Reverse mathematics, hypergraphs, and edge representation

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Reverse mathematics

Reverse mathematics uses a hierarchy of axioms of second order arithmetic to measure the strength of theorems.

The language has variables for natural numbers and sets of natural numbers.

The base system, RCA_0, includes

- arithmetic axioms of $\mathbb{N}$ ($+$, $\cdot$, work as expected)
- an induction scheme restricted to Σ^0_1 formulas, and
- recursive comprehension:
  sets with computable characteristic functions exists

ACA_0

The system ACA_0 adds to RCA_0 the comprehension scheme for arithmetically definable sets:

$$\exists X \forall n (n \in X \leftrightarrow \psi(n))$$

where $\psi(n)$ is a formula whose quantifiers are restricted to natural numbers and in which X does not occur freely.

Theorem: The following are provably equivalent over RCA_0.

(1) ACA_0

(2) If g is an injective mapping from $\mathbb{N}$ to $\mathbb{N}$, then $\mathrm{Range}(g) = \{n : \exists m\, g(m) = n\}$ exists.

(3) Every bounded sequence of real numbers has a convergent subsequence.

(4) Every countable commutative ring has a maximal ideal.

(5) König's lemma: Every infinite, finitely branching tree has an infinite path.

WKL_0

The system WKL_0 adds Weak König's Lemma to RCA_0.

Weak König's Lemma: If $T \subseteq 2^{<\mathbb{N}}$ is an infinite tree, then T has an infinite path.

Theorem: The following are provably equivalent over RCA_0.

(1) WKL_0

(2) If f and g are injective functions from $\mathbb{N}$ into $\mathbb{N}$ and $\mathrm{Range}(f) \cap \mathrm{Range}(g) = \emptyset$, then there is a set X such that $\mathrm{Range}(f) \subset X$ and $X \cap \mathrm{Range}(g) = \emptyset$.

(3) Every covering of a compact metric space by a sequence of open sets has a finite subcovering.

(4) Every continuous real-valued function on $[0, 1]$ is Riemann integrable.

(5) Every countable commutative ring has a prime ideal.

Hypergraphs

A *hypergraph* consists of a set of vertices $V = \{v_i \mid i \in \mathbb{N}\}$ and a collection of edges E. Edges of a hypergraph may contain any amount of vertices, finite or infinite.

The edges may be presented in different ways, depending on the cardinality of the edges.

Hypergraphs

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The edges may be presented in different ways, depending on the cardinality of the edges.

If an edge is finite, it may be encoded by a single number. If all edges are finite, then E may be either a set or a sequence of the codes for the edges.

Finite edges and infinite edges can be represented by a sequence of characteristic functions $\langle \chi_i \rangle_{i \in \mathbb{N}}$, where $\chi_i(n) = 1$ if and only if v_n is in the i^{th} edge.

Representations of finite edges

For hypergraph with finite edges, changing the representation of the edges requires different set comprehension.

$$\begin{array}{c} \text{Set of codes of the edges} \\ \mathsf{RCA}_0 \ \downarrow \uparrow \ \mathsf{ACA}_0 \\ \text{Sequence of codes of the edges} \\ \mathsf{RCA}_0 \ \downarrow \uparrow \ \mathsf{ACA}_0 \\ \text{Sequence of characteristic functions} \end{array}$$

Working in ACA_0 or any stronger system, we may assume the edges are presented in any manner.

In the absence of arithmetical comprehension, the presentation choice matters.

Coloring finite edges

A vertex coloring of a hypergraph is *proper* if no edge with more than one vertex is monochromatic.

Theorem: (RCA_0) For $k \geqslant 2$, the following are equivalent.

(1) WKL_0

(2) Let G be a graph. If every partial graph of of G has a proper k-coloring, then G has a proper k-coloring. (Hirst [4])

(3) Let H be a hypergraph with a sequence of finite edges. For $k \geqslant 2$, if every finite partial hypergraph of H has a proper k-coloring, then H has a proper k-coloring.

Statement (3) generalizes statement (2) to the hypergraph setting, for hypergraphs with sequences (or sets) of finite edges.

Coloring finite edges

A vertex coloring of a hypergraph is *strong* if the coloring is injective on each edge.

Theorem: (RCA_0) The following are equivalent.

(1) WKL_0

(2) Let H be a hypergraph with any edge representation. If for some k every finite partial hypergraph of H has a strong k-coloring, then H has a strong k -coloring.

(3) Let H be a hypergraph with a set of finite sets for edges. If every finite partial hypergraph of H has a strong 3-coloring, then H has a strong k-coloring for some k.

(4) Let H be a hypergraph with a sequence of finite sets for edges. If every finite partial hypergraph of H has a strong 2-coloring, then H has a strong k-coloring for some k.

Coloring finite edges

Vertex colorings of hypergraphs do differ from graphs.

Theorem: (RCA_0) For $k \geqslant 2$, the following are equivalent.

(1) ACA_0

(2) Let H be a hypergraph with finite edges presented as a sequence of characteristic functions. If every finite partial hypergraph of H has a proper k-coloring, then H has a proper k-coloring.

Theorem: (RCA_0) For $k \geqslant 2$, the following are equivalent.

(1) WKL_0

(2) Let G be a graph with finite edges presented as a sequence of characteristic functions. If every finite partial graph of G has a proper k-coloring, then H has a proper k-coloring.

Coloring infinite edges

Finite vertex colorings of hypergraph with infinite edges are not arithmetically definable.

Theorem: (RCA_0) Fix $k \geqslant 2$. The following are equivalent.

(1) Π^1_1-CA_0, the comprehension scheme for Π^1_1 definable sets.

(2) If $\langle H_i \rangle_{i \in \mathbb{N}}$ is a sequence of hypergraphs, then there is a function $f : \mathbb{N} \to 2$ such that $f(i) = 1$ if and only if H_i has a proper k-coloring.

Conflict-free colorings

A vertex coloring is called *conflict-free* if each edge contains a color that appears only once in that edge.

Every conflict-free coloring is proper. While hypergraphs with infinite edges may have a finite proper coloring, a finite conflict-free coloring may not exist.

The $\mathcal{M}$-*graph* (the Matroshka graph) is the hyergraph with vertex set $\mathbb{N}$ and edges $\{E_j : j \in \mathbb{N}\}$, where $E_j = \{k : j \leqslant k\}$.

Every finite partial subhypergraph of the $\mathcal{M}$-graph has a conflict-free 2-coloring.

However, RCA_0 can prove that no finite coloring of the $\mathcal{M}$-graph is conflict-free. See section 4 of [2] and section 1 of [1].

Enumerated hypergraphs

An *enumerated hypergraph* is a set $V \subseteq \mathbb{N}$ of vertices and a sequence $\langle e_i \rangle_{i \in \mathbb{N}}$ of enumerations of edges such that $e_i : \mathbb{N} \to V \cup \{\#\}$ for each i.

The vertices of the edge represented by e_i are those $v \in V$ such that $\exists n \, e_i(n) = v$.

Edges of a hypergraph presented by a characteristic function corresponds to *computable* sets.

The edges of an e-hypergraph correspond to *computably enumerable* sets.

Enumerated hypergraphs

Constructing a hypergraph with edges presented by characteristic function when given an e-hypergraph, or vice versa, requires different amounts of set comprehension.

Sequence of characteristic functions

RCA_0 $\downarrow$ $\uparrow$ ACA_0

Sequence of enumeration functions

Enumerated hypergraphs

Constructing a hypergraph with edges presented by characteristic function when given an e-hypergraph, or vice versa, requires different amounts of set comprehension.

Sequence of characteristic functions

RCA_0 ↓ ↑ ACA_0

Sequence of enumeration functions

This provides a motivation to investigate colorings of e-hypergraphs and comparing to the previous results.

Coloring e-hypergraphs

Theorem: (RCA_0) The following are equivalent:

(1) WKL_0.

(2) Let H be an e-hypergraph. If every e-hypergraph fragment of H has a strong k-coloring, then H has a strong k-coloring.

(3) Let H be an e-hypergraph. If every e-hypergraph fragment of H has a strong 2-coloring, then H has a strong k-coloring for some k.

Theorem: (RCA_0) The following are equivalent.

(1) ACA_0

(2) Let H be a e-hypergraph such that the size of each edge is bounded by some function. If for some k every fragment of H has a proper k-coloring, then H has a proper k-coloring.

(3) Statement (2) with "proper" replaced by "conflict-free."

At most b edges

A hypergraph with edges represented by a sequence of characteristic functions $\langle \chi_i \rangle_{i \in \mathbb{N}}$ has *at most b edges* if for any $b+1$ functions there are indices i and j such that for all n $\chi_i(n) = \chi_j(n)$.

Theorem: (RCA_0) Suppose H is a hypergraph with at most b edges. There is a finite sequence $n_0, \ldots, n_k$ with $k < b$ such that every edge of H occurs exactly once in the list $\chi_{n_0}, \ldots, \chi_{n_k}$.

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An e-hypergraph with a sequence of edge enumerations $\langle e_i \rangle_{i \in \mathbb{N}}$ has *at most b edges* if for any collection of $b+1$ enumerations there are indices i and j such that $\forall m \exists n (e_i(m) = e_j(n) \wedge e_j(m) = e_i(n))$.

Theorem: (RCA_0) The following are equivalent.

(1) $I\Sigma^0_2$, the induction scheme for Σ^0_2 formulas.

(2) Let H be an e-hypergraph with at most b disjoint edges. There is a finite sequence $n_0, \ldots, n_k$ with $k < b$ such that every edge of H occurs exactly once in the list $e_{n_0}, \ldots, e_{n_k}$.

At most b edges

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(1) $I\Sigma^0_2$, the induction scheme for Σ^0_2 formulas.

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Question: Does the equivalence still hold is the edges are not necessarily disjoint?

Response: If so, it is not obvious.

An open problem

Conjecture: (RCA_0) The following are equivalent.

(1) $B\Pi^0_2$: Let ψ be a Π^0_2 formula. For any fixed u,

$$\forall x \leqslant u \, \exists y \, \psi(x, y) \longrightarrow \exists v \, \forall x \leqslant u \, \exists y \leqslant v \, \psi(x, y)$$

(2) Let H be an e-hypergraph with at most b edges. There is a finite sequence $n_0, \ldots, n_k$ with $k < b$ such that every edge of H occurs exactly once in the list $e_{n_0}, \ldots, e_{n_k}$.

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