RESEARCH STATEMENT

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My research goal is to develop the theory for weak solutions to nonlinear PDEs in gas dynamics: existence and compactness, inviscid limit, and non-uniqueness. I describe my major contributions as follows.

My joint work with Vasseur (Invent.2016) was devoted to the existence theory for weak solutions to compressible Navier-Stokes equations with linear shear viscosities. In particular, the viscosity coefficients are given by $\mu = \nu\rho$ and $\lambda = 0$, where ν is a constant and ρ is the density of flow. The challenge is to deal with the degenerate viscosity, together with the lack of integrability on the velocity. The key idea is to construct *renormalized* estimate for the *velocity* variable in the compressible Navier-Stokes. It was done by generalizing the DiPerna and Lions's renormalizated technique. This provides global existence of weak solutions for the compressible Navier-Stokes with large initial data possibly vanishing on the vacuum. The work has been featured at the " Seminaire Bourbaki " in IHP, Paris, June 2017. This work treats the case when the viscous coefficients $\mu = \nu\rho,\ \lambda = 0$, which can be viewed as a typical example enjoying a particular structure (named BD entropy). In 2D, this model recovers the classical viscous shallow water equations. Thus, this also solves the Lions' open problem on the shallow water equations.

Meanwhile, the new *renormalization* technique introduced in my joint work (Invent. 2016) still leaves open the existence problem for a large class of model equations satisfying the BD entropy, especially the ones with nonlinear shear viscosities. Compared to the linear shear viscosity case, for the general nonlinear shear viscosity, there are substantial difficulties to obtain the existence theory. Our main goal is to fill this gap so as to have a full understanding of the existence theory for the compressible Navier-Stokes with the BD structure. In my joint work with Bresch and Vasseur (to appear in JEMS), thanks to a technique of *renormalized* solutions for the *velocity* variable which was based on our technique of *renormalized* estimate, we can avoid the problem of concentration in velocity. It is not necessary to derive the Mellet-Vasseur type inequality: **this allows us to cover any adiabatic constant** $\gamma > 1$, where the pressure meets the law $P = \rho^\gamma$. This new result provides an answer to a longstanding mathematical question on compressible NSE with density dependent viscosities as mentioned for instance by F. Rousset in the Bourbaki 69ème année, 2016–2017, no 1135. This is reminiscent to showing the existence of global weak solutions to the compressible NSE with constant viscous coefficients when $1 < \gamma \leq \frac{3}{2}$. This is a long standing open problem proposed by Lions in 1998. The techniques developed in our recent works shed new insight to this problem.

In a recent work with M. Chen and A. Vasseur(Adv. Math., minor revisions), we have shown that for any initial data belonging to a dense subset of the energy space, there exist infinitely many global-in-ime admissible weak solutions to the isentropic Euler system whenever $1 < \gamma \leq 1 + \frac{2}{n}$, where n is dimension. This result shows that the Euler equations are severely ill-posed for a dense set of initial data in the finite energy space. The new contribution is to allow for a broad class of initial data from which ill-posedness can be established.

We choose to construct energy sub-solutions using regularization of any weak inviscid limit of a Navier–Stokes model with degenerate viscosities. Since our existence for such models allows $\gamma > 1$, not like the standard theory which requires $\gamma > \frac{3}{2}$ for the constant viscosities. The regularization of this limit solves the Euler-Reynolds system. A new convex integration scheme for oscillatory perturbations was developed, which was inspired by the work of De Lellis and Szekelyhidi for the incompressible equations. This allows, even in the compressible case, to convex integrate any smooth positive Reynolds stress. The different structure of energy inequality for the compressible equations needs a restrict on γ. A large family of sub-solutions can be considered. Applying convex integration to sub-solutions, infinitely many global weak solutions with energy inequality can be generated.

In the next few years, we will further develop the renormalized techniques to investigate the existence of weak solutions to the more general nonlinear PDEs. We will also develop convex integration to handle the shock phenomena in the conservation laws. This will narrow down further the class of weak solutions to single out physically relevant solutions of the compressible Euler equations for the uniqueness. It will have important impacts in the shock theory of conservation laws.