Name: Solutions 10/11/22 MAC 2313 Quiz 5 (L14-16)

For questions 1 and 2, consider the function

$$f(x, y, z) = xy^2 + 3x^2 z$$

1. [2 points] Find the directional derivative of $f(x, y, z)$ at the point $(1, 3, -1)$ in the direction of $\vec{u} = \langle 2, -1, 2 \rangle$

$\nabla f = \langle y^2 + 6xz,\ 2xy,\ 3x^2 \rangle,$

$\nabla f(1,3,-1) = \langle 9-6,\ 6,\ 3 \rangle = \langle 3, 6, 3 \rangle$

$$D_{\hat{u}} f = \frac{\nabla f \cdot u}{|u|} = \frac{\langle 3,6,3 \rangle \cdot \langle 2,-1,2 \rangle}{\sqrt{4+1+4}} = \frac{6}{\sqrt{9}} = \frac{6}{3} = \boxed{2}$$

2. [2 points] Find the maximum rate of change of $f(x, y, z)$ at the same point $(1, 3, -1)$, and the direction in which it occurs.

$$\text{max r.o.c} = |\nabla f(1,3,-1)| = |\langle 3,6,3 \rangle| = \sqrt{9+36+9} = \boxed{\sqrt{54} = 3\sqrt{6}}$$

Direction: just the gradient. $\boxed{\nabla f = \langle 3,6,3 \rangle}$

or parallel, $c\nabla f$ for $c > 0$.

So if you normalized, $\dfrac{\nabla f}{|\nabla f|} = \left\langle \dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}} \right\rangle$

CHECK BACK

3. [4 points] Find equations for both the tangent plane *and* the normal line to the surface given by the equation

$$z^3 = xy^2 - yz + 1$$

at the point $(1, 1, 1)$.

Let $f = z^3 - xy^2 + yz - 1$, Surface is the level curve $f(x, y, z) = 0$.

Normal vector is $\nabla f = \langle -y^2, -2xy + z, 3z^2 + y \rangle$,

at point $(1,1,1)$ is $\nabla f(1,1,1) = \langle -1, -2+1, 3+1 \rangle$

$= \langle -1, -1, 4 \rangle = \vec{n}$

Tangent plane: $\vec{n} \cdot \langle x-1, y-1, z-1 \rangle = 0$,

$\langle -1, -1, 4 \rangle \cdot \langle x-1, y-1, z-1 \rangle = 0$

$-x + 1 - y + 1 + 4z - 4 = 0$

$\boxed{-x - y + 4z - 2 = 0}$ or equivalent.

Normal line: $\vec{r}(t) = \langle 1, 1, 1 \rangle + t \langle -1, -1, 4 \rangle$

$\boxed{\vec{r}(t) = \langle 1-t, 1-t, 1+4t \rangle}$

Or, in symmetric form, $\boxed{\dfrac{x-1}{-1} = \dfrac{y-1}{-1} = \dfrac{z-1}{4}}$

or $n = \langle 1, 1, -4 \rangle$, then tangent plane is $\boxed{x + y - 4z + 2 = 0}$

and normal line is $r(t) = \langle 1,1,1 \rangle + t \langle 1, 1, -4 \rangle$

$= \boxed{\langle 1+t, 1+t, 1-4t \rangle}$

in vector form, or

$\boxed{x-1 = y-1 = \dfrac{z-1}{-4}}$

in symmetric form