

Lecture 36: Section 5.5

Multiple Angles and ~~Product-to-Sum Formulas~~

Double-angle formulas

Power reducing formulas

Half-angle formulas

~~Product-to-sum formulas~~

~~Sum-to-product formulas~~

Double-Angle Formulas

* 1. $\sin 2u = 2 \sin u \cos u$

$$2. \cos 2u = \cos^2 u - \sin^2 u$$

$$\begin{aligned} &= 1 - 2 \sin^2 u \\ &= 2 \cos^2 u - 1 \end{aligned}$$

$$\sin^2 u + \cos^2 u = 1$$

$$3. \tan 2u = \frac{2 \tan u}{1 - \tan^2 u} \text{ OR } \frac{\sin 2u}{\cos 2u}$$

Proof.

$$\begin{aligned} \sin 2u &= \sin(u+u) = \sin u \cos u + \cos u \sin u \\ &= 2 \sin u \cos u \end{aligned}$$

$$\begin{aligned} \cos 2u &= \cos(u+u) = \cos u \cos u - \sin u \sin u \\ &= \cos^2 u - \sin^2 u \end{aligned}$$

$$\begin{aligned} \tan 2u &= \tan(u+u) = \frac{\tan u + \tan u}{1 - \tan u \tan u} \\ &= \frac{2 \tan u}{1 - \tan^2 u} \end{aligned}$$

$$\sin(2u) = 2 \sin u \cos u$$

$$\cos(2u) = \cos^2 u - \sin^2 u$$

NOTE: The double-angle formulas are not restricted to angles u and $2u$. For example:

$$\sin 6\theta = \sin(2(3\theta)) = 2 \sin(3\theta) \cos(3\theta)$$

$$\cos 4\theta = \cos(2(2\theta)) = \cos^2(2\theta) - \sin^2(2\theta)$$

ex. Graph $y = 4 \sin x \cos x = 2 \cdot 2 \sin x \cos x$

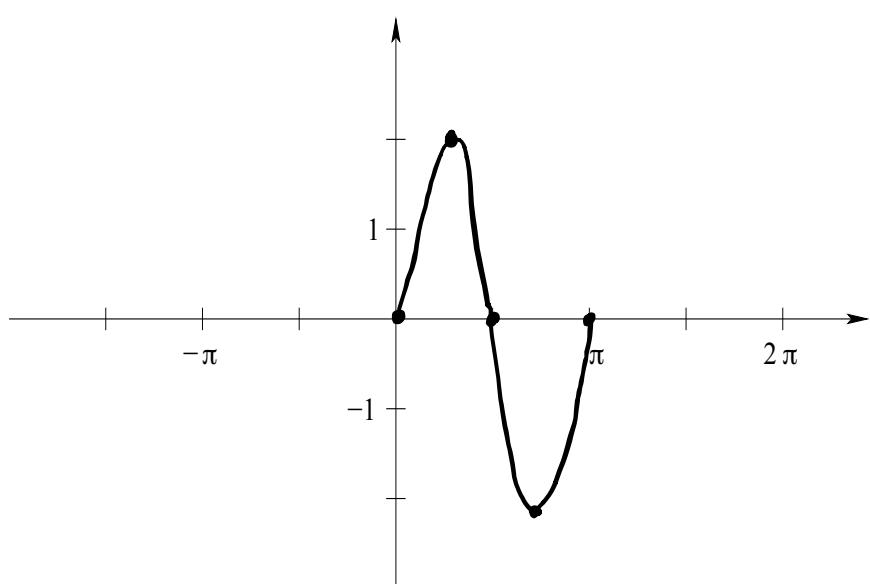
NEW START: $2x = 0$
 $x = 0$

$$= 2(\sin(2x))$$

$$= 2\sin(2x)$$

NEW END: $2x = 2\pi$
 $x = \pi$

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	0	2	0	-2	0
	0	1	0	-1	0



Checkpoint: Lecture 36, problem 1

$$(-2)^2 + y^2 = 3^2$$

$$\cos x = -\frac{2}{3} = \frac{\text{Adj}}{\text{Hyp}}$$

$$y = \pm \sqrt{5}$$

$$y = \sqrt{5}$$

↑
POSITIVE
B/C QII

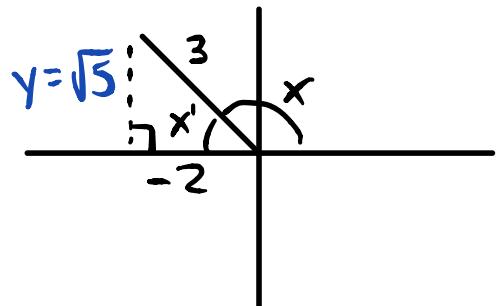
ex. If $\cos x = -\frac{2}{3}$ and x is in quadrant II, find

$$1) \sin 2x$$

$$= 2 \sin x \cos x$$

$$= 2 \left(\frac{\sqrt{5}}{3} \right) \left(-\frac{2}{3} \right)$$

$$= \boxed{-\frac{4\sqrt{5}}{9}}$$



$$\sin x = \frac{\text{Opp}}{\text{Hyp}} = \frac{\sqrt{5}}{3}$$

$$\tan x = \frac{\text{Opp}}{\text{Adj}} = -\frac{\sqrt{5}}{2}$$

$$2) \tan 2x$$

$$= \frac{2 \tan x}{1 - \tan^2 x}$$

$$= \frac{2 \left(-\frac{\sqrt{5}}{2} \right)}{1 - \left(-\frac{\sqrt{5}}{2} \right)^2} = \frac{-\sqrt{5}}{1 - \frac{5}{4}} = \frac{-\sqrt{5}}{\frac{4}{4} - \frac{5}{4}} = \frac{-\sqrt{5}}{-\frac{1}{4}}$$

$$= -\frac{\sqrt{5}}{1} \cdot -\frac{4}{1}$$

Checkpoint: Lecture 36, problem 2

$$= \boxed{4\sqrt{5}}$$

*OR: $\tan 2x = \frac{\sin 2x}{\cos 2x}$

ex. Write $\sin(2 \arccos x)$ as an algebraic expression.

$$\text{LET } \theta = \arccos x \Rightarrow \cos \theta = x = \frac{\text{ADJ}}{\text{HYP}}$$

$$\begin{aligned} \sin(2 \arccos x) &= \sin(2\theta) = 2\sin\theta\cos\theta = 2(\sqrt{1-x^2})(x) \\ &= \boxed{2x\sqrt{1-x^2}} \end{aligned}$$

$$y = \sqrt{1-x^2} \quad \sin\theta = \frac{\text{OPP}}{\text{HYP}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

$$x^2 + y^2 = 1 \Rightarrow y = \pm \sqrt{1-x^2} = \sqrt{1-x^2}$$

Checkpoint: Lecture 36, problem 3

USE POSITIVE B/C FOR
 $\arccos x$, OUR
RANGE IS $[0, \pi]$

A Triple-Angle Formula



ex. Write $\cos 3x$ in terms of $\cos x$.

$$\begin{aligned} \cos(3x) &= \cos(x+2x) = \cos x \overbrace{\cos 2x}^{2\cos^2 x - 1} - \sin x \overbrace{\sin 2x}^{2\sin x \cos x} \\ &= \cos x (2\cos^2 x - 1) - \sin x (2\sin x \cos x) \\ &= 2\cos^3 x - \cos x - 2 \overbrace{\sin^2 x}^{1-\cos^2 x} \cos x \\ &= 2\cos^3 x - \cos x - 2(1-\cos^2 x)\cos x \\ &= 2\cos^3 x - \cos x - 2\cos x (1-\cos^2 x) \\ &= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x \\ &= \boxed{4\cos^3 x - 3\cos x} \end{aligned}$$

Power-Reducing Formulas

$$\cos 2u = 1 - 2\sin^2 u$$

SOLVE FOR $\sin^2 u$

$$\Rightarrow \sin^2 u = \frac{1 - \cos 2u}{2}$$

$$1. \sin^2 u = \frac{1 - \cos 2u}{2}$$

$$2. \cos^2 u = \frac{1 + \cos 2u}{2}$$

$$3. \tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u} = \frac{\sin^2 u}{\cos^2 u} = \frac{1 - \cos 2u}{1 + \cos 2u}$$

SOLVE FOR $\cos^2 u$

ex. Express $\sin^2 x \cos^2 x$ in terms of the first power of cosine.

$$\begin{aligned}\sin^2 x \cos^2 x &= \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right) \quad \text{FOIL} \\ &= \frac{1 - \cos^2(2x)}{4} \\ &= \frac{1}{4} \left(1 - \cos^2(2x) \right) \quad \text{USE #2 FROM ABOVE} \\ &= \frac{1}{4} \left[1 - \left(\frac{1 + \cos 4x}{2} \right) \right] \\ &= \frac{1}{4} - \left(\frac{1 + \cos 4x}{8} \right) = \frac{1}{4} - \left(\frac{1}{8} + \frac{\cos 4x}{8} \right) \\ &\stackrel{\text{L36 - 6}}{=} \frac{1}{4} - \frac{1}{8} - \frac{\cos 4x}{8} \\ &= \frac{2}{8} - \frac{1}{8} = \frac{1}{8} \\ &= \boxed{\frac{1}{8} - \frac{\cos 4x}{8}}\end{aligned}$$

Half-Angle Formulas

$$1. \sin \frac{u}{2} = \pm \sqrt{\frac{1 - \cos u}{2}}$$

$$2. \cos \frac{u}{2} = \pm \sqrt{\frac{1 + \cos u}{2}}$$

$$3. \tan \frac{u}{2} = \frac{1 - \cos u}{\sin u} = \frac{\sin u}{1 + \cos u}$$

The choice of the + or - sign depends on the quadrant in which $u/2$ lies.

ex. Find the exact value of $\sin 22.5^\circ$.

22.5° is

in QI

$\begin{array}{|c|} \hline (+,+) \\ \hline \end{array}$

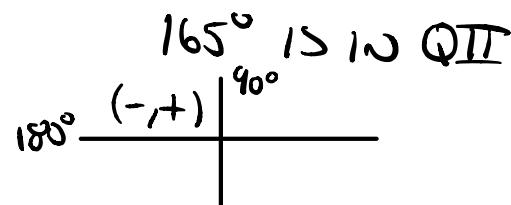
$$\sin(22.5^\circ) = \sin\left(\frac{45^\circ}{2}\right)$$

$$= \sqrt{\frac{1 - \cos(45^\circ)}{2}} = \sqrt{\left(\frac{1 - \frac{\sqrt{2}}{2}}{2}\right) \frac{2}{2}}$$

$$= \boxed{\sqrt{\frac{2 - \sqrt{2}}{4}}} \quad \text{OR:} \quad \frac{\sqrt{2 - \sqrt{2}}}{\sqrt{4}}$$

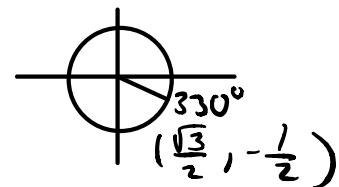
$$= \boxed{\frac{\sqrt{2 - \sqrt{2}}}{2}}$$

$$\cos\left(\frac{4}{2}\right) = \pm \sqrt{\frac{1 + \cos 4}{2}}$$



ex. Find the exact value of $\cos 165^\circ$.

$$\cos(165^\circ) = \cos\left(\frac{330^\circ}{2}\right) = -\sqrt{\frac{1 + \cos(330^\circ)}{2}}$$



$$= -\sqrt{\left(\frac{1 + \frac{\sqrt{3}}{2}}{2}\right)\left(\frac{2}{2}\right)}$$

$$= -\sqrt{\frac{2 + \sqrt{3}}{4}} = -\frac{\sqrt{2 + \sqrt{3}}}{\sqrt{4}} = \boxed{-\frac{\sqrt{2 + \sqrt{3}}}{2}}$$

ex. Find $\tan \frac{x}{2}$ if $\sin x = \frac{2}{5}$ and x is in quadrant II.

$$\tan\left(\frac{x}{2}\right) = \frac{1 - \cos x}{\sin x}$$

$$= \frac{1 - \left(-\frac{\sqrt{21}}{5}\right)}{\frac{2}{5}}$$

$$= \left(\frac{1 + \frac{\sqrt{21}}{5}}{\frac{2}{5}}\right) \frac{5}{5}$$

$$= \boxed{\frac{5 + \sqrt{21}}{2}}$$

$$\cos x = \frac{\text{ADJ}}{\text{HYP}}$$

$$= -\frac{\sqrt{21}}{5}$$

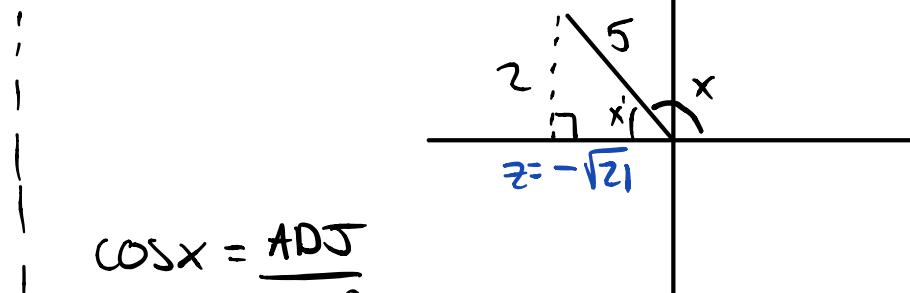
$$\sin x = \frac{2}{5} = \frac{\text{OPP}}{\text{HYP}}$$

$$z^2 + z^2 = \frac{4}{5}z^2$$

$$z = \pm \sqrt{2}$$

$$z = -\sqrt{2}$$

↑ NEGATIVE
B/C QII



Checkpoint: Lecture 36, problem 4

Solving Trigonometric Equations

ex. Solve the equation in the interval $[0, 2\pi)$.

$$1) \sin 2x - \cos x = 0$$

$$2 \sin x \cos x - \cos x = 0$$

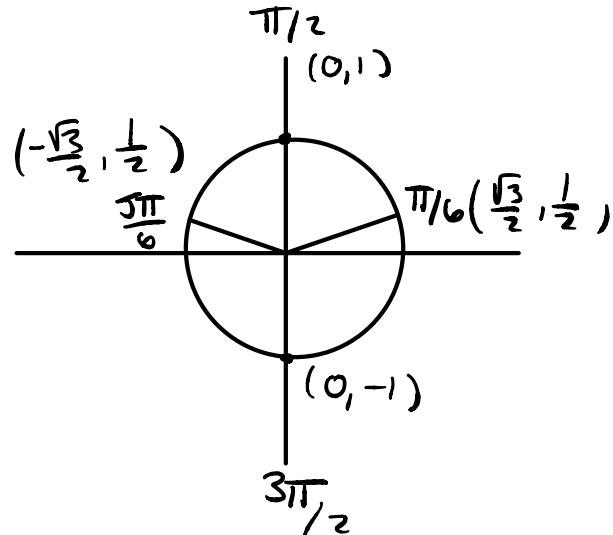
$$\cos x (2 \sin x - 1) = 0$$

$$\cos x = 0$$

$$2 \sin x - 1 = 0$$

$$\sin x = \frac{1}{2}$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$



$$2) \tan \frac{x}{2} - \sin x = 0 \quad \text{MUST CHECK DOMAIN}$$

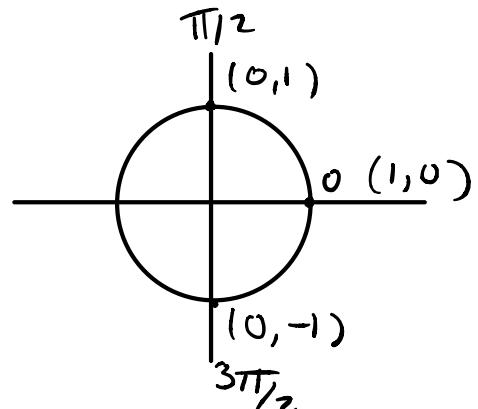
$$\left(\frac{1 - \cos x}{\sin x} - \sin x = 0 \right) \sin x$$

$$1 - \cos x - \sin^2 x = 0$$

$$1 - \cos x - (1 - \cos^2 x) = 0$$

$$1 - \cos x - 1 + \cos^2 x = 0$$

$$\cos^2 x - \cos x = 0 \Rightarrow \cos x (\cos x - 1) = 0$$



Checkpoint: Lecture 36, problem 5 $\cos x = 0 \quad \cos x - 1 = 0$

$$\cos x = 1$$

$$x = 0, \frac{\pi}{2}, \frac{3\pi}{2}$$



CHECK SOLUTIONS TO MAKE SURE IN DOMAIN:

• $x=0$:

$$\tan\left(\frac{0}{2}\right) = \tan 0 = \frac{0}{1} \quad \checkmark$$

• $x=\frac{\pi}{2}$:

$$\tan\left(\frac{\pi/2}{2}\right) = \tan\left(\frac{\pi}{4}\right) = 1 \quad \checkmark$$

• $x=\frac{3\pi}{2}$:

$$\tan\left(\frac{3\pi/2}{2}\right) = \tan\left(\frac{3\pi}{4}\right) = -1 \quad \checkmark$$

$$\Rightarrow \boxed{x=0, \frac{\pi}{2}, \frac{3\pi}{2}}$$