RESEARCH STATEMENT

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My main research interests center around nonlinear partial differential equations that correspond to the important models of transport phenomena. I have published several papers [9, 10, 11, 12, 15] that are related to my Ph.D thesis. In the past several years, I have obtained several results in [1, 3, 4, 6, 7, 8, 14, 16]. Very recently, I have finished another four papers [2, 5, 13, 17] currently in submission. The following sections describe my major contributions in last few years.

1. Existence theory

1.1. **Background.** We have worked on the following compressible Navier-Stokes equations

$$\begin{aligned} &\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0 \\ &(\rho\mathbf{u})_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \mathrm{div}\sigma = 0 \end{aligned} \tag{1}$$

with initial data

$$\rho(0,x) = \rho_0(x), \quad (\rho\mathbf{u})(0,x) = \mathbf{m}_0(x),$$

where $(t,x)\in[0,\infty)\times\mathbb{R}^3$, $\sigma = 2\mu(\rho)\,\mathbb{D}(\mathbf{u}) + \big(\lambda(\rho)\mathrm{div}\mathbf{u} - P(\rho)\big)\,\mathrm{Id}$, Id is an identical matrix. $P=\rho^\gamma$ denotes the pressure, adiabatic constant $\gamma>1$. Note that the most interesting regime in physics is $\gamma\in(1,\frac{5}{3}]$. ρ is the density of fluid, $\mathbf{u}\in\mathbb{R}^3$ stands for the velocity of fluid, $\mathbb{D}\mathbf{u} = \frac{1}{2}[\nabla\mathbf{u}+\nabla^T\mathbf{u}]$ is the strain tensor. The viscosity coefficients $\mu(\rho)$ and $\lambda(\rho)$ may dependent on the density. When the density is a constant, system (1) reduces to the incompressible Navier-Stokes equations. In the particular case $\gamma=2$ with $\mu=\rho$ and $\lambda=0$, (1) becomes the shallow water equation, where $\rho(t,x)$ stands for the height of the water and $\mathbf{u}(t,x)$ is the fluid velocity in $\mathbb{R}^2$.

The existence theory of global solutions to the Navier-Stokes equations (NSE) was established by Leray in 1934. His notion of weak solution preceded both introduction of the Sobolev spaces (1936) and the generalized derivatives (Schwartz 1944). Despite considerable progress made on these equations in recent years, many important questions remain open. The existence theory to the compressible NSE is a fundamental problem, and has a long-standing history ever since the work of J. Serrin (1959) and J. Nash (1962). When the viscosity coefficients are constants, the existence of weak solution has been established by P.-L. Lions (1995-1998) for $\gamma\geq\frac{9}{5}$ and Feireisl-Novotný-Petzeltová (2001) for $\gamma>\frac{3}{2}$. Improving the range of γ is a very important open question. When the viscosity coefficients depend on the density, on the other hand, are mathematically challenging in their own right and they may occur in many other situations such as in the shallow water system. In fact, the shallow water model is proposed as an open problem by P.-L. Lions in 1998. The main difficulties lie in the degeneracy of the diffusion effect and the concentration in convection term.

The basic energy inequality does not provide the enough uniform bounds for the weak stability of solutions. To obtain improved estimates for (1), more than a decade ago Bresch–Desjardins introduced an entropy conservation (BD entropy) for the case when the viscous coefficients depend on the density variable, in particular, they have to verify $\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho))$. The BD entropy provides extra bounds on the gradient of the density, and therefore yields the compactness of the density in L^p space. However, the estimates on the velocity field are still not strong enough to conclude the compactness of $\rho\mathbf{u}\otimes\mathbf{u}$. Hence there is a notable difference between the non-degenerate and degenerate cases. Utilizing the BD entropy, Mellet and Vasseur managed to prove formally that an $L(0,T;LlogL)$ property on $\rho|\mathbf{u}|^2$ may propagate in time if initially it is controlled, and thus the weak stability follows, for $1<\gamma<3$ in 3D and for any $\gamma>1$ in 2D.

But how to build an approximation system for which the existence of solutions can be proved through the uniform bounds from the BD entropy and the Mellet-Vasseur (MV) inequality? The classical way to construct global weak solutions would consist of constructing smooth approximation solutions, verifying the priori estimates, including the BD entropy, and the MV inequality. However, those extra estimates impose a lot of structure on the approximating system. Bresch and Desjardins proposed a nice construction of approximations, controlling both the usual energy and BD entropy. This allows the construction of weak solutions, when additional terms – such as drag terms, or cold pressure, for instance – are added. However, their construction does not provide the control of $\rho|\mathbf{u}|^2$ in the space of $L\log L$.

1.2. **Linear shear viscosity.** In [6], we have constructed a global weak solution of the compressible NSE based on our approximated solutions in [7] when the viscous coefficients verifying $\mu(\rho)=\rho$ and $\lambda(\rho)=0$. Note that, the approximated scheme has additional terms, such as drag forces and the Bohm potential $\kappa\rho\nabla(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}})$, controlling both the basic energy inequality and BD entropy. Due to the Bohm potential, there is no chance to get the MV inequality for $\kappa>0$. On other hand, this term can bring us higher regularity of density. Our main contribution of [6] is to derive the MV type inequality for the weak solutions, even it is not verified by the first levels of approximation. This provides existence of global solutions, for any $\gamma>1$ in 2D and for $1<\gamma<3$ in 3D, in both case with large initial data possibly vanishing on the vacuum. This solves an open problem proposed by Lions. The work has been featured at the "Seminaire Bourbaki " in Institut Henri Poincaré, Paris, June 2017.

Building up from the result [7], we establish the logarithmic estimate for the weak solutions similar to the MV inequality. To this end, we first derive a *renormalized* estimate on $\rho\varphi(\mathbf{u})$, for φ nice enough, for solutions of [7] with the additional drag forces. It is showed to be independent on the strength of those drag forces, allowing to pass into the limit when those forces vanish. Since this estimate cannot be derived from the approximation scheme of [7], it has to be carefully derived on weak solutions. After passing into the limit κ goes to 0, we can recover the logarithmic estimate, taking a suitable function φ. This logarithmic estimate allows us to recover the global weak solutions by passing to the limits.

1.3. **Nonlinear shear viscosity.** The results obtained in [6] treat the case when the viscous coefficients $\mu=\rho,\ \lambda(\rho)=0$, which can be viewed as a special example enjoying the BD structure. In 2D, this model recovers the classical viscous shallow water equations. On the other hand, the new *renormalization* technique introduced in [6] still leaves open the existence problem for a large class of model equations satisfying the BD structure, especially the ones with nonlinear shear viscosities. Compared to the linear shear viscosity case, for the general nonlinear shear viscosity, there are substantial difficulties to obtain the existence theory. Our main goal is to fill this gap so as to have a fully understanding of the existence theory for the compressible NSE with the BD structure.

In our very recent work [1], we extend considerably the global existence results of weak solutions to compressible NSE with density dependent viscosities obtained in [6]. More precisely, we are able to consider a physical symmetric viscous stress tensor $\sigma=2\mu(\rho)\,\mathbb{D}(\mathbf{u})+\big(\lambda(\rho)\mathrm{div}\mathbf{u}-P(\rho)\big)\,\mathrm{Id}$, where $\mathbb{D}(\mathbf{u})=[\nabla\mathbf{u}+\nabla^T\mathbf{u}]/2$ with a shear and bulk viscosities (respectively $\mu(\rho)$ and $\lambda(\rho)$) satisfying the BD relation $\lambda(\rho)=2(\mu'(\rho)\rho-\mu(\rho))$ and a pressure law $P(\rho)=a\rho^\gamma$ (with $a>0$ a given constant) for any adiabatic constant $\gamma > 1$. The nonlinear shear viscosity $\mu(\rho)$ satisfies some lower and upper bounds for low and high densities. The nonlinear viscous coefficients may degenerate on the vacuum. Our new result in [1] provides an answer to a longstanding mathematical question on compressible NSE with density dependent viscosities as mentioned for instance by F. Rousset in the Bourbaki 69ème année, 2016–2017, no 1135.

The method is based on the consideration of a multiple level approximated system with an extra pressure quantity, appropriate non-linear drag terms and appropriate capillarity terms. This generalizes the Quantum-Navier-Stokes system with quadratic drag terms considered in [6, 7]. First we prove that weak solutions of multiple level approximated system are *renormalized* solutions for the *velocity* variable. This technique was introduced by Lacroix-Violet and Vasseur based on our argument *renormalized* estimate for the *velocity* variable in [6]. Then we pass to the limit with respect to several artificial terms to get renormalized solutions of the compressible NSE. The final step concerns the proof that a renormalized solution of the compressible NSE is a global weak solution of the compressible NSE. Note that, thanks to the technique of *renormalized* solutions for the *velocity* variable, it is not necessary to derive the Mellet-Vasseur type inequality in [1]: **this allows us to cover the full range $\gamma > 1$.** This is reminiscent to showing the existence of global weak solutions to the compressible NSE with constant viscous coefficients when $1 < \gamma \leq \frac{3}{2}$. However, it is a long standing open problem proposed by Lions in 1998. The techniques were developed in [1, 6] shed new insight to this problem.

1.4. **Others.** When the viscous coefficients are constants, the global existence theory of the compressible NSE was established by Lions in 1998 for the pressure law in terms of the density, in particular, $P = a\rho^\gamma$. The range of γ was improved by Feireisl-Novotny-Petzeltova in 2001. We have extended the Lions-Feireisl theory to the pressure laws depending on two phases in [8], such as $P = a\rho^\gamma + bn^\alpha$, where ρ and n are two phase densities that they both satisfy the conservation of mass equations. Our key idea of [8] is to make variable reduction and control nonzero remainder in the nonlinear pressure form.

We also have existence result [17] on the equations with temperature dependent viscosity coefficient. The viscosity may degenerate on absolute zero temperature. We relied on De Giorgi method to show that the temperature is bounded away from zero. This provides the uniform bounds based on which we can build the existence result. In [13], we extend the results of Chen-Glimm on the vanishing viscosity limit in the periodic domain to the whole space. In particular, if the weak solutions of the Navier-Stokes equations satisfy the Kolmogorov hypothesis, we can recover a weak solution to the Euler equations by vanishing viscosity coefficient in the whole space.

2. Infinitely many solutions

The aforementioned results all focus on the existence theory of weak solutions via suitably designed approximation schemes and delicate functional analytical techniques on the convergence. On the other hand, it is well-known that the weak solutions thus obtained may not be unique. In fact, in the context of incompressible Euler and Navier-Stokes equations, the recent works by De Lellis and Szekelyhidi proved that the systems can admit *infinitely many* rough solutions. We aim to address this issue for another class of PDEs, with the hope to be able to connect this "wild" phenomenon to some reasonable physical interpretations.

The model we study here is the isentropic Euler equations

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0 \\ (\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u} \otimes \mathbf{u}) + \nabla P = 0, \end{cases} \tag{2}$$

where ρ is the density, $\mathbf{u}$ is the velocity, and $P = \rho^\gamma$ is the pressure.

It is well known that, even starting from extremely regular initial data, the compressible Euler systems may develop singularities in finite time. Thus, we consider the notion of admissible weak solutions. In [2], we have shown that for any initial datum belonging to a dense subset of the energy space, there exist infinitely many global-in-time admissible weak solutions to the isentropic Euler system whenever $1 < \gamma \leq 1 + \frac{2}{n}$. This result shows that the Euler equations are severely ill-posed for a dense set of initial data in the finite energy space. The new contribution is to allow for a broad class of initial data from which ill-posedness can be established.

We choose to construct energy sub-solutions from the weak inviscid limit of Navier–Stokes equations. Note that the standard existence theory requires $\gamma > 3/2$. For this reason, we are using instead a Navier–Stokes model with degenerate viscosities constructed in [1, 6, 7] which allows $\gamma > 1$. We then modify the inviscid limit obtained from this model to ensure that the density and the Reynolds stress R are smooth enough. They solve the so-called "Euler–Reynolds" system

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + P I_n + \rho R) = 0, \end{cases} \tag{3}$$

where $R(t, x)$ is a positive semi-definite symmetric matrix for every x, t. Note that, some compensating potential energy should be pumped into the Euler–Reynolds system. Such an energy requirement imposes the constraint on the adiabatic exponent $\gamma \leq 1 + \frac{2}{n}$.

To the best of our knowledge, in the context of compressible fluids, the convex integration technique used so far allows to deal with only *diagonal* Reynold stresses. Such a method is a variant from the incompressible case. Here, we have developed a convex integration scheme that is able to handle "general Reynolds stresses" so that the method can treat bigger families of perturbations. Applying convex integration to sub-solutions, the infinity many global weak solutions with energy inequality can be generated.

3. **Energy equality for weak solutions**

Given that the solution to the fluid equations is sufficiently smooth, it is easy to see that the total kinetic energy of the flow is conserved. However, the global existences of smooth solution for the fluid equations, for example Euler equations and Navier-Stokes equations, are long-standing open problems. Naturally, of particular interest is the question: How badly behaved can a solution be while still satisfying energy conservation? For the Euler equations, this possibility goes by the name of the "Onsager conjecture": non-conservation of energy in the three-dimensional Euler equations would be related to the loss of regularity. Specifically, Onsager conjectured that every weak solution to the Euler equations with Hölder continuity exponent $\alpha > \frac{1}{3}$ conserves energy; and anomalous dissipation of energy occurs when $\alpha < \frac{1}{3}$.

3.1. **Energy equality.**

The objective of the works [3, 15] is to address the relation between the energy conservation and the degree of regularity of the solutions for fluid equations with variable density. In particular we provide sufficient conditions on the regularity of solutions to ensure the conservation of the total energy. Our approach is in the spirit of Constantin-E-Titi in 1994 and [6], with additional care to the density term ρ. We choose to work with the unknowns $\mathbf{u}$ and ρ. Due to the density effect, the main difficulties are **time regularity of ρ, the term of pressure, and continuity at** $t = 0$. Handling them constitutes the main contribution of our works [3, 15]. Note that many of the ideas have been successfully developed in [6] to the case of *compressible* Navier-Stokes equations, wherein the main purpose was to derive a priori estimates rather than energy equality. Further development of the method does lead to the energy conservation of such flows [15]. Note that our method allows us to deal with the vacuum issue. We see that this tool also works well for incompressible Euler flows in [3]. In particular, we are able to avoid assuming additional time regularity on the velocity field $\mathbf{u}$, and hence can recover the classical result of Constantin-E-Titi.

Our main tools are weak convergence method, renormalized techniques and convex integration. We will further develop original ideas and novel techniques to investigate new problems in our research program.

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