An introduction to reverse mathematics

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Preliminaries

An **axiom system** is a collection of statements taken as true.
Using these axioms, we then deduce mathematical theorems.

If we can prove a theorem **Thm** from an axiom system $\mathfrak{T}$, we write

$$\mathfrak{T} \vdash \textbf{Thm}$$

If an additional axiom **Ax** is needed to prove **Thm**, we write

$$\mathfrak{T} + \textbf{Ax} \vdash \textbf{Thm} \quad \text{or} \quad \mathfrak{T} \vdash \textbf{Ax} \longrightarrow \textbf{Thm}$$

Motivation

Suppose we are working in a weak axiom system $\mathfrak{B}$ that proves $\mathbf{Thm}_1$ but not $\mathbf{Thm}_2$:

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we can say that $\mathbf{Thm}_2$ requires more "strength" to prove.

This is not a precise measurement of logical strength.
$\mathbf{Ax}_1$ may be *too* powerful to give any useful insight in how to properly compare $\mathbf{Thm}_1$ and $\mathbf{Thm}_2$.

Improving this measure

Since $\mathfrak{B} + \mathbf{Ax}_1 \vdash \mathbf{Thm}_2$, then

$$\mathfrak{B} \vdash \mathbf{Ax}_1 \longrightarrow \mathbf{Thm}_2$$

Suppose we can show $\mathfrak{B} + \mathbf{Thm}_2 \vdash \mathbf{Ax}_1$; that is,

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We call this *reversing* $\mathbf{Thm}_2$ to $\mathbf{Ax}_1$ and we conclude that $\mathbf{Ax}_1$ and $\mathbf{Thm}_2$ are *provably equivalent* over $\mathfrak{B}$.

$$\mathfrak{B} \vdash \mathbf{Ax}_1 \longleftrightarrow \mathbf{Thm}_2$$

Improving this measure

Let's consider another theorem $\mathbf{Thm}_3$.

Suppose we find another axiom $\mathbf{Ax}_2$ and show

$$\mathfrak{B} \vdash \mathbf{Ax}_2 \longleftrightarrow \mathbf{Thm}_3$$

What can we say about the relationship between $\mathbf{Thm}_1$, $\mathbf{Thm}_2$, and $\mathbf{Thm}_3$?

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What can we say about the relationship between $\mathbf{Thm}_1$, $\mathbf{Thm}_2$, and $\mathbf{Thm}_3$?

For the relationship between $\mathbf{Thm}_2$ and $\mathbf{Thm}_3$, we need the comparison of $\mathbf{Ax}_1$ and $\mathbf{Ax}_2$.

Reverse Mathematics

Goal: Determine which set existence axioms are needed to prove familiar theorems.

Method: Prove results of the form

$$\mathsf{RCA}_0 \vdash \textbf{Thm} \quad \text{and} \quad \mathsf{RCA}_0 \vdash \textbf{Ax} \longleftrightarrow \textbf{Thm}$$

where:

- RCA_0 is a weak axiom system,
- **Ax** is a set existence axiom, and
- **Thm** is a familiar non-set theoretic theorem.

What's the point?

Work in reverse mathematics can:

- categorize the logical strength of theorems in a precise manner.
- distinguish between different proofs of theorems.
- provide insight into the foundation of mathematics.
- utilize and contribute to disciplines of mathematical logic – including computability theory, proof theory, models of arithmetic, etc.

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Natural number variables: x, y, z Set variables: X, Y, Z

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  where φ is any formula in Z_2.
- Set comprehension:

  $$\exists X \forall n\ (n \in X \longleftrightarrow \varphi(n))$$

  where $\varphi(n)$ is any formula of Z_2 in which X does not occur freely.

Subsystems of Second Order Arithmetic

A **subsystem** of second order arithmetic is an axiom system consisting of the arithmetic axioms, the induction and comprehension schemas restricted to certain formulas, or has theorems of Z_2 as axioms.

There are five subsystems of particular interest, each with an increasing strength of set comprehension:

$$\mathsf{RCA}_0 \quad \mathsf{WKL}_0 \quad \mathsf{ACA}_0 \quad \mathsf{ATR}_0 \quad \Pi^1_1\text{-}\mathsf{CA}_0$$

In reverse mathematics, RCA_0 is (usually) chosen as the *base system*, the weakest subsystem we will work in. The other four subsystems are given by amending RCA_0 with additional set existence axioms.

Some notation

Formulas of Z_2 can be categorized into two forms.

- Formulas that do not quantify over sets (called *arithmetic*)
  - Σ^0_0 formulas are constructed from atomic formulas and bounded quantifiers.
  - Σ^0_k formulas: $\exists n_1 \forall n_2 \ldots \exists n_k \theta$, where $\theta \in \Sigma^0_0$
  - Π^0_k formulas: $\forall n_1 \exists n_2 \ldots \forall n_k \theta$, where $\theta \in \Sigma^0_0$
- Formulas that do quantify over sets:
  - Σ^1_k formulas: $\exists X_1 \forall X_2 \ldots \exists X_k \varphi$, where φ is arithmetic
  - Π^1_k formulas: $\forall X_1 \exists X_2 \ldots \forall X_k \varphi$, where φ is arithmetic

Recursive Comprehension and RCA_0

RCA_0 is the subsystem of Z_2 whose axioms are:

- Arithmetic axioms of $\mathbb{N}$
- Σ^0_1 induction
  $(\varphi(0) \wedge \forall n\,(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n\,(\varphi(n))$,
  where φ is any Σ^0_1 formula
- Recursive comprehension
  If $\varphi \in \Sigma^0_1$ and $\psi \in \Pi^0_1$, then

$$\forall n\,(\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X \forall n\,(n \in X \leftrightarrow \varphi(n))$$

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- RCA_0 is able to prove the arithmetic necessary for pairing functions.
- Functions and countable sequences correspond to sets of pairs.

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- Elements of countable collections of objects can be identified with natural numbers.
- RCA_0 is able to prove the arithmetic necessary for pairing functions.
- Functions and countable sequences correspond to sets of pairs.
- RCA_0 can encode the integers, rational numbers, real numbers, countable Abelian groups, continuous real-valued functions, and many other mathematical objects.

What can RCA_0 prove?

Theorem: *The following are provable in* RCA_0.

(1) The system $\mathbb{Q}, +, -, \cdot, 0, 1, <$ is an ordered field.

(2) The intermediate value theorem: If f is continuous on $[0, 1]$ and $f(0) < 0 < f(1)$, then there exists an x such that $0 < x < 1$ and $f(x) = 0$.

(3) If $f : \mathbb{N} \to 2$, then there is an infinite set X such that f is constant on X.

(4) Every finite graph with maximum degree 2 and no cycles of odd length is bipartite (i.e. can be 2-colored).

(5) Van der Waerden's theorem: For any c and k there exists n such that if $f : n \to c$ then there is a homogenous arithmetic progression of length k.

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The subsystem WKL_0 is RCA_0 plus weak König's lemma.

There is an infinite computable binary tree with no infinite computable path, so $\mathsf{RCA}_0 \nvdash \mathsf{WKL}_0$.

Some Reverse Mathematics!

Theorem: (RCA_0) *The following are equivalent:*

(1) WKL_0

(2) If f and g are one-to-one functions from $\mathbb{N}$ to $\mathbb{N}$ and $\text{Range}(f) \cap \text{Range}(g) = \emptyset$, then there exists a set X such that $\text{Range}(f) \subset X$ and $X \cap \text{Range}(g) = \emptyset$.

(3) Every continuous function on $[0, 1]$ attains a supremum.

(4) The Heine-Borel theorem for $[0, 1]$.

(5) If every finite subgraph of G can be 2-colored, then G can be 2-colored.

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The “standard trick” to show that **Thm** reverses to WKL_0 is to use (2); that is, show that $RCA_0 +$ **Thm** can find a separating set of two arbitrary injections with disjoint ranges.

Arithmetical Comprehension

ACA_0 is RCA_0 plus the arithmetical comprehension scheme:

For any arithmetical formula $\varphi(n)$, then

$$\exists X \forall n \ (n \in X \longleftrightarrow \varphi(n))$$

Equivalently, the set $\{n \in \mathbb{N} \mid \varphi(n)\}$ exists.

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However, $\mathsf{RCA}_0 \vdash \mathsf{ACA}_0 \longleftrightarrow \Sigma^0_1$ comprehension, so Σ^0_1-CA_0 would be equivalent to ACA_0.

Mathematics in ACA_0

Theorem: (RCA_0) *The following are equivalent:*

(1) ACA_0

(2) If $f : \mathbb{N} \to \mathbb{N}$ is one-to-one, then Range(f) exists.

(3) Every Cauchy sequence converges.

(4) The Bolzano-Weierstraß theorem: Every bounded sequence of real numbers contains a convergent subsequence.

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General rule of thumb: ACA_0 suffices for undergraduate math.

An example

Theorem: (RCA_0) *The following are equivalent:*

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(2) Suppose H is a hypergraph with finite edges presented as a sequence of characteristic functions. If every finite partial hypergraph of H has a proper 2-coloring, then H has a proper 2-coloring.

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The idea of the proof:
(1)$\to$(2): For every m, there is a least 2-coloring of the vertices $v_0, \ldots, v_m$ that can be extended to a proper 2-coloring of every finite partial hypergraph. Nesting these least 2-colorings yields a 2-coloring of all of H that is arithmetically definable.

The reversal: 2-coloring implies ACA_0

Given an arbitrary injection f, we want to construct H so that a proper 2-coloring of H encodes the range of f.

Example: Suppose $f(0) = 1$, $f(2) = 0$, and $2 \notin \text{Range}(f)$.

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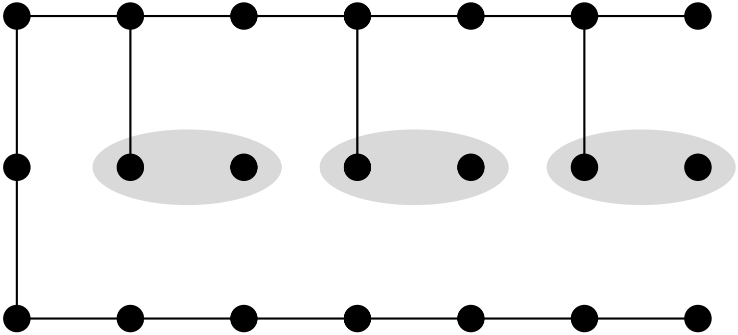

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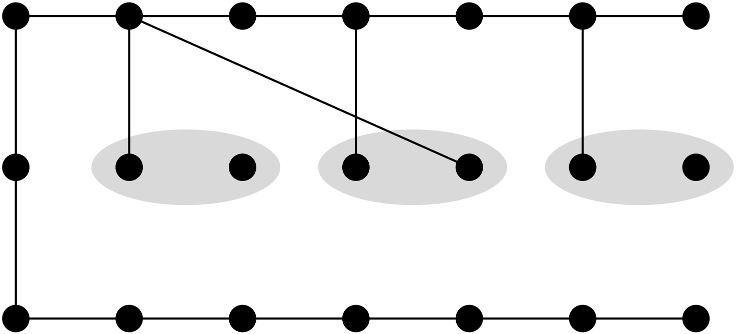

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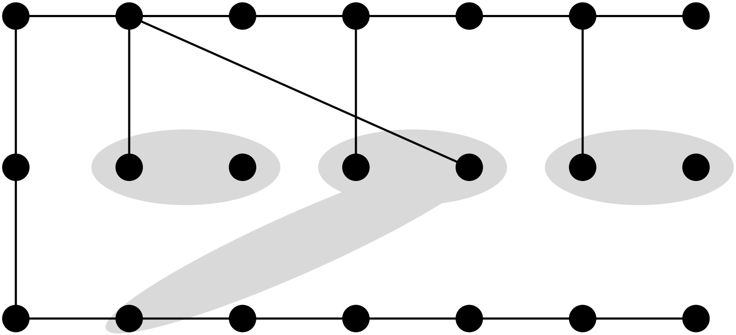

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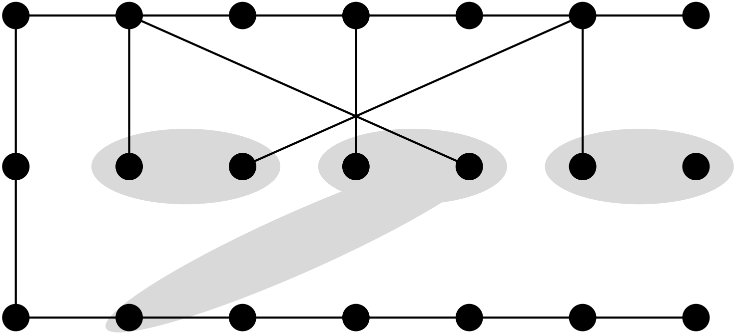

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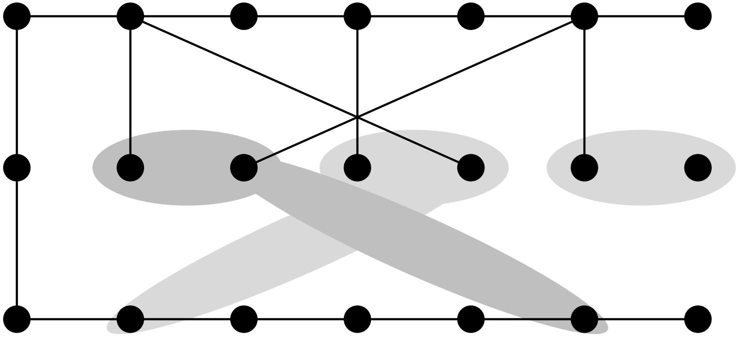

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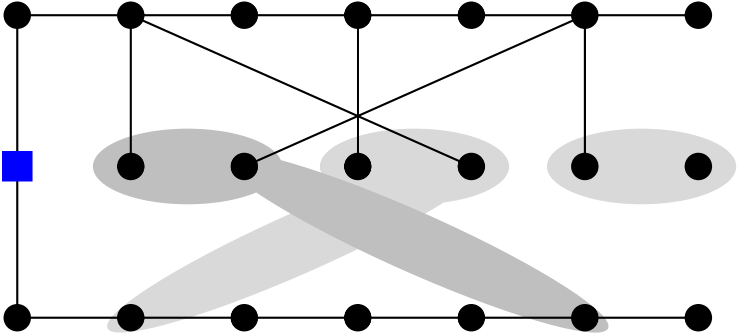

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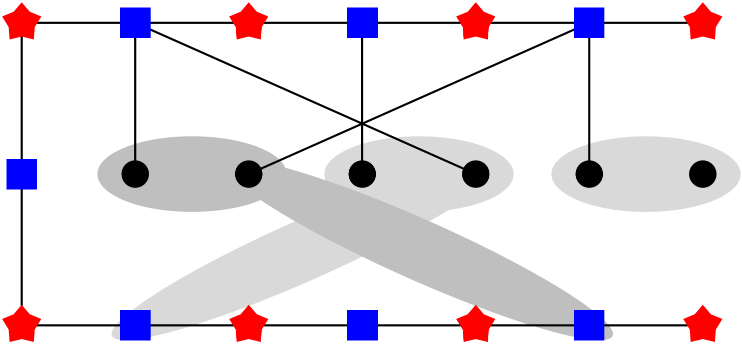

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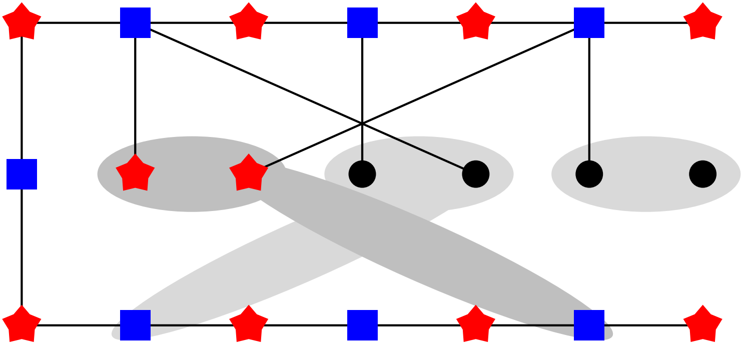

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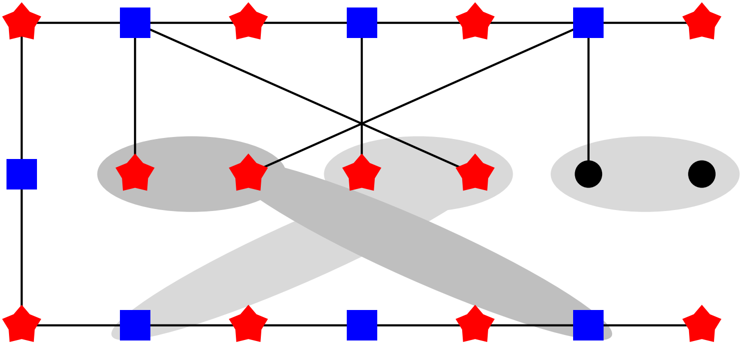

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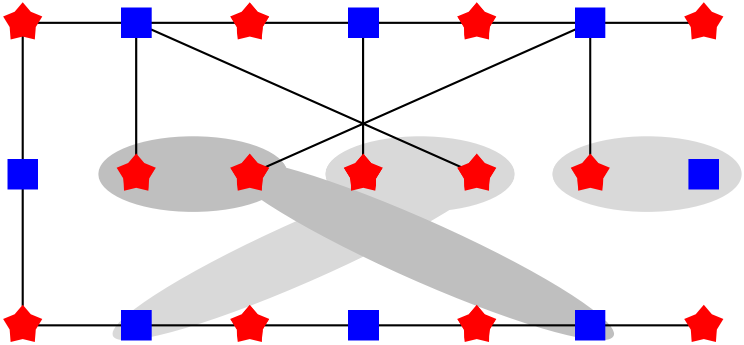

Arithmetical Transfinite Recursion

ATR_0 is the subsystem given by adding to RCA_0 axioms that allow for iteration of arithmetical comprehension along any well-ordering.

A tool for proofs:
Theorem: (ATR_0) If $\psi(X)$ is a Σ^1_1 formula that is only satisfied by well-ordered sets, then there is a well-ordering β such that $\psi(X)$ implies $X < \beta$.

Theorem: (RCA_0) *The following are equivalent:*

(1) ATR_0

(2) If α and β are well-orderings, then $\alpha \leqslant \beta$ or $\beta \leqslant \alpha$.

(3) Lusin's separation theorem: Any two disjoint analytic sets can be separated by a Borel set.

(4) Every countable reduced Abelian p-group has an Ulm resolution.

Π^1_1 comprehension

Π^1_1-CA_0 is RCA_0 plus the Π^1_1 comprehension scheme:

We can assert the existence of the set

$$\{n \in \mathbb{N} \mid \psi(n)\},$$

where ψ is a Π^1_1 formula.

Theorem: (RCA_0) *The following are equivalent:*

(1) Π^1_1-CA_0

(2) If $\langle T_n \rangle_{n\in\mathbb{N}}$ is a sequence of trees in $\mathbb{N}^{<\mathbb{N}}$, then there is a function $f : \mathbb{N} \to 2$ such that $f(n) = 1$ if and only if T_n contains an infinite path.

(3) Every countable Abelian group is the direct sum of a divisible group and a reduced group.

To reverse **Thm** to Π^1_1-CA_0, the usual trick is to first show $\mathsf{RCA}_0 \vdash \textbf{Thm} \to \mathsf{ACA}_0$, then show $\mathsf{ACA}_0 \vdash \textbf{Thm} \to \Pi^1_1$-$\mathsf{CA}_0$.

An abbreviated list of references

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