

Lecture 35: Section 5.4

Sum and Difference Formulas

Sum and difference formulas for sine, cosine, and tangent

Formulas for sine:

$$\sin(u + v) = \sin u \cos v + \cos u \sin v$$

$$\sin(u - v) = \sin u \cos v - \cos u \sin v$$

Formulas for cosine:

$$\cos(u + v) = \cos u \cos v - \sin u \sin v$$

$$\cos(u - v) = \cos u \cos v + \sin u \sin v$$

Formulas for tangent:

$$\begin{aligned}\tan(u + v) &= \frac{\tan u + \tan v}{1 - \tan u \tan v} &= \frac{\sin(u+v)}{\cos(u+v)} \\ \tan(u - v) &= \frac{\tan u - \tan v}{1 + \tan u \tan v} &= \frac{\sin(u-v)}{\cos(u-v)}\end{aligned}$$

ex. Find the exact value.

$$\begin{aligned} 1) \sin 75^\circ &= \sin(30^\circ + 45^\circ) \\ &= \sin(30^\circ)\cos(45^\circ) + \cos(30^\circ)\sin(45^\circ) \\ &= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\ &= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \boxed{\frac{\sqrt{2} + \sqrt{6}}{4}} \end{aligned}$$

$$2) \cos \frac{\pi}{12} \text{ EASIER TO CHANGE TO DEGREES:}$$

$$\frac{\pi}{12} \cdot \frac{180^\circ}{\pi} = \frac{180^\circ}{12} = 15^\circ$$

COULD USE $60^\circ - 45^\circ = 15^\circ$ OR $45^\circ - 30^\circ = 15^\circ$

$$\begin{aligned} \cos(15^\circ) &= \cos(60^\circ - 45^\circ) \\ &= \cos(60^\circ)\cos(45^\circ) + \sin(60^\circ)\sin(45^\circ) \\ &= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\ &= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \boxed{\frac{\sqrt{2} + \sqrt{6}}{4}} \end{aligned}$$

NOTE: $\sin(90^\circ - \theta) = \cos(\theta)$, so FROM ABOVE WE SEE
 $\sin(75^\circ) = \sin(90^\circ - 15^\circ) = \cos(15^\circ)$

$$3) \tan 15^\circ$$

$$\begin{aligned} &= \tan(60^\circ - 45^\circ) = \frac{\tan(60^\circ) - \tan(45^\circ)}{1 + \tan(60^\circ)\tan(45^\circ)} \\ &= \frac{\sqrt{3} - 1}{1 + (\sqrt{3})(1)} = \left(\frac{\sqrt{3} - 1}{1 + \sqrt{3}} \right) \left(\frac{1 - \sqrt{3}}{1 - \sqrt{3}} \right) = \frac{\sqrt{3} - 3 - 1 + \sqrt{3}}{1 - 3} \\ &= \frac{2\sqrt{3} - 4}{-2} \\ &= \frac{2(\sqrt{3} - 2)}{-2} \\ &= -(\sqrt{3} - 2) \\ &= \boxed{-\sqrt{3} + 2} \end{aligned}$$

Checkpoint: Lecture 35, problem 1

ex. Find the exact value.

$$1) \sin 20^\circ \cos 40^\circ + \cos 20^\circ \sin 40^\circ$$

$$\begin{aligned} &= \sin(20^\circ + 40^\circ) \\ &= \sin(60^\circ) \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

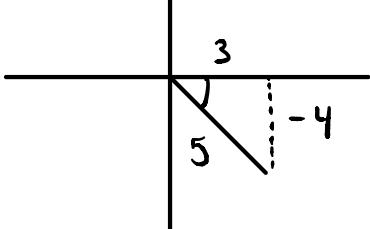
$$2) \frac{\tan 73^\circ - \tan 13^\circ}{1 + \tan 73^\circ \tan 13^\circ}$$

$$\begin{aligned} &= \tan(73^\circ - 13^\circ) \\ &= \tan(60^\circ) \\ &= \frac{\sqrt{3}/2}{1/2} = \frac{\sqrt{3}}{2} \cdot \frac{2}{1} = \sqrt{3} \end{aligned}$$

ex. If $\sin x = -\frac{4}{5}$, $\cos y = \frac{5}{13}$ and both x and y are in quadrant IV, find the exact value of

$$\sin x = \frac{\text{OPP}}{\text{HYP}} = -\frac{4}{5}$$

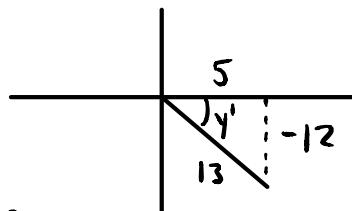
$$\cos y = \frac{\text{ADJ}}{\text{HYP}} = \frac{5}{13}$$



$$(-4)^2 + x^2 = 5^2 \Rightarrow x = 3$$

$$\Rightarrow \cos x = \frac{3}{5}$$

$$\Rightarrow \tan x = -\frac{4}{3}$$



$$5^2 + y^2 = 13^2 \Rightarrow y = -12$$

$$\Rightarrow \sin y = -\frac{12}{13}$$

$$\Rightarrow \tan y = -\frac{12}{5}$$

↑ NEGATIVE B/L
QIV

$$1) \sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$= \left(-\frac{4}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right)$$

$$= -\frac{20}{65} - \left(-\frac{36}{65}\right)$$

$$= -\frac{20}{65} + \frac{36}{65} = \boxed{\frac{16}{65}}$$

$$2) \cot(x + y) = \frac{1}{\tan(x+y)} = \frac{1}{\frac{\tan x + \tan y}{1 - \tan x \tan y}}$$

$$= \frac{1 - \tan x \tan y}{\tan x + \tan y} = \frac{1 - \left(-\frac{4}{3}\right)\left(-\frac{12}{5}\right)}{\left(-\frac{4}{3}\right) + \left(-\frac{12}{5}\right)} = \frac{1 - \frac{48}{15}}{-\frac{56}{15}} = \frac{-\frac{33}{15}}{-\frac{56}{15}}$$

Checkpoint: Lecture 35, problem 2

$$= \left(-\frac{33}{15}\right)\left(-\frac{15}{56}\right)$$

$$x^2 + ?^2 = 1^2$$

$$?^2 = 1 - x^2$$

$$? = \sqrt{1-x^2}$$

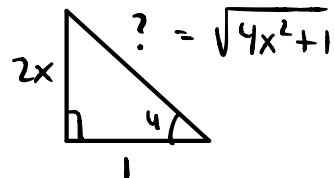
$$?^2 = (2x)^2 + 1^2$$

$$?^2 = 4x^2 + 1$$

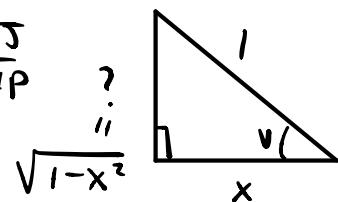
$$? = \sqrt{4x^2+1}$$

ex. Write $\sin(\arctan 2x - \arccos x)$ as an algebraic expression.

LET $u = \arctan(2x)$, so $\tan(u) = 2x = \frac{2x}{1} = \frac{\text{OPP}}{\text{ADJ}}$



LET $v = \arccos(x)$, so $\cos(v) = x = \frac{x}{1} = \frac{\text{ADJ}}{\text{HYP}}$



$$\sin u = \frac{\text{OPP}}{\text{HYP}} = \frac{2x}{\sqrt{4x^2+1}}$$

$$\cos u = \frac{\text{ADJ}}{\text{HYP}} = \frac{1}{\sqrt{4x^2+1}}$$

$$\sin v = \frac{\text{OPP}}{\text{HYP}} = \frac{\sqrt{1-x^2}}{1}$$

$$\cos v = \frac{\text{ADJ}}{\text{HYP}} = \frac{x}{1} = x$$

$$\sin(u-v) = \sin u \cos v - \cos u \sin v$$

$$= \left(\frac{2x}{\sqrt{4x^2+1}} \right) (x) - \left(\frac{1}{\sqrt{4x^2+1}} \right) \left(\sqrt{1-x^2} \right)$$

$$= \frac{2x^2}{\sqrt{4x^2+1}} - \frac{\sqrt{1-x^2}}{\sqrt{4x^2+1}} = \boxed{\frac{2x^2 - \sqrt{1-x^2}}{\sqrt{4x^2+1}}}$$

Checkpoint: Lecture 35, problem 3

ex. Prove the cofunction identity $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$.

$$\begin{aligned} \text{LHS} &= \sin\left(\frac{\pi}{2} - \theta\right) = \sin\left(\frac{\pi}{2}\right)\cos\theta - \cos\left(\frac{\pi}{2}\right)\sin\theta \\ &= (1)\cos\theta - (0)\sin\theta \\ &= \cos\theta - 0 \\ &= \cos\theta = \text{RHS} \end{aligned}$$

ex. Verify $\cos(\theta + 2\pi) = \cos\theta$.

$$\begin{aligned} \text{LHS} &= \cos(\theta + 2\pi) = \cos\theta\cos(2\pi) - \sin(\theta)\sin(2\pi) \\ &= \cos\theta(1) - \sin(\theta)(0) \\ &= \cos\theta - 0 \\ &= \cos\theta \\ &= \text{RHS} \end{aligned}$$

ex. Verify the identity $\frac{1 + \tan x}{1 - \tan x} = \tan\left(\frac{\pi}{4} + x\right)$.

$$\begin{aligned} \text{RHS} &= \tan\left(\frac{\pi}{4} + x\right) = \frac{\tan(\pi/4) + \tan x}{1 - \tan(\pi/4)\tan(x)} \\ &= \frac{1 + \tan x}{1 - (1)(\tan x)} = \frac{1 + \tan x}{1 - \tan x} = \text{LHS} \end{aligned}$$

ex. If $f(x) = \cos x$, verify the identity used in calculus:

$$\frac{f(x+h) - f(x)}{h} = \cos x \left(\frac{\cos h - 1}{h} \right) - \sin x \left(\frac{\sin h}{h} \right)$$

where $h \neq 0$.

$$\begin{aligned}
 \text{LHS} &= \frac{\cos(x+h) - \cos x}{h} = \frac{\cos x \cosh - \sin x \sinh - \cos x}{h} \\
 &= \frac{\cos x \cosh - \cos x - \sin x \sinh}{h} \\
 &= \frac{\cos x \cosh - \cos x}{h} - \frac{\sin x \sinh}{h} \\
 &= \frac{\cos x (\cosh - 1)}{h} - \sin x \left(\frac{\sinh}{h} \right) \\
 &= \cos x \left(\frac{\cosh - 1}{h} \right) - \sin x \left(\frac{\sinh}{h} \right) \\
 &= \text{RHS}
 \end{aligned}$$

Solving a Trigonometric Equation

ex. Find all solutions of

$$\sin\left(x + \frac{\pi}{6}\right) - \sin\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$$

in the interval $[0, 2\pi)$.

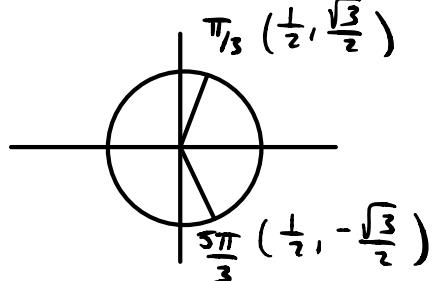
$$\left(\sin x \cos\left(\frac{\pi}{6}\right) + \cos x \sin\left(\frac{\pi}{6}\right)\right) - \left(\sin x \cos\left(\frac{\pi}{6}\right) - \cos x \sin\left(\frac{\pi}{6}\right)\right) = \frac{1}{2}$$

$$\cancel{\sin x\left(\frac{\sqrt{3}}{2}\right)} + \cos x\left(\frac{1}{2}\right) - \cancel{\sin x\left(\frac{\sqrt{3}}{2}\right)} + \cos x\left(\frac{1}{2}\right) = \frac{1}{2}$$

$$\frac{\cos x}{2} + \frac{\cos x}{2} = \frac{1}{2}$$

$$\frac{2\cos x}{2} = \frac{1}{2}$$

$$\cos x = \frac{1}{2}$$



$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

Checkpoint: Lecture 35, problem 4