# 1. Integration(Revision)

**1.1. u-Substitutions.** Evaluate the following definite integrals.

$$\int (x^2-3)^3.2x\,dx$$

$$\int \frac{x}{\sqrt{x-1}}\,dx$$

**1.2. Integration by Parts.** Evaluate the following definite integrals.

$$\int xe^{2x}\,dx$$

$$\int x^2\sin x\,dx$$

**1.3. Trigonometric Integrals.** Problem-Solving Strategy: Integrating Products and Powers of $\sin x$ and $\cos x$

$$\int \cos^j x\sin^k x\,dx$$

To find the above integral, use the following strategies.

(1) If $k$ is odd, rewrite $\sin^k x = \sin^{k-1}\sin x$ and use the identity $\sin^2 x = 1-\cos^2 x$ to rewrite $\sin^{k-1} x$ in terms of $\cos x$. Integrate using the substitution $u = \cos x$.
(2) If $j$ is odd, rewrite $\cos^j x = \cos^{j-1}\cos x$ and use the identity $\cos^2 x = 1-\sin^2 x$ to rewrite $\cos^{j-1} x$ in terms of $\sin x$. Integrate using the substitution $u = \sin x$. (Note: If both $j$ and $k$ are odd, either strategy 1 or strategy 2 may be used.)
(3) If both $j, k$ are even, then use the formulas $\sin^2 x = \frac{1}{2}-\frac{1}{2}\cos 2x$ and $\cos^2 x = \frac{1}{2}+\frac{1}{2}\cos 2x$. After applying these formulas, simplify and reapply strategies 1 through 3 as appropriate.

Evaluate the following definite integrals.

$$\int \cos^8 x\sin^5 x\,dx$$

$$\int \cos^4 x\sin^4 x\,dx$$

# Lecture 3

**The Dot Product**

Given $\vec{u} =< u_1, u_2, u_3 >$ and $\vec{v} =< v_1, v_2, v_3 >$, the dot product $\vec{u}.\vec{v}$ is defined by

$$\vec{u}.\vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

*Properties of the Dot Product.*

(1) $\vec{u}.\vec{u} = {u_1}^2 + {u_2}^2 + {u_3}^2 = |\vec{u}|^2$
(2) $\vec{u}.\vec{v} = \vec{v}.\vec{u}$
(3) $\vec{u}.(\vec{v} + \vec{w}) = \vec{u}.\vec{v} + \vec{u}.\vec{w}$
(4) $(\vec{v} + \vec{w}).\vec{u} = \vec{v}.\vec{u} + \vec{w}.\vec{u}$
(5) $c(\vec{u}.\vec{v}) = (c\vec{u}).\vec{v} = \vec{u}.(c\vec{v})$
(6) $\vec{0}.\vec{u} = \vec{0}$

.

**Theorem:**

If $\theta$ is the angle between the nonzero vectors $\vec{u}$ and $\vec{v}$, then

$$\vec{u}.\vec{v} = |\vec{u}||\vec{v}|\cos\theta, \text{ or}$$

$$\cos\theta = \frac{\vec{u}.\vec{v}}{|\vec{u}||\vec{v}|}.$$

**Corollary:**

The angle $\theta(0 \leq \theta \leq \pi)$ between two nonzero vectors $\vec{u}$ and $\vec{v}$ is given by

$$\theta = \arccos\left(\frac{\vec{u}.\vec{v}}{|\vec{u}||\vec{v}|}\right)$$

**Notes:**

(1) Two nonzero vectors are orthogonal if the angle between them is $\theta = \frac{\pi}{2}$.
(2) Two nonzero vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u}.\vec{v} = 0$.
(3) The zero vector $\vec{0}$ is orthogonal to all other vectors.

**Orthogonal Projections.**

*Scalar projection of $\vec{u}$ onto $\vec{v}$.*

$$comp_{\vec{v}}\vec{u} = |\vec{u}|\cos\theta = \frac{\vec{u}.\vec{v}}{|\vec{v}|}$$

*Vector projection of $\vec{u}$ onto $\vec{v}$.*

$$\begin{aligned} proj_{\vec{v}}\vec{u} = |\vec{u}|\cos\theta\frac{\vec{v}}{|\vec{v}|} &= \left(|\vec{u}|\frac{\vec{u}.\vec{v}}{|\vec{u}||\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|} \\ &= \left(\frac{\vec{u}.\vec{v}}{|\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|} \end{aligned}$$

**Work Done.** Suppose a constant force $\vec{F}$ is applied to an object resulting in displacement $\vec{D}$. If $\theta$ is the angle between $\vec{F}$ and $\vec{D}$, then the work done by the force is given by

$$W = (|\vec{F}|\cos\theta)|\vec{D}| = \vec{F}.\vec{D}$$

**The Cross Product.** Given two vectors $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$, the cross product $\vec{u}\times\vec{v}$ is given by

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

**Note:**
In general, $(\vec{u}\times\vec{v})\times\vec{w} \neq \vec{u}\times(\vec{v}\times\vec{w})$.

**Theorem 1:**
The vector $\vec{u}\times\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

**Theorem 2:**
If $\theta$ is the angle between two nonzero vectors $\vec{u}$ and $\vec{v}$, then

$$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta.$$

**Corollary 3:**
two nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u}\times\vec{v} = \vec{0}$.

**Note:**
The area of a parallelogram determined by the vector $\vec{a}$ and $\vec{b}$ is given by $A = |\vec{u}\times\vec{v}|$.

*Properties of the Cross Product.*

(1) $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$
(2) $\vec{u}\times\vec{u} = \vec{0}$
(3) $(c\vec{u})\times\vec{v} = c(\vec{u}\times\vec{v}) = \vec{u}\times(c\vec{v})$
(4) $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$
(5) $(\vec{u}+\vec{v})\times\vec{w} = \vec{u}\times\vec{w} + \vec{v}\times\vec{w}$
(6) $\vec{u}.(\vec{v}\times\vec{w}) = (\vec{u}\times\vec{v}).\vec{w}$

.

**Note:**
The cross product is not associative. i.e., In general, $(\vec{u}\times\vec{v})\times\vec{w} \neq \vec{u}\times(\vec{v}\times\vec{w})$.

**Triple Product.** The product $\vec{a}.(\vec{b}\times\vec{c})$ that occurs in Property 6 above is called the scalar triple product of the vectors $\vec{a}, \vec{b}$ and $\vec{c}$. We can write the scalar triple product as a determinant:

$$\vec{a}.(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

**Note:**
The geometric significance of the scalar triple product can be seen by considering the parallelepiped determined by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$. The volume of the parallelepiped is

$$V = \vec{a}.(\vec{b}\times\vec{c}).$$

**Lines in Space.** A line $L$ in three-dimensional space is determined when we know a point $P(x_0, y_0, z_0)$ on $L$, and a vector $\vec{v}$ that is parallel to the line. If $P(x, y, z)$ is an arbitrary point on the line, the vector equation of the line $L$ is given by

$$\vec{r} = \vec{r_0} + t\vec{v}$$

,where $\vec{r}$ and $\vec{r_0}$ are position vectors of $P$ and $P_0$ respectively.

If the vector $\vec{v}$ is given by $\vec{v} = \langle a, b, c \rangle$, the above equation becomes

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle.$$

*The parametric equation for the line $L$.*

$$x = x_0 + at \ , \ y = y_0 + bt \ , \ z = z_0 + ct.$$

*The symmetric equation for the line $L$.*

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

**Planes.** A plane in space is determined by a point $P(x_0, y_0, z_0)$ in the plane and a vector $\vec{n}$ that is orthogonal to the plane. If $P(x, y, z)$ is an arbitrary point on the plane.

*Vector equation of the plane.*

$$(\vec{r} - \vec{r_0}).\vec{n} = 0.$$
$$\langle x - x_0, y - y_0, z - z_0 \rangle.\langle a, b, c \rangle = 0$$

*Scalar equation of the plane.*

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$
$$ax + by + cz + d = 0.$$

**Definition** Two planes are said to be parallel if their normal vectors are parallel (i.e.,$\vec{n_1} = t\vec{n_2}$.) Two planes are said to be orthogonal if their normal vectors are orthogonal(i.e.,$\vec{n_1}.\vec{n_2} = 0$.) If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors.

**Distances.** The distance from point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is given by

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

## Problems

(1) Find the angle between a diagonal of a cube and one of it's edges.

(2) Find the area of the parallelogram with the vertices $A(-3,0), B(-1,6), C(9,5)$ and $D(7,-1)$.

(3) True or False

   (a) $(\vec{u} \times \vec{u}).\vec{u} = 0$

   (b) If $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$ and $\vec{u} \neq \vec{0}$, then $\vec{v} = \vec{w}$.

(4) Find the parametric equation and symmetric equation for the line of intersection of the planes $x + 2y + 2z = 5$ and $-x - 2y + z = 1$.

(5) Find the distance from the point $(1, 0, -1)$ to the line $x = -1 + t, \quad y = 1 + t, \quad z = t$.

(6) Find an equation of the plane that passes through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$ and is perpendicular to the plane $x + y - 2z = 1$.

# Lecture 6

**Cylinders.** Given a curve $C$ in a plane $P$, and a line $L$ not in the plane $P$, a cylinder is the surface consisting of all lines parallel to $L$ and passing through $C$.

**Review of Conic Sections.**

$$\begin{aligned} y &= x^2 \quad \text{Parabola} \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \quad \text{Ellipse} \\ \frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 \quad \text{Hyperbola} \end{aligned}$$

**Surfaces in 3D.** The graph of a 3-variable equation which can be written in the form $F(x, y, z) = 0$ or $z = f(x, y)$ is a surface in 3D.

**Quadratic Surfaces.** A quadratic surface is a 3D surface whose equation is of the second degree. The general equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0,$$

given that

$$A^2 + B^2 + C^2 + D^2 + E^2 + F^2 \neq 0.$$

**Basic Quadratic Surfaces.**

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= 1 \quad \text{Ellipsoid} \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} &= 1 \quad \text{Hyperboloid of One Sheet} \\ -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} &= 1 \quad \text{Hyperboloid of Two Sheets} \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} &= 0 \quad \text{Elliptic Cone} \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} &= z \quad \text{Elliptic Paraboloid} \\ \frac{y^2}{b^2} - \frac{x^2}{a^2} &= z \quad \text{Hyperbolic Paraboloid} \end{aligned}$$

# Lecture 7

**Vector Valued Functions.** A vector valued function is a function of the form

$$\vec{r}(t) =< f(t), g(t), h(t) >,$$

where the component functions $f(t), g(t), h(t)$ are real-valued functions of the parameter $t$.

**Space Curves.** A space curve $C$ is the set of all ordered triplets $(f(t), g(t), h(t))$ together with their defining parametric equations

$$x = f(t), \quad y = g(t), \quad z = h(t).$$

**Limits.** The limit of the vector valued function $\vec{r}(t) =< f(t), g(t), h(t) >$ is defined as

$$\lim_{t \to a} \vec{r}(t) =< \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) >$$

given that the limits of the component functions exist.

**Continuity.** A vector valued function $\vec{r}(t)$ is continuous at $a$ if

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a).$$

**Derivatives.** The derivative $\vec{r'}$ of the vector valued function $\vec{r}(t)$ is defined by

$$\frac{d\vec{r}}{dt} = \vec{r'}(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

**Note:** $\vec{r'}(t)$ gives the rate of change of the function $\vec{r}(t)$ at $t = a$.

**Unit Tangent Vector.** We define the 'unit tangent vector' to the curve $\vec{r}(t)$ by

$$\hat{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|}.$$

**Differentiation Rules.** Differentiation rules for vector valued functions work the way you think they do! Please refer to Lecture Notes for formulas.

**Integrals.** An antiderivative of the vector function $\vec{r}(t) =< f(t), g(t), h(t) >$ is a function $\vec{R}(t) =< F(t), G(t), H(t) >$ such that

$$\vec{R'}(t) = \vec{r}(t).$$

The indefinite integral of $\vec{r}(t)$ is defined by

$$\int \vec{r}(t) dt = \vec{R}(t) + \vec{C}.$$

The definite integral of $\vec{r}(t)$ over $[a, b]$ is defined by

$$\int_a^b \vec{r}(t) dt = \vec{R}(b) - \vec{R}(a).$$

**Arc Length.** Consider a curve $\vec{r}(t) =< f(t), g(t), h(t) >$ , $a \leq t \leq b$, where $f', g'$ and $h'$ are continuous. If the curve is traversed exactly once as $t$ increases from $a$ to $b$, then its length is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt = \int_a^b |\vec{r'}(t)|$$

**Arc Length Function.** Suppose $C$ is a piecewise smooth curve given by a vector-valued function $\vec{r}(t) =< f(t), g(t), h(t) >$, $a \leq t \leq b$. The arc length functions is defined by

$$s(t) = \int_a^t |\vec{r'}(u)| du.$$

The fundamental theorem of calculus tells us that $s$ is a differentiable function of $t$ and

$$\frac{ds}{dt} = |\vec{r'}(t)|.$$

**Curvature.** The curvature of a curve is defined by

$$\kappa = \left| \frac{d\hat{T}}{ds} \right|.$$

We use the following two formulas to calculate the curvature.

$$\kappa(t) = \frac{|\hat{T}(t)|}{|\vec{r'}(t)|}$$

$$\kappa(t) = \frac{|\vec{r'}(t) \times \vec{r''}(t)|}{|\vec{r'}(t)|^3}$$

**Problems.**

(1) If $r(t) =< ln(2t), t, t^2 >$, find the following.

(a) $\vec{r'}(t)$.

(b) $\vec{r''}(t)$.

(c) $|\vec{r'}(t) \times \vec{r''}(t)|$.

(d) $\hat{T}(t)$.

(e) $\hat{T}(1)$.

(f) The curvature of the curve $r(t) =< \ln(2t), t, t^2 >$.

(2) Find the length of the curve $r(t) =< 2t, t^2, \frac{1}{3}t^2 >$, $0 \le t \le 1$.

(3) Find the unit tangent vector $\hat{T}(t)$, and the unit normal vector $\vec{N}(t)$ of the curve $r(t) =< 3\cos(t), 3\sin(t), 6 >$ at the point $P(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 6)$.

(4) Find an equation for the surface consisting of all points that are equidistant from the point $(2, 0, 0)$ and the plane $x = -2$.

(5) Find the limit

$$\lim_{t\to 0} \left\langle \frac{3e^t - 3}{t}, \frac{\sqrt{t+9} - 3}{t}, \frac{2}{t+1} \right\rangle.$$

# Lecture 11

**Limits.** Intuitively, $\lim_{(x,y)\to(a,b)} f(x,y) = L$ means that as $(x,y)$ gets very close to $(a,b)$, $f(x,y)$ gets very close to $L$.

**Strategy for Evaluating Limits.** Limits can be a little daunting because there are many things you can try. My advice is to follow the steps below. Step (1) should always be your first, and step (5) should probably be your last. The rest can be done in any order:

**1) Check for continuity at $(a,b)$.** If $f(x,y)$ is continuous at $(a,b)$, then $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. Practically, in most cases this means that if $f(x,y)$ is NOT a piecewise function and there is no problem with plugging in $(a,b)$, then the limit is $f(a,b)$.

**2) Convert to polar coordinates.** Only works if $(x,y)$ approaches zero. Then you can convert the limit to $\lim_{r\to 0} f(r\cos(\theta), r\sin(\theta))$. Don't forget that $x^2 + y^2 = r^2$. You may or may not have to use the Squeeze Theorem to finish.

**3) Try algebraic manipulations.** Try factoring, rationalizing denominator, or simplifying in some way.

**4) Approach $(a,b)$ along various paths.** If $f(x,y)$ approaches different things (or nothing) as $(x,y) \to (a,b)$ along various paths, then the limit doesn't exist. For the moment, assume $(a,b) = (0,0)$. The paths you should use are:

(1) Along the $x$-axis. In this case $y$ is always zero, so the limit becomes $\lim_{x\to 0} f(x,0)$.
(2) Along the $y$-axis. In this case $x$ is always zero, so the limit becomes $\lim_{y\to 0} f(0,y)$.
(3) Along the line $y = mx$, where $m$ is a constant. The limit becomes $\lim_{x\to 0} f(x, mx)$.
(4) Along the parabola $y = cx^2$, or along the parabola $x = cy^2$, where $c$ is a constant. The limit becomes $\lim_{x\to 0} f(x, cx^2)$ in the first case, and $\lim_{y\to 0} f(cy^2, y)$ in the second case.

For general $(a,b)$, you can transform the paths above from going through $(0,0)$ to going through $(a,b)$. The corresponding limits are
$(1)'$ $\lim_{x\to a} f(x,b)$
$(2)'$ $\lim_{y\to b} f(a,y)$
$(3)'$ $\lim_{x\to a} f(x, m(x-a)+b)$
$(4)'$ $\lim_{x\to a} f(x, c(x-a)^2+b)$ or $\lim_{y\to b} f(c(y-b)^2+a, y)$

**5) Use the Squeeze Theorem.** If you can find two functions $g(x,y), h(x,y)$ that squeeze $f(x,y)$ (that is, $g(x,y) \le f(x,y) \le h(x,y)$) and $L = \lim_{(x,y)\to(a,b)} g(x,y) = \lim_{(x,y)\to(a,b)} h(x,y)$, you can conclude that $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

**Continuity.** As I already alluded to, $f(x,y)$ is continuous at $(a,b)$ means that $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and equals $f(a,b)$.

## Exercises

1) Find the domain and range of $f(x,y) = \ln(x) + \ln(y-1)$.

2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{x^2+2xy+5}$

3) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$

4) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x-y}{\sqrt{x}-\sqrt{y}}$

5) $\lim\limits_{(x,y)\to(0,1)} \dfrac{\arccos(x/y)}{1+xy}$

6) $\lim\limits_{(x,y)\to(0,0)} \dfrac{1}{x+y}$

7) $\lim\limits_{(x,y)\to(0,0)} \dfrac{5x^2y}{x^2+y^2}$

8) Can $f(x,y) = \dfrac{xy}{(x^2+y^2)^2}$ be defined at $(0,0)$ so that it becomes continuous on $\mathbb{R}^2$?

9) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{x^2-y^2}$

10) Can $\lim\limits_{(x,y)\to(0,0)} \dfrac{4x^2y^2}{x^2+y^2}$ be defined at $(0,0)$ so that it becomes continuous on $\mathbb{R}^2$?

## Extra Problem

Find the domain and range of $g(x,y) = \frac{|x|+|y|}{1+|x|+|y|}$.

## Lecture 12

**Partial Derivatives**
Consider the function $f(x, y)$. The partial derivative with respect to $x$ of $f$ is denoted by

$$\frac{\partial f}{\partial x} = f_x = D_x f = D_1 f$$

and similarly for the partial w.r.t. $y$. The partials are functions of $x$ and $y$, so you could put $(x, y)$ next to each of them to emphasize that. But we often don't. If the function is written as $z = f(x, y)$, you can may see the $f$ replaced with a $z$.

In this class, we will pretty much always use the first two notations.

**Geometric Interpretation**
Consider the function $z = f(x, y)$ and the point $(a, b)$ in the domain. Intersect the graph of $f(x, y)$ with the plane $y = b$. In the plane, you will see a curve, which has input $x$ and output $z$. The slope of the tangent line to that curve at $x = a$ is precisely $f_x(a, b)$.

**Higher Order Partial Derivatives**
Partial derivatives are themselves functions of $x$ and $y$, so we can take *their* partials. Remember that in the different notations, the order of the variables is reversed. So

$$\frac{\partial^4 f}{\partial x^2 \partial y \partial x} = f_{xyxx}.$$

Thess both mean that you first take partial w.r.t. $x$, then $y$, then $x$, then $x$ again.
It turns out that (under some technical assumptions which, unless you have a piecewise function, you don't usually worry about), the mixed partial derivatives are equal. I.e. $f_{xy} = f_{yx}$.

**Tangent Planes and Linear Approximations**
The tangent plane of a function $f(x, y)$ at the point $(a, b)$ is

$$z = L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Suppose we want to approximate $f(x, y)$ at a point $(\bar{a}, \bar{b})$ that is close to $(a, b)$. Then we could just compute $L(\bar{a}, \bar{b})$. That is,

$$f(\bar{a}, \bar{b}) \approx L(\bar{a}, \bar{b}).$$

You may also see the following (same idea/equation, different notation):

$$f(\bar{a}, \bar{b}) \approx f(a, b) + df$$

where

$$df = f_x dx + f_y dy.$$

Here, the partials $f_x, f_y$ are evaluated at $(a, b)$ and $dx = \bar{a} - a$, and $dy = \bar{b} - b$.

**Theorems**
So far, we discussed partial derivatives of a function $f(x, y)$ at a point $(a, b)$. There actually a different thing called the derivative of $f(x, y)$ at a point $(a, b)$. While the partial derivatives are ultimately numbers, the derivative is actually a matrix. The formal definition of the derivative is a little bit technical, so it wasn't given in the notes, and you don't have to worry about it. The informal definition is more simple: $f(x, y)$ is differentiable at $(a, b)$ if you can draw a tangent plane to the graph at $(a, b, f(a, b))$.

**Theorem 1 (Conditions for Differentiability):** Consider $f(x, y)$. If $f_x, f_y$ exist near $(a, b)$ (including at $(a, b)$ itself) and they are continuous at $(a, b)$, then $f$ is differentiable at $(a, b)$.

**Theorem 2: (Differentiability Implies Continuity)** If $f(x, y)$ is differentiable at $(a, b)$, then it is continuous at $(a, b)$.

There are the tools you'll need to tackle questions of the type "Is $f(x, y)$ differentiable at $(a, b)$?" You have two possibilities:

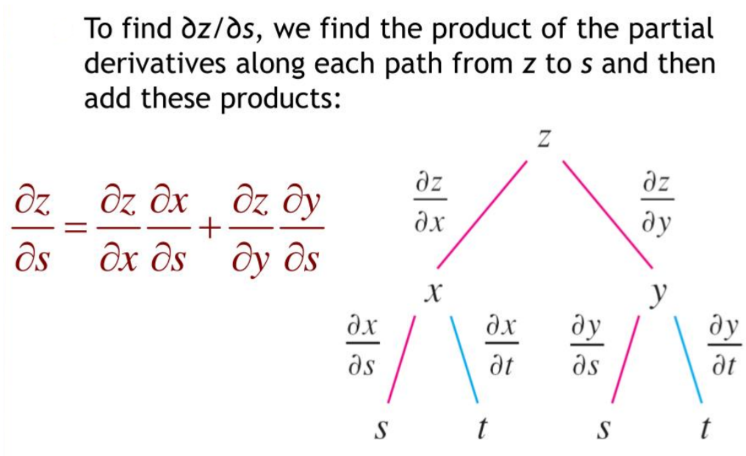
$a$) If you can show that $f$ is discontinuous at $(a, b)$, you can conclude that it's not differentiable, by citing the statement of theorem 2.

$b$) If you can show that the partials exist near $(a, b)$ and are continuous at $(a, b)$ then you can conclude $f$ is differentiable, by citing the statement of theorem 1.

# Lecture 14

**Chain Rule**

Here, it is just best to follow the tree diagram. Using it, you can find the derivatives of any combination of functions. Here's an example:




**Implicit Differentiation**

We can use partial derivatives to solve implicit differentiation problems. If $F(x, y) = 0$ and $y$ is a function of $x$, then

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

provided $F_y \neq 0$. If $F(x, y, z) = 0$ and $z$ is a function of $(x, y)$, then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

**Gradient and Partial Derivative**
Everything in this section applies to functions of $3, 4$, or any number of variables, but we will just concentrate on functions of two variables $z = f(x, y)$.

The gradient of $f(x, y)$ is the vector $\langle f_x(x, y), f_y(x, y)\rangle$ and it is denoted by $\nabla f(x, y)$.

If $\vec{v} = \langle v_1, v_2\rangle$ is a unit vector, the directional derivative in the direction of $v$ is defined by

$$D_{\vec{v}}f(a, b) = \lim_{t\to 0} \frac{f(a + tv_2, b + tv_2) - f(a, b)}{t}.$$

Notice that if $\vec{v} = \langle 1, 0\rangle$ or $\vec{v} = \langle 0, 1\rangle$ then $D_{\vec{v}}f(a, b)$ is $f_x(a, b)$ or $f_y(a, b)$, respectively.

**Theorem:** If $f$ is a differentiable function of $x$ and $y$, then $f$ has a directional derivative in the direction of any unit vector $\hat{v} = \langle v_1, v_2\rangle$ and

$$D_{\vec{v}}f(a, b) = f_x(a, b)v_1 + f_y(a, b)v_2$$

or

$$D_{\vec{v}}f(a, b) = \nabla f(a, b) \cdot \vec{v}.$$

**Geometric Interpretation of Gradient**
Suppose you're standing at a point $(a, b, f(a, b))$ on the graph of $f(x, y)$, and you want to climb to a higher point as quickly as possible. Which direction should you go in? The answer is $\nabla f(a, b)$! Notice that for a function $f(x, y)$, the gradient is a vector with two components. So it gives you the direction you should follow in the *domain*, not in 3D space.

If you want to go to a lower point as quickly as possible, you should go in the direction $-\nabla f(a, b)$. If you want to maintain the same altitude, you should go in a direction perpendicular to $\nabla f(a, b)$.

**Theorem:** Let $f$ be differentiable at the point $P(a, b)$ with $\nabla f(a, b) \neq 0$.

- $f$ has its maximum rate of increase at $P$ in the direction of the gradient $\nabla f(a, b)$. The rate of increase in this direction is $|\nabla f(a, b)|$.
- $f$ has its maximum rate of decrease at $P$ in the direction of the gradient $-\nabla f(a, b)$. The rate of decrease in this direction is $-|\nabla f(a, b)|$.
- $D_{\vec{v}}f(a, b) = 0$ in any direction orthogonal to $\nabla f(a, b)$.

**Gradients and Tangent Planes**

The main concept of this lecture is that the gradient is always perpendicular to level curves. That is, if you have a level curve $L$ which goes through a point $(a, b)$, then $\nabla f(a, b)$ is perpendicular to the tangent line (or tangent plane) of $L$ at $(a, b)$. This fact is very useful for finding the tangent lines/planes of curves/surfaces.

**Tangent Planes**

- If a surface in $\mathbf{R}^3$ has the form $z = f(x, y)$, then the tangent plane to the surface at the point $P(a, b, f(a, b))$ can be found using
$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b).$$
- If a surface in $\mathbf{R}^3$ has the form $F(x, y, z) = k$, then the tangent plane to the surface at the point $P(a, b, c)$ can be found using
$$\nabla F(a, b, c).\langle x - a, y - b, z - c\rangle = 0.$$

**Normal Lines**

The normal line to the surface $F(x, y, z) = 0$ at $P(a, b, c)$ is given by
$$\frac{x - a}{F_x(a, b, c)} = \frac{y - b}{F_y(a, b, c)} = \frac{z - c}{F_z(a, b, c)},$$
given that $\nabla F(a, b, c) \neq 0$.

## Lecture 17

**Critical Points** An interior point $(a, b)$ in the domain of $f$ is a critical point if either
- $f_x(a, b) = f_y(a, b) = 0$, or
- $f_x$ or $f_y$ does not exist at $(a, b)$.

**Theorem:** If $f$ has a local max or min value at $(a, b)$ and $f_x$ and $f_y$ exist near $(a, b)$, then $f_x(a, b) = f_y(a, b) = 0$.

**Important!** The converse of the above theorem not true!

**Saddle Points** A function $f$ has a saddle point at a critical point $(a, b)$ if $f_x(a, b) = f_y(a, b) = 0$ but $f$ does not have a local extremum at $(a, b)$.

Therefore, if $(a, b)$ is a critical point of $f$, then $(a, b)$ is either a local min, a local max, or a saddle point.

**Second Derivative Test**

Suppose $(a, b)$ is a critical point of $f(x, y)$. Let $D$ be the determinant of the matrix

$$\begin{bmatrix} f_{xx}(a,b) & fyx(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{bmatrix}$$

called the hessian matrix. That is, $D = f_{xx}(a, b)f_{yy}(a, b) - f_{yx}(a, b)f_{xy}(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{yx}(a, b)^2$. $D$ is called the discriminant.

(1) If $D > 0$ and $f_{xx}(a, b) < 0$ then $(a, b)$ is a local max.
(2) If $D > 0$ and $f_{xx}(a, b) > 0$ then $(a, b)$ is a local min.
(3) If $D < 0$ then $(a, b)$ is a saddle point.
(4) If $D = 0$ the test is inconclusive.

**Extreme Value Theorem**
The Extreme Value Theorem tells us that if we have a function on a closed and bounded domain (such as the unit disk, for example) then that function will achieve a global (i.e. absolute) minimum and a maximum, and these will be achieved either at a critical point or on the boundary.

**Lagrange's Theorem.** Let $f$ and $g$ have first partial derivatives such that $f$ has an extremum at a point $(a, b)$ on the smooth constraint curve $g(x, y) = c$. If $\nabla g(a, b) \neq \vec{0}$, then there is a real number $\lambda$ such that

$$\nabla f(a, b) = \lambda \nabla g(a, b),$$

where the scalar $\lambda$ is called a *Lagrange multiplier.*

**Method of Lagrange Multipliers.** To find the maximum and minimum values of $f(x, y)$ subject to the constraint $g(x, y) = k$ (assuming that these extreme values exist and $\nabla g \neq \vec{0}$ on the surface $g(x, y) = k$):

(1) Find all values of $x, y$ and $\lambda$ such that

$$f(x, y) = \lambda \nabla g(x, y)$$

and

$$g(x, y) = k.$$

(2) Evaluate $f$ at all the points $(x, y)$ that result from step (1). The largest of these values is the maximum value of $f$ and the smallest is the minimum value of $f$.

**Note:** The same method can be applied for a function with three variables(or more than three).

**Note:** If you want to find maximum and minimum values of $f(x, y, z)$ subject to two constraints, say $g(x, y, z) = C_1$ and $h(x, y, z) = C_2$, then you need to solve

$$f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$$
$$g(x, y, z) = C_1$$
$$g(x, y, z) = C_1.$$

# Lecture 19

## Double Integrals over Rectangles

**Definition.** The double integral of $f(x,y)$ over the rectangle $R$ is

$$\int\int_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}{}^* y_{ij}{}^*)\Delta A.$$

**Note:** If $f(x,y) \geq 0$, then the volume $V$ of the solid that lies above the rectangle $R$ and below the surface $z = f(x,y)$ is

$$V = \int\int_R f(x,y)dA.$$

**Fubini's Theorem.** If $f$ is continuous on the rectangle $R = \{(x,y)|a \leq x \leq b, c \leq y \leq d\}$, then

$$\int\int_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy.$$

**Note:** If $f(x,y) = g(x)h(y)$ on $R = [a,b] \times [c,d]$, then

$$\int\int_R f(x,y)dA = \int\int_R g(x)h(y)dxdy = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right).$$

# Lecture 20

## Double Integrals over General Regions

Consider a function and a general plane region $D$. To evaluate the integral

$$\int\int_D f(x,y)dA,$$

we classify the region $D$ into two types:

**Type I:** $D = \{(x,y)|a \le x \le b,\ g_1(x) \le y g_2(x)\}$. In this case,

$$\int\int_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx.$$

**Type II:** $D = \{(x,y)|c \le y \le d,\ h_1(y) \le x h_2(y)\}$. In this case,

$$\int\int_D f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy.$$

**Notes:**

(1) If we integrate the constant function $f(x,y) = 1$ over a region $D$, we get the area of $D$.

$$A(D) = \int\int_D 1dA$$

(2) The techniques developed in this lecture can also be used to determine the volume between two continuous surfaces $z_1 = f(x,y)$ and $z_2 = g(x,y)$ with $g(x,y) \le f(x,y)$ on a region $D$ in the xy-plane.

$$V = \int\int_D [f(x,y) - g(x,y)]\,dA.$$

**Exercises:**

(1) Sketch the region of integration and change the order of integration.

(a)
$$\int_0^1 \int_0^y f(x,y)dxdy.$$

(b)
$$\int_{-2}^2 \int_0^{\sqrt{4-y^2}} f(x,y)dxdy.$$

(c)
$$\int_0^2 \int_{x^2}^4 f(x,y)dydx.$$

(2) Evaluate
$$\int_0^1 \int_x^1 \sin(y^2)dydx.$$

(3) Evaluate
$$\int_0^1 \int_{x^2}^1 \sqrt{y}\sin(y)dydx.$$

(4) Find the volume in the first octant bounded by $y^2 = 4 - x$ and $y = 2z$.

(5) Find the volume of the solid bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0$, $z = 0$ in the first octant.

LECTURE 21

**Polar Coordinates**
*Polar to rectangular*:

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

*Rectangular to Polar*:

$$r = \sqrt{x^2+y^2} \qquad \tan(\theta) = \frac{x}{y}$$

**Cylindrical Coordinates**
*Cylindrical to rectangular*:

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad z = z$$

*Rectangular to Cylindrical*

$$r = \sqrt{x^2+y^2}\tan(\theta) = \frac{x}{y} \qquad z = z$$

**Spherical Coordinates**
*Spherical to rectangular*:

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi$$

*Rectangular to spherical*:

$$\rho = \sqrt{x^2+y^2+z^2} \qquad \tan(\theta) = \frac{y}{x} \qquad \cos\phi = \frac{z}{\rho}$$

LECTURE 22

**Jacobian** Suppose $x = g(u,v), y = h(u,v)$ is a transformation from the $uv$ plane to the $xy$ plane. The **Jacobian** (or more accurately, Jacobian Determinant) is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Similarly, if we have a transformation $x = g(u,v,w), y = h(u,v,w), z = k(u,v,w)$ of $uvw$ space to $xyz$ space, then the Jacobian is

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

**Change of Variables**

Suppose $x = g(u,v), y = h(u,v)$ is a transformation that maps a region $S$ in the $uv$ plane onto a region $R$ in the $xy$ plane. Then

$$\iint_R f(x,y)dA = \iint_S f(g(u,v),h(u,v))|J(u,v)|dA$$

Similarly, suppose $x = g(u,v,w), y = h(u,v,w), z = k(u,v,w)$ is a transformation that maps a region $S$ in $uvw$ space onto a region $R$ in $xyz$ space. Then

$$\iiint_A f(x,y,z)dV = \iint_S f(g(u,v,w),h(u,v,w),k(u,v,w))|J(u,v,w)|dV$$

**Double Integrals Over Polar Regions**

Here, we apply the Change of Variables Theorem to the transformation of polar coordinates. Suppose you can write a region $R$ in the $xy$ plane using polar coordinates as $a \leq \theta \leq b$, $g(\theta) \leq r \leq h(\theta)$ where $0 < b - a \leq 2\pi$. Then

$$\iint_R f(x,y)dA = \int_a^b \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta) r dr d\theta$$

**Areas and Volumes Over Polar Regions**

If a solid is bounded by the surface $z = f(r,\theta)$ over a polar region $D = \{(r,\theta)|a \leq \theta \leq b, 0]leg(\theta) \leq r \leq h(\theta)\}$, the volume of the solid is

$$V = \int_a^b \int_{g(\theta)}^{h(\theta)} f(r,\theta) r dr d\theta$$

and the area of the polar region $D$ is

$$A = \int_a^b \int_{g(\theta)}^{h(\theta)} r dr d\theta$$

EXERCISES

(1) Describe the set $S = \{(x,y)|0 \leq x \leq 8, 0 \leq y \leq \sqrt{8x - x^2}\}$ in polar coordinates.

(2) Describe the set $S = \{(x,y,z)|x = 0, y \geq 0, -\sqrt{1-y^2} \leq z \leq \sqrt{1-y^2}\}$ in spherical coordinates.

(3) Find
$$\iint_R (x-y)dA$$
where $R$ is the parallelogram joining the points $(1,2)$, $(3,4)$, $(4,3)$, and $(6,5)$ using the change of variable $x = \frac{3u-v}{2}, y = \frac{u-v}{2}$.

(4) Using $x = v$ and $y = \sqrt{u+v}$, evaluate
$$\iint_R y\sin(y^2 - x)dA$$
where $r$ is the region bounded by $y = \sqrt{x}, x = 2$, and $y = 0$.

(5) Find the volume of the solid that lies under the paraboloid $z = 1 - x^2 - y^2$ and above the unit circle in the $xy$ plane.

(6) A pizza slice has an angle of 30 degrees, a radius of 5 inches, and an average height of 0.5 inches. Approximate its volume using integrals.

## Lecture 24

**Volume and Triple Integrals** If $S$ is a solid in $3D$ space, its volume is given by

$$\iiint_S 1 \, dV$$

**Fubini's Theorem** If $f$ is continuous on $B = [a,b] \times [c,d] \times [r,s]$ then

$$\iiint_B f(x,y,z)dV = \int_a^b \int_c^d \int_r^s f(x,y,z)dzdydx$$

**How to Calculate Triple Integrals** Given an integrand $f(x,y,z)$ and a region $S$, you have to write $S$ in one of the 6 different types. For example, one of the types is $S = \{(x,y,z)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x), h_1(x,y) \leq z \leq h_2(x,y)\}$. Using this type, we have

$$\iiint_S f(x,y,z) = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)dzdydx.$$

## Lecture 25

**When to Transform to Cylindrical Coordinates** A good rule of thumb is that you want to transform whenever the integrand has lots of $(x^2+y^2)$'s and/or when the region of integration involves a cylinder, or some other solid of revolution.

**How to Transform to Cylindrical Coordinates** You want to describe the region if integration as $a \leq \theta \leq b, g_1(\theta) \leq r \leq g_2(\theta), h_1(r,\theta) \leq z \leq h_2(r,\theta)$. If you are successfully, you can do

$$\iiint f(x,y,z)dV = \int_a^b \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r,\theta)}^{h_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)rdzdrd\theta$$

## Lecture 26

**When to Transform to Spherical Coordinates** You want to transform the spherical if the integrand contains $x^2+y^2+z^2$ and/or the region of integration has to do with a sphere or a cone.

**How to Transform to Spherical Coordinates** You want to describe the region if integration as $a \leq \theta \leq b, g_1(\theta) \leq \phi \leq g_2(\theta), h_1(\theta,\phi) \leq \rho \leq h_2(\theta,\phi)$. If you are successfully, you can do

$$\iiint f(x,y,z)dV = \int_a^b \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(\theta,\phi)}^{h_2(\theta,\phi)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\theta)\rho^2\sin\phi d\rho d\phi d\theta$$

## Exercises

(1) Evaluate
$$\iiint_E 2x + 5y + 7z dV$$
where $E = \{(x, y, z)|0 \leq x \leq 1, 0 \leq y \leq -x + 1, 1 \leq z \leq 2\}$.

(2) Consider the solid enclosed by $x^2 + y^2/9 = 4$, $z = -10$, $z = 10$. Write it's volume as an integral in 3 different ways, using $dxdydz$, $dydxdz$, and $dzdxdy$.

(3) Consider
$$\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z) dzdydx.$$
Change the order of integration to $dxdzdy$.

(4) Evaluate
$$\iiint_E \sqrt{x^2 + z^2} dV$$
where $E$ is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

(5) Find the volume of the solid $E$ that lies under the plane $2x + y + z = 8$ and whose projection in the $xy$ plane is bounded by $y = 0, x = \pi/2$, and $y = \sin(x)$.

(6) Let $E$ be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. Set up an integral to find the volume using cylindrical coordinates.

(7) Set up an integral to find the volume of the region bounded by the cone $z = \sqrt{(x^2 + y^2)}$ and the hemisphere $z = \sqrt{4 - x^2 - y^2}$

# Lecture 27

## Vector Fields

**Definition(Vector Field)**

- Let $D$ be a set in $\mathbb{R}^2$ (a plane region). A vector field on $\mathbb{R}^2$ is a function $\vec{F}$ that assigns to each point $(x, y)$ in $D$ a two-dimensional vector $\vec{F}(x, y)$.
- Let $E$ be a set in $\mathbb{R}^3$ (a plane region). A vector field on $\mathbb{R}^3$ is a function $\vec{F}$ that assigns to each point $(x, y, z)$ in $E$ a three-dimensional vector $\vec{F}(x, y, z)$.

**Note:** Suppose $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$. The vector field $\vec{F}$ is continuous or differentiable on $D$ if $P$ and $Q$ are continuous and differentiable on $D$ (similarly for 3-D).

**Definition(Gradient Fields)**

A vector field $\vec{F}$ is said to be a conservative (or gradient) field if there exists a differentiable function $f$ such that $\vec{F} = \nabla f$. The function $f$ is called the potential function for $\vec{F}$.

**Cross Partial Property**

Let $P, Q$ and $R$ have continuous first partial derivatives. If the vector field $\vec{F} = \langle P, Q, R\rangle$ is conservative, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \text{ and } \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

**Note:** $\vec{F}$ does not have the cross-partial property $\Longrightarrow$ $\vec{F}$ is not conservative.

**Theorem**

Let $P, Q$ and $R$ have continuous first partial derivatives on an open simply connected region $D$. The vector field $\vec{F} = \langle P, Q, R\rangle$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \text{ and } \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} \text{ on } D.$$

**Note:** $\mathbb{R}^2$ and $\mathbb{R}^3$ are open simply connected.

## Lecture 28

## Line Integrals

**Scalar Line Integrals in the Plane**
If $f$ is defined on a smooth curve $C$, then the line integral of $f$ along $C$ is is

$$\int_C f(x,y)ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i,y_i)\Delta s$$

,if this limit exists.

**Theorem**
If the curve $C$ is given by $\vec{r}(t) = \langle x(t), y(t)\rangle, a \leq t \leq b$, then

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt = |\vec{r'}(t)|dt.$$

Therefore,

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))|\vec{r'}(t)|dt.$$

**Scalar Line Integrals in Space**
If the curve $C$ is given by $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle, a \leq t \leq b$, then

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))|\vec{r'}(t)|dt.$$

**Vector Line Integrals**
The line integral of a vector field $\vec{F} = \langle P, Q, R\rangle$ over an oriented curve $C$ parametrized by $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle, a \leq t \leq b$ can be expressed as

$$\begin{aligned}\int_C \vec{F}.\vec{T}ds &= \int_a^b \vec{F}.\vec{r'}(t)dt\\ &= \int_a^b \Big(Px'(t) + Qy'(t) + Rz'(t)\Big)dt\\ &= \int_C Pdx + Qdy + Rdz\\ &= \int_C \vec{F}.d\vec{r}.\end{aligned}$$

**Application**
Consider an object moving along a path $C$ in the vector field $\vec{F}$. The total work done by the particle is given by

$$W = \int_C \vec{F}.\vec{T}ds.$$

## Lecture 29

## The Fundamental Theorem for Line Integrals

**Definitions**

- A curve is 'closed' if it intersects itself at its endpoints.
- A curve is called 'simple' if it intersects itself only at its endpoints.
- A region $D$ is 'open' if it does not contain any of its boundary points.
- An open region $D$ is connected if any two points in $D$ can be connected by a continuous curve lying entirely in $D$.
- An open region $D$ is 'simply connected' if every closed curve in $D$ enclosed only points lying in $D$.

**The Fundamental Theorem for Line Integrals**

Let $C$ be a smooth curve with parametrization $\vec{r}(t), a \leq t \leq b$ lying in an open connected region $D$. Let $f$ be a function of two or three variables with first-order partial derivatives that exist and are continuous on $C$. Then,

$$\int_C \nabla f.d\vec{r} = f|_{\vec{r}(a)}^{\vec{r}(b)} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

**Notes:**

- If $\vec{F}$ is a conservative vector field and $f$ is a potential function of $F$, then
$$\int_C \vec{F}.d\vec{r} = \int_C \nabla f.d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$
- If $\vec{F}$ is a conservative vector field and $C$ is a closed curve, then
$$\int_C \vec{F}.d\vec{r} = 0.$$

**Theorem (Independence of Path)**

If Let $\vec{F}$ is continuous on an open connected region, then the line integral $\int_C \vec{F}.d\vec{r}$ is independent of path if and only if $\vec{F}$ is conservative.

**Note:**

Let $\vec{F}$ be continuous on an open simply connected region $D$. The following statements are equivalent.

(1) $\vec{F}$ is conservative on $D$.
(2) $\int_C \vec{F}.d\vec{r}$ is independent of path $C$ in $D$.
(3) $\oint_C \vec{F}.d\vec{r} = 0$ for every closed path $C$ in $D$.
(4) Cross-partial property holds.

## Lecture 30

## Green's Theorem

**Green's Theorem**

Let $D$ be an open simply connected region with a boundary curve $\delta D$ that is a piecewise smooth curve oriented positively. Let $\vec{F} = \langle P, Q \rangle$ be a vector field with component functions that have continuous partial derivatives on $D$. Then,

$$\oint_{\delta D} \vec{F}.d\vec{r} = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

**Theorem**

If $D$ is a plane region bounded by a piecewise smooth simple closed curve $C$, oriented counter-clockwise, then the area of the region $D$ is

$$A = \int_C x dy = -\int_C y dx = \frac{1}{2} \int_C (x dy - y dx)$$

**Exercises**

(1) Determine if $\vec{F} = \langle y+z, x+z, x+y \rangle$ is conservative and if so, find its potential function.

(2) An object moves from $(1,1,1)$ to $(2,4,8)$ along the path $\vec{r}(t) = \langle t, t^2, t^3 \rangle$, subject to the force $\vec{F}(x,y,z) = \langle \sin x, \sin y, \sin z \rangle$. Find the work done.

(3) Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F}(x,y,z) = \langle 2x\ln(y), \frac{x^2}{y} + z^2, 2yz \rangle$ and $C$ is a curve with parametrization $\vec{r}(t) = \langle t^2, t, t \rangle$, $1 \le t \le e$.

(4) Use Green's Theorem to evaluate $\oint_C y^2 dx + x^2 dy$, where $C$ is the positive oriented boundary of the region determined by $y = x^2$ and $y = 1$.

(5) Use Green's Theorem to evaluate $\int_C \vec{F}.d\vec{r}$, where $\vec{F}(x,y) = \langle y\cos(x) - xy\sin(x), xy + x\cos(x) \rangle$ and $C$ is the triangle from $(0,0)$ to $(0,4)$ to $(2,0)$ $(0,0)$.

## Lecture 31

### Parametric Surfaces

In the past, we've written curves parametrically as $r(t) = \langle x(t), y(t), z(t)\rangle$. Similarly, we can write surfaces as $r(s,t) = \langle x(s,t), y(s,t), z(s,t)\rangle$.

**Partial Derivatives** The partial derivatives are defined as

$$\vec{r}_s(s,t) = \langle x_s(s,t), y_s(s,t), z_s(s,t)\rangle$$
$$\vec{r}_t(s,t) = \langle x_t(s,t), y_t(s,t), z_t(s,t)\rangle$$

$\vec{r}_s(s,t), \vec{r}_t(s,t)$ are vectors tangent to the surface at the point $r(s,t)$. If $\vec{r}_s \times \vec{r}_t$ is never 0, then the surface is smooth, and it has a tangent plane at every point.

**Surface Area** Suppose $(r,t)$ come from a region $D$ in the $st$ plane. The area of the surface $r(s,t)$ is

$$\text{Surface Area} = \iint_S dS = \iint_D |\vec{r}_s \times \vec{r}_t| dA$$

as long as every point of $S$ corresponds exactly to one point of $D$.
If we have a surface described by $z = f(x,y)$, then it can be parametrized by $(x, y, f(x,y))$, and we can use the formula above to get

$$\text{Surface Area} = \iint_D \sqrt{f_x(x,y) + f_y(x,y) + 1} dA$$

In the past, we've integrated a function $f(x, y, z)$ over a curve. Conceptually, what we did was we divided the curve into $n$ small pieces $p_i$, found the function value at a point $P_i$ of $p_i$, found $\sum_{i=1}^{n} f(P_i) \cdot \text{length}(p_i)$, and let $n \to \infty$. We can apply a similar concept to define the integral of $f(x, y, z)$ over a surface $S$. Divide $S$ into $n$ small pieces $S_i$, pick a point $P_i$ from each $S_i$, find $\sum_{i=1}^{n} f(P_i) \cdot \text{area}(S_i)$, and let $n \to \infty$.
It turns out that the integral of $f(x, y, z)$ over $S$ as defined above is

$$\iint_S f(x, y, z) dS = \iint_D f(r(s, t)) |r_s(s, t) \times r_t(s, y)| dA$$

**Orientation of a Surface** Most surfaces have 2 sides. We can pick which side we want to be the "positive side" and which side to be the "negative side". If the surface is closed (like a ball), then by convention, we always pick the outside as the positive side, and the inside as the negative side. Now suppose we have a parametrization $r(s, t)$ of $S$. If the normal vector $\vec{r_s} \times r_t$ is always pointing towards the positive side, then the parametrization is positive. If the normal vector is pointing to the negative side, then the parametrization is negative.

**Integrating a Vector Field Over a Surface** Let $F$ be a vector field. We define the integral of $F$ over $S$ as

$$\iint_S F \cdot (\vec{r_s}(s, t) \times r_t(s, t)) dS$$

The interpretation is that if we have a gas in $3D$ with velocity field $F$, then the rate at which gas passes through $S$ is the integral of $F$ over $S$. This is called the *flux*.

# LECTURE 33

Let $F(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$ be a vector field.

$$\text{curl } F = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$$

Note that curl $F = \nabla \times F$. If curl $F = 0$, then $F$ is *irrotational.*

**Theorem** If $F$ is conservative, then curl $F = 0$. If the domain is simply connected, then the converse holds as well. That is, if curl $F = 0$, then $F$ is conservative.

**Stoke's Theorem** (without the technical assumptions) Let $S$ be a surface whose boundary $\partial S$ has positive orientation. Then

$$\int_{\partial S} F \cdot dr = \iint_S \text{curl } F \cdot \hat{n} dS$$

## Exercises

(1) Give a parametrization of the cone $x^2 + y^2 = z^2$ lying on or above the plane $z = -2$.
(2) Show that the surface area of the sphere $x^2 + y^2 + z^2 = r^2$ is $4\pi r^2$
(3) Show that the surface area of the open cylinder $x^2 + y^2 = r^2$, $0 \le z \le h$ is $2\pi rh$.
(4) Calculate $\iint x + y^2 dS$ where $S$ is the cylinder $x^2 + y^2 = 4$, $0 \le z \le 3$.
(5) Calculate $\iint_S F \cdot \hat{n} dS$ where $F = \langle -y, x, 0\rangle$ and $S$ is the surface with parametrization $r(u, v) = \langle u, v^2 - u, u + v\rangle$ with $0 \le u \le 3$ and $0 \le v \le 4$.
(6) Verify Stoke's Theorem for the vector field $F = \langle y, 2z, x^2\rangle$ and the surface $S$, where $S$ is the paraboloid $z = 4 - x^2 - y^2$, $z \ge 0$.
(7) Use Stoke's Theorem to calculate $\int_C F \cdot dr$ where $F = \langle xy, x^2 + y^2 + z^2, yz\rangle$ and $C$ is the boundary of the parallelogram with vertices $(0, 0, 1), (0, 1, 0), (2, 0, -1)$, and $(2, 1, -2)$, going counterclockwise.