AND  $-\sqrt{2} < -1$

Checkpoint: Lecture 34, problem 2

\* [NOTE: Do not cancel the factor  $\sin \theta$  from both sides! This will eliminate some solutions.] \*

(DON'T DIVIDE BY A VARIABLE, CAN LOSE SOLUTIONS)

## Trigonometric Equations of Quadratic Types

Quadratic Equations

$$2x^2 - 3x + 1 = 0$$

$$(2x - 1)(x - 1) = 0$$

Quadratic Type Equations

$$2 \sin^2 x - 3 \sin x + 1 = 0$$

$$(2 \sin(x) - 1)(\sin(x) - 1) = 0$$

OR: LET  $u = \sin(x)$ , THEN  $u^2 = \sin^2(x)$

$$\text{so, } 2u^2 - 3u + 1 = 0$$

$$(2u - 1)(u - 1) = 0 \quad \begin{matrix} \text{SUB} \\ u = \sin(x) \end{matrix}$$

$$(2 \sin(x) - 1)(\sin(x) - 1) = 0$$

ex. Find all solutions in the interval  $[0, 2\pi)$ .

$$1) 2\sin^2 x - 3\sin x + 1 = 0$$

$$(2\sin(x) - 1)(\sin(x) - 1) = 0$$

$$2\sin(x) - 1 = 0$$

$$\sin(x) = \frac{1}{2}$$

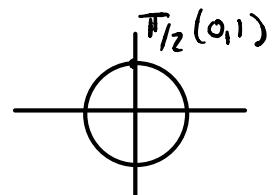
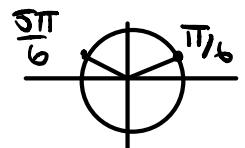
$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\sin(x) - 1 = 0$$

$$\sin(x) = 1$$

$$x = \frac{\pi}{2}$$

$$\Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}$$



$$2) 2\tan^2 x - 5\tan x + 2 = 0$$

$$(2\tan(x) - 1)(\tan(x) - 2) = 0$$

$$2\tan(x) - 1 = 0$$

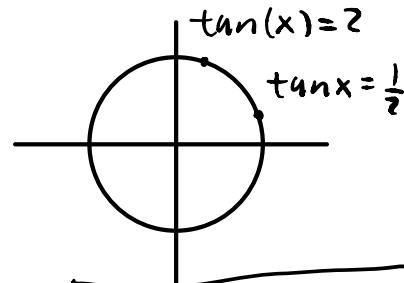
$$\tan(x) = \frac{1}{2}$$

$$x = \arctan(\frac{1}{2})$$

$$\tan(x) - 2 = 0$$

$$\tan(x) = 2$$

$$x = \arctan(2)$$



$$\boxed{x = \arctan(\frac{1}{2}), x = \arctan(2)}$$

\*WANT:  
ONE TRIG.  
FUNCTION

$$3) 1 + \sin x = 2\cos^2 x$$

$$1 + \sin x = 2(1 - \sin^2 x)$$

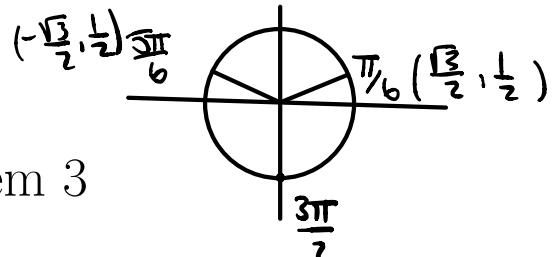
$$1 + \sin x = 2 - 2\sin^2 x$$

$$2\sin^2 x + \sin x - 1 = 0$$

$$(2\sin x - 1)(\sin x + 1) = 0$$

$$\sin x = \frac{1}{2} \quad \sin x = -1$$

$$\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}}$$



Checkpoint: Lecture 34, problem 3

\* WHEN YOU RAISE BOTH SIDES TO AN EVEN POWER,  
YOU MUST CHECK FOR EXTRANEOUS SOLUTIONS

ex. Find all solutions in the interval  $[0, 2\pi)$ .

$$1) (\sin \theta + \cos \theta)^2 = (1)^2$$

$$\underline{\sin^2 \theta} + 2\sin \theta \cos \theta + \underline{\cos^2 \theta} = 1$$

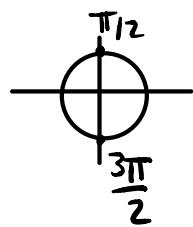
$$= 1$$

$$1 + 2\sin \theta \cos \theta = 1$$

$$\frac{2\sin \theta \cos \theta}{2} = \frac{0}{2}$$

$$\sin \theta \cos \theta = 0$$

$$\sin \theta = 0 \quad \cos \theta = 0$$



$$\theta = 0, \pi$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

CHECK!  $\sin \theta + \cos \theta = 1$

$$\theta = 0$$

$$\sin(0) + \cos(0) ? = 1$$

$$0 + 1 = 1 \checkmark$$

$$\theta = \pi$$

$$\sin(\pi) + \cos(\pi) ? = 1$$

$$0 + (-1) = 1$$

$-1 \neq 1 \Rightarrow \theta = \pi$  NOT A  
SOLUTION

$$\theta = \frac{\pi}{2}$$

$$\sin(\frac{\pi}{2}) + \cos(\frac{\pi}{2}) ? = 1$$

$$1 + 0 = 1 \checkmark$$

$$\theta = \frac{3\pi}{2}$$

$$\sin(\frac{3\pi}{2}) + \cos(\frac{3\pi}{2}) ? = 1$$

$$-1 + 0 = 1$$

$-1 \neq 1 \Rightarrow \theta = \frac{3\pi}{2}$  NOT A  
SOLUTION!

$$\theta = 0, \frac{\pi}{2}$$

**NOTE:** If we perform an operation on an equation that may introduce new roots, such as squaring both sides, then we must **check** that the solutions obtained are not extraneous.

MUST  
CHECK SOLUTIONS!

$$2) \tan \theta + \sec \theta = 1$$

$$\frac{\sin \theta}{\cos \theta} + \frac{1}{\cos \theta} = 1$$

$$\left( \frac{\sin \theta + 1}{\cos \theta} \right)^2 = (1)^2$$

$$\frac{\sin^2 \theta + 2\sin \theta + 1}{\cos^2 \theta} = 1$$

$$\sin^2 \theta + 2\sin \theta + 1 = \cos^2 \theta$$

$$\sin^2 \theta + 2\sin \theta + 1 = 1 - \sin^2 \theta$$

$$[2\sin^2 \theta + 2\sin \theta = 0] \frac{1}{2}$$

$$\sin^2 \theta + \sin \theta = 0$$

$$\sin \theta (\sin \theta + 1) = 0$$

$$\sin \theta = 0$$

$$\theta = 0, \pi$$

$$\sin \theta + 1 = 0$$

$$\begin{aligned} \sin \theta &= -1 \\ \theta &= \frac{3\pi}{2} \end{aligned}$$

CHECK!  $\tan \theta + \sec \theta = 1$

$$\theta = 0:$$

$$\begin{aligned} \tan(0) + \sec(0) &= 1 \\ \frac{0}{1} + \frac{1}{1} &= 1 \\ 1 &= 1 \checkmark \end{aligned}$$

$$\theta = \pi:$$

$$\begin{aligned} \tan(\pi) + \sec(\pi) &= 1 \\ \frac{0}{-1} + \frac{1}{-1} &= 1 \\ 0 + -1 &= 1 \\ -1 &\neq 1 \end{aligned}$$

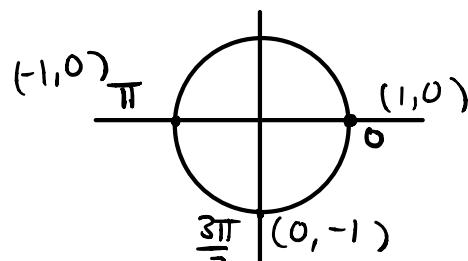
$$\theta = \frac{3\pi}{2}:$$

$$\begin{aligned} \tan\left(\frac{3\pi}{2}\right) + \sec\left(\frac{3\pi}{2}\right) &= 1 \\ -\frac{1}{0} + \frac{1}{0} &= 1 \end{aligned}$$

↑ CANNOT DIVIDE BY 0!

SO,  $\theta = \frac{3\pi}{2}$  NOT A SOLUTION

⇒  $\theta = 0$  IS ONLY SOLUTION



Checkpoint: Lecture 34, problem 4

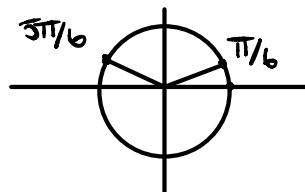
# Trigonometric Functions of Multiple Angles

When solving trigonometric equations that involve functions of multiples of angles, we first solve for the multiple of the angle, then divide to solve for the angle.

ex. Consider the equation  $2 \sin 3x - 1 = 0$ .

1) Find all solutions of the equation.

$$\sin(3x) = \frac{1}{2} \quad \text{"WHAT ANGLES HAVE A SIGN VALUE OF } \frac{1}{2} \text{ ? "}$$



$$\Rightarrow \left( 3x = \frac{\pi}{6} + n(2\pi) \right) \frac{1}{3} \quad \text{AND} \quad \left( 3x = \frac{5\pi}{6} + n(2\pi) \right) \frac{1}{3}$$

$$x = \frac{\pi}{18} + \frac{2n\pi}{3}$$

AND

$$x = \frac{5\pi}{18} + \frac{2n\pi}{3}$$

2) Find the solutions in the interval  $[0, 2\pi)$ .

$$x = \frac{\pi}{18} + \frac{2n\pi}{3}$$

AND

$$x = \frac{5\pi}{18} + \frac{2n\pi}{3}$$

$n = -1$  YOU GET A NEGATIVE,  
SO START WITH  $n=0$ :

$$\cdot n=0: x = \frac{\pi}{18} + 0\left(\frac{2\pi}{3}\right) = \boxed{\frac{\pi}{18}}$$

$$\cdot n=1: x = \frac{\pi}{18} + 1\left(\frac{2\pi}{3}\right) \quad \leftarrow \boxed{+ \frac{12\pi}{18}}$$

$$= \frac{\pi}{18} + \frac{12\pi}{18} = \boxed{\frac{13\pi}{18}}$$

$$\cdot n=2: x = \frac{\pi}{18} + 2\left(\frac{2\pi}{3}\right) \quad \leftarrow \boxed{+ \frac{12\pi}{18}}$$

$$= \frac{\pi}{18} + \frac{4\pi}{3}$$

$$= \frac{\pi}{18} + \frac{24\pi}{18} = \boxed{\frac{25\pi}{18}}$$

$$n=3: x = \frac{\pi}{18} + 3\left(\frac{2\pi}{3}\right)$$

$$= \frac{\pi}{18} + \frac{6\pi}{3} \quad \text{TOO BIG!}$$

$$= \frac{\pi}{18} + \frac{36\pi}{18} = \frac{37\pi}{18}$$

SO,

$$x = \frac{\pi}{18}, \frac{13\pi}{18}, \frac{25\pi}{18}, \frac{5\pi}{18}, \frac{17\pi}{18}, \frac{29\pi}{18}$$

$n = -1$  YOU GET A NEGATIVE,  
SO START WITH  $n=0$ :

$$\cdot n=0: x = \frac{5\pi}{18} + 0\left(\frac{2\pi}{3}\right) = \boxed{\frac{5\pi}{18}}$$

$$\cdot n=1: x = \frac{5\pi}{18} + 1\left(\frac{2\pi}{3}\right) = \boxed{\frac{17\pi}{18}}$$

$$\cdot n=2: x = \frac{5\pi}{18} + 2\left(\frac{2\pi}{3}\right) = \boxed{\frac{29\pi}{18}}$$

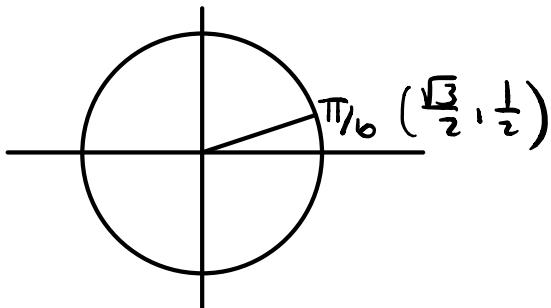
$$\cdot n=3: x = \frac{5\pi}{18} + 3\left(\frac{2\pi}{3}\right) = \frac{41\pi}{18}$$

↑  
TOO BIG!

ex. Consider the equation  $\sqrt{3} \tan \frac{x}{2} - 1 = 0$ .

1) Find all solutions of the equation.

$$\tan\left(\frac{x}{2}\right) = \frac{1}{\sqrt{3}}$$



$$\frac{x}{2} = \frac{\pi}{6} + n\pi$$

$$x = \frac{2\pi}{6} + 2n\pi \quad \text{OR} \quad \boxed{x = \frac{\pi}{3} + 2n\pi}$$

2) Find the solutions in the interval  $[0, 4\pi)$ .

$n = -1$  YOU GET A NEGATIVE, SO START WITH  $n=0$ :

$$\cdot n=0: x = \frac{\pi}{3} + 0(2\pi) = \frac{\pi}{3}$$

$$\cdot n=1: x = \frac{\pi}{3} + 1(2\pi) = \frac{\pi}{3} + \frac{6\pi}{3} = \frac{7\pi}{3}$$

$$\cdot n=2: x = \frac{\pi}{3} + 2(2\pi) = \frac{\pi}{3} + 4\pi = \frac{\pi}{3} + \frac{12\pi}{3} = \frac{13\pi}{3} \quad (\text{TOO BIG SINCE } \frac{13\pi}{3} > 4\pi)$$

so, 
$$\boxed{x = \frac{\pi}{3}, \frac{7\pi}{3}}$$

Checkpoint: Lecture 34, problem 5

## Using Inverse Trigonometric Functions

ex. Solve the equation  $\underbrace{\sec^2 x - \tan x - 3}_{{1+\tan^2 x}} = 0$ .

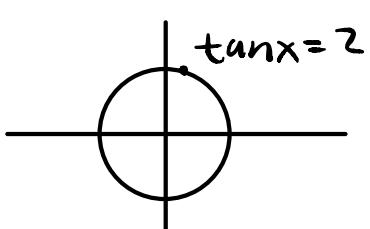
$$1 + \tan^2 x - \tan x - 3 = 0$$

$$\tan^2 x - \tan x - 2 = 0$$

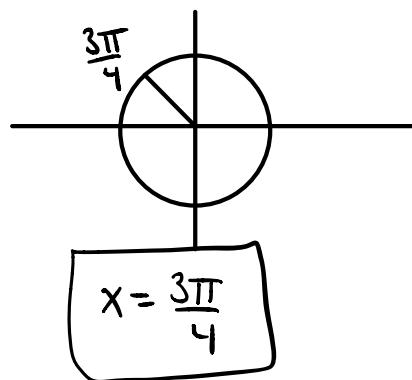
$$(\tan x - 2)(\tan x + 1) = 0$$

$$\tan x - 2 = 0 \quad \tan x + 1 = 0$$

$$\tan x = 2 \quad \tan x = -1$$



$$x = \arctan(2)$$



$$\text{So, } x = \frac{3\pi}{4} + n\pi$$

$$x = \arctan(2) + n\pi$$