Lecture 26: Representation of functions as Power Series(II)

<u>**ex.**</u> **Use the geometric series** (L25)

$$(*) \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

to find power series representations and its radius and interval of convergence of the following functions:

1. $f(x) = \arctan x$

   The key is to remember that $\arctan x$ is an antiderivative of $1/(1+x^2)$ which we can easily find a power series representation by substituting $(-x^2)$ for x in the geometric series in (*).

   Substituting $x = 0$ and noting that $\arctan 0 = 0$, the two sides of this equation agree provided we choose $C = 0$,

   The power series converges for $|x| < 1$. Testing the endpoints separately, we find that:

2. $f(x) = \dfrac{1}{(1-x)^2}$

First, we realize that $\dfrac{1}{(1-x)^2}$ is a derivative of $\dfrac{1}{1-x}$ (with possibly some coefficient) which has a power series as in (*).

We proceed to take the derivative of both sides of (*):

and the radius and interval of convergence are

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3. Use $\ln(1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n}$ over $[-1, 1)$, (derived in L25), evaluate the following indefinite integral as a power series and determine the radius and interval of convergence?

$$\int \frac{\ln(1-t)}{t} dt$$

To start, $\ln(1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n}, R = 1.$

$$\frac{\ln(1-t)}{t} =$$

We now take the integral of both sides,

check end points:

NYTI: 1. Find the power series representation of the following functions:

a. $\dfrac{4}{x+2}$, b. $\dfrac{x}{4+9x^2}$ c. $\dfrac{x}{(1-x)^3}$

2. (a)Evaluate $f(x) = \displaystyle\int \frac{1}{1+x^5}\, dx$ as a power series.

(b) Use the first 2 terms of the power series to approximate the value of $\displaystyle\int_0^{0.2} \frac{1}{1+x^5}\, dx$.

3. Find a power series representations for

(a) $\arctan(2x)$

(b) $\ln(5-x)$

4. Use the power series representation for $\arctan x$,

(a) first approximate $\arctan 1$ and $\arctan\left(\frac{1}{\sqrt{3}}\right)$,

(b) then get an approximation for π.

(c) Why does one approximation take a fewer terms than the other to get a good approximation?

5. Find a power series and radius and interval of convergence for $f(x) = \dfrac{x}{(1-2x)^2}$

First, we note that $\dfrac{1}{(1-2x)^2}$ is a derivative of

$\dfrac{1}{1-2x}$ (with possibly some coefficient).

Geometric Series Formula

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \ |x| < 1$$

$$R = 1, \quad IOC = (-1, 1)$$

We can find power series representation for other functions by manipulating the Geometric Series Formula via Substitution, Integration and Differentiation.

- $\frac{1}{1+4x} \underbrace{=}_{(sub.)} \sum_{n=0}^{\infty} (-4x)^n = \sum_{n=0}^{\infty} (-1)^n 4^n x^n,$

  <u>Must check the Radius of convergence</u> by checking

  $$| -4x| < 1, \rightarrow |x| < 1/4,$$
  $$R = 1/4, \quad I.O.C = (-1/4, 1/4)$$

- $\ln(1-x) \underbrace{=}_{(int.)} -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\sum_{n=1}^{\infty} \frac{x^n}{n}$

  <u>Must check 'end points'</u> for convergence

  $$R = 1, \quad I.O.C = [-1, 1)$$

- $$\frac{1}{(1-x)^2} \underbrace{=}_{(diff.)} -\sum_{n=1}^{\infty} nc^{n-1} = -\sum_{n=1*}^{\infty} \frac{x^n}{n}$$

$$R = 1, \quad I.O.C = (-1, 1)$$

Note: If $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^{\boxed{\text{n}}} = 1 + x + x^2 + \cdots$

$$\frac{d}{dx}\left(\sum_{n=\boxed{0}}^{\infty} x^{\boxed{\text{n}}}\right) = 1 + 2x + 3x^2 + \cdots$$

$$= \sum_{n=\boxed{1}}^{\infty} nx^{n-1}$$

On the other hand,

$$\text{If } \sum_{n=\boxed{0}}^{\infty} x^{\boxed{2n+1}} = x + x^3 + x^5 + \cdots$$

$$\frac{d}{dx}\left(\sum_{n=\boxed{0}}^{\infty} x^{\boxed{2n+1}}\right) = 1+3x^2+5x^4+\cdots$$

$$= \sum_{n=\boxed{0}}^{\infty} (2n+1)x^{2n}$$

One must be careful with the new power series index when differentiating a power series. It may be a good idea to write out a few terms of the power series, before and after differentiating, to make sure the indexing is correct.