ADVANCED CALCULUS 2, DR. BLOCK, SAMPLE FINAL EXAM (WITH ANSWERS)

PART ONE. PROOFS.

1. (6 points) Prove the following Theorem: (Alternating Series Test) Suppose that $\{a_n\}$ is a decreasing sequence of positive real numbers, and $\lim_{n\to\infty} a_n = 0$. Then the alternating series $\sum_{k=1}^{\infty}(-1)^{k+1} a_k$ converges.

2.(6 points) Suppose that $\{a_n\}$ is the sequence defined by

$$a_n = \begin{cases} \dfrac{-1}{n^2} & \text{if } n \text{ is even} \\ \dfrac{1}{n} & \text{if } n \text{ is odd} \end{cases}$$

Prove that the altenating series $\sum a_k$ diverges, even thought its terms tend to zero.

PART TWO. PROBLEMS: Justify your answer in each problem.

3. (5 points) Determine whether the given series converges or diverges.

$$\sum_{k=1}^{\infty} \sin(\ln(1+\frac{1}{k^2}))$$

Answer: converges
Suggested method: Apply the limit comparison test, and compare with the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

4. (5 points) Determine whether the given series converges or diverges.

$$\sum_{k=2}^{\infty} \frac{\int_1^k e^{-x^2}\,dx}{k^3-5}$$

Answer: converges
Suggested method: Apply the direct comparison test, and compare with the series

$$\sum_{k=1}^{\infty} \frac{c}{k^3-5},$$

where c is the positive real number, $\int_1^\infty e^{-x^2}\,dx$.

5. (5 points) Determine whether the given series converges or diverges.

$$\sum (-1)^k \frac{2^k\,k!}{k^k}$$

Answer: converges
Suggested method: Apply the ratio test and see that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{2}{e} < 1.$$

So the series converges absolutely by the ratio test.

6. (5 points) Determine whether the given series converges or diverges. Find the sum if the series converges.

$$\sum_{k=1}^{\infty} \frac{1}{4k^2-1}$$

Answer: converges and the sum is $\frac{1}{2}$.
Suggested method: Use partial fractions to see that the given series is a telescoping series. By looking at the partial sums, show that the series converges and the sum is $\frac{1}{2}$.

7. (5 points) Determine whether the given series converges absolutely, converges conditionally, or diverges.

$$\sum (-1)^k a_k$$

where

$$a_k = \begin{cases} \dfrac{1}{2^k} & \text{if } k \text{ is even} \\ \dfrac{1}{3^k} & \text{if } k \text{ is odd} \end{cases}$$

Answer: converges absolutely
Suggested method: Apply the root test (Theorem 7.3.8 of the text).

8. (5 points) Determine whether the improper integral converges or diverges. Evaluate the integral if the integral converges.

$$\int_0^1 (\ln x)^2 \, dx$$

Answer: the improper integral converges and

$$\int_0^1 (\ln x)^2 \, dx = 2.$$

Suggested method: Use integration by parts twice to find an antiderivative, and then compute the appropriate limit.

9. (8 points) Find all values of x for which the series

$$\sum_{k=2}^{\infty} \frac{(x+5)^k}{k \ln k}$$

a. converges absolutely.
b. converges conditionally.
c. converges.
d. diverges.

Answers:
a.$-6 < x < -4$
b. $x = -6$
c. $-6 \leq x < -4$
d. $x < -6$ or $x \geq -4$
Suggested method: Use the ratio test first, and then use the integral test for $x = -4$ and the alternating series test for $x = -6$.