AN ALGEBRAIC PERSPECTIVE ON SEMI-RETRACTIONS AND THE RAMSEY PROPERTY

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1. Introduction

Our current subject is the Ramsey property for substructures and mechanisms that transfer this property between classes. Given a first-order structure $\mathcal{A}$ in any signature, let $\mathcal{K} := \mathrm{age}(\mathcal{A})$ be the class of all finitely generated substructures of $\mathcal{A}$. Given an element $A \in \mathcal{K}$, we may refer to all substructures of $\mathcal{A}$ isomorphic to A as the *A-substructures of $\mathcal{A}$*. We say that $\mathcal{K}$ has the Ramsey property if it has a certain partition property as stated in Definition 2.1, below: $\mathcal{K}$ has the *Ramsey property (RP)* if for all $A, B \in \mathcal{K}$ and integers $k \geq 1$ there exists $C \in \mathcal{K}$ such that for any coloring of the A-substructures of C, there exists a copy B' of B in C such that the coloring is constant on the A-substructures of B'. There has been much recent work in structural Ramsey theory to understand the full landscape of classes of structures with the Ramsey property. The Nešetřil-Rödl and Abramson-Harrington theorem gave general classes of structures with the Ramsey property. [1, 13]. Several years later, this work has been extended to first order structures in signatures that are not purely relational. [6]. Our context is also structures in signatures that are not purely relational.

The second author was motivated to pursue this work by a desire to better understand a particular tool in model theory, the *$\mathcal{I}$-indexed (generalized) indiscernible sequence* defined in Definition 2.10. This is a sequence of elements from some power of a model that is homogeneous in some way: the model does not make any more distinctions than the atomic formulas in $\mathcal{I}$. This tool gives streamlined proofs of the equivalence of certain dividing lines in model theory, for example that a theory is unstable if and only if it has the independence property or the strict order property, or that a theory has the tree property if and only if it has the tree property of either the first or second kind. [18] A property stated for a specific class of trees in [4] was referred to as the *modeling property* in [15] and proved to be equivalent to $\mathrm{age}(\mathcal{I})$ having the Ramsey property, under certain assumptions. In [16], this result was improved to include all cases when $\mathcal{I}$ is locally finite and ordered, modulo an additional property which was eliminated in [17] as a result of conversations with the first author. The complete dictionary theorem is stated in Theorem 2.17, below.

It was hoped that the dictionary theorem could better translate results between classification theory in model theory and structural Ramsey theory. In fact, an argument in [16] that relied on the theory of generalized indiscernible sequences deduced the Ramsey property for a certain class of trees in a functional language, that was later found to be due to Leeb. [10] In [17], the notion of *semi-retraction* is introduced, which is a pair of maps between two structures $\mathcal{A}, \mathcal{B}$ in possibly different signatures (see Definition 2.6, below.) This notion, together with the dictionary theorem, yielded a theorem detailing when a semi-retraction transfers the Ramsey property from $\mathcal{B}$ to $\mathcal{A}$, stated in Corollary 2.21 below. Since the

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notion of semi-retraction is essentially algebraic, the first author suggested an investigation into a "formula-free" proof of the same result (one that does not specifically use the tools of first-order logic), with the hope that this could lead to further insights into the dynamical consequences of the Ramsey property.

The paper is organized as follows. In Section 2, we outline our basic definitions and notational conventions as well as overview prior results. In Section 3, we explore a categorical characterization of semi-retractions. In Section 4, we state a characterization of semi-retractions under certain conditions, Theorem 4.4, and give several examples and non-examples of semi-retractions. In Section 5, we present our "formula-free" argument for how semi-retractions transfer the Ramsey property. Theorem 5.9 gives this result: if $\mathcal{A}$ is locally finite and the finitely-generated substructures of $\mathcal{B}$ are rigid, then if $\mathcal{B}$ has RP and $\mathcal{A}$ is a semi-retract of $\mathcal{B}$, $\mathcal{A}$ must have RP. Moreover, Theorem 5.5 eliminates the assumption of rigidity in the case of relational structures. The same technique yields a result on transfer of finite small Ramsey degrees in Theorem 5.13.

2. Preliminaries

We present our basic notation around sequences. Given a tuple $\overline{a} = (a_0, \ldots, a_{n-1})$ and a function f, by $f(\overline{a})$ we mean $(f(a_0), \ldots, f(a_{n-1}))$ and ran $\overline{a} := \{a_i \mid i < n\}$. For two tuples $\overline{a}, \overline{b}$, by $\overline{a} \subseteq \overline{b}$ we mean that ran $\overline{a} \subseteq$ ran $\overline{b}$ (even if $\overline{a}$ enumerates a structure, this inclusion does not carry the connotation of the substructure relation.) For an integer $k \geq 0$, $k := \{0, 1, 2, \ldots, k-1\}$. For an integer $k \geq 1$ a **k-coloring** of a set X is any function $c : X \to k$. All tuples $\overline{a}$ are assumed to be finite unless said otherwise. Given a function $f : X^n \to Y$ and $\overline{\imath}' := (\overline{\imath}_k : k < s) \in {}^s(X^n)$, we define $f(\overline{\imath}') = (f(\overline{\imath}_k) : k < s)$.

A signature L is a list of symbols that must be interpreted in any L-structure as either relations or functions of the specified arity. We do not assume that signatures are either finite or relational, unless explicitly stated. Given a structure $\mathcal{A}$, $L_{\mathcal{A}}$ refers to the signature of $\mathcal{A}$ and $|\mathcal{A}|$ refers to the underlying set of $\mathcal{A}$. Th($\mathcal{A}$) is the set of all $L_{\mathcal{A}}$-sentences true in $\mathcal{A}$. An n-ary L-formula is a first-order formula with free variables included in the list $(x_0, \ldots, x_{n-1})$. Given $\overline{a}$ from $\mathcal{A}$, $\langle \overline{a} \rangle_{\mathcal{A}}$ denotes the substructure generated by $\overline{a}$ in $\mathcal{A}$ (it will be the closure of $\overline{a}$ under the n-ary function symbols of $L_{\mathcal{A}}$ for all $n \geq 0$.) Given $\overline{x}$ a (possibly infinite) tuple in 1-1 correspondence with elements of $\mathcal{A}$, by $\mathrm{Diag}_{\mathcal{A}}(\overline{x})$ we mean the set of all basic relations $R(\overline{x})$ true of the enumeration $\overline{a}$ from $\mathcal{A}$ corresponding to $\overline{x}$. For the basics of formulas and structures, the reader is referred to [11, 5].

For structures $\mathcal{A}, \mathcal{B}$, $\mathcal{A} \subseteq \mathcal{B}$ always denotes that $\mathcal{A}$ is a substructure of $\mathcal{B}$, in which case they are structures in the same signature. The age of a structure $\mathcal{A}$, age($\mathcal{A}$) is the set of all finitely-generated substructures of $\mathcal{A}$, modulo $L_{\mathcal{A}}$-isomorphism. We will use roman letters A, B for finitely generated substructures of a given structure $\mathcal{A}$. For structures $\mathcal{A}, \mathcal{A}'$, $\mathcal{A} \cong \mathcal{A}'$ means that the structures are isomorphic (and thus, they are in the same signature, L.) To emphasize the shared signature, we might write $\mathcal{A} \cong_L \mathcal{A}'$. If there exists a structure $\mathcal{A}$ such that $A, A' \subseteq \mathcal{A}$, we might write $A \cong_{\mathcal{A}} A'$ to emphasize not only that the two structures A, A' are in the signature $L_{\mathcal{A}}$, but that they are substructures of $\mathcal{A}$ (i.e., we are working in $\mathcal{A}$, for the purposes of verifying a partition property of age($\mathcal{A}$).)

Fix a structure $\mathcal{A}$.

- An *n-type over $\emptyset$ in $\mathcal{A}$* is a set of n-ary formulas that is consistent with Th($\mathcal{A}$). This type is *complete* if for every $L_{\mathcal{A}}$-formula φ, either φ or $\neg\varphi$ is in the type.
- An n-type p over $\emptyset$ in $\mathcal{A}$ is *realized (in $\mathcal{A}$)* if there exists $\overline{a} \in \mathcal{A}^n$ such that $\mathcal{A} \models \varphi(\overline{a})$ for all $\varphi \in p$.

- We define $S^{\mathcal{M}}_n(\emptyset)$ to be the space of all n-types over $\emptyset$ in $\mathcal{A}$ with the usual Stone topology, with basic open sets $[\psi] := \{p \in S^{\mathcal{M}}_n(\emptyset) \mid \psi \in p\}$.
- A *quantifier-free type in* $\mathcal{A}$ is a type in $\mathcal{A}$ that contains only quantifier-free formulas. This type is a *complete quantifier-free type* if for every quantifier-free $L_{\mathcal{A}}$-formula θ, either θ or $\neg\theta$ is in the type.
- Given a tuple $\bar{\imath} \in \mathcal{A}^n$, $\mathrm{tp}^{\mathcal{A}}(\bar{\imath})$ is the complete type of $\bar{\imath}$ in $\mathcal{A}$ and $\mathrm{qftp}^{\mathcal{A}}(\bar{\imath})$ is the complete quantifier-free type of $\bar{\imath}$ in $\mathcal{A}$.
- For same-length tuples $\overline{a}, \overline{a}'$ from $\mathcal{A}$, we will use $\overline{a} \sim_{\mathcal{A}} \overline{a}'$ to mean that $\mathrm{qftp}^{\mathcal{A}}(\overline{a}) = \mathrm{qftp}^{\mathcal{A}}(\overline{a}')$, and $\overline{a} \equiv_{\mathcal{A}} \overline{a}'$ to mean that $\mathrm{tp}^{\mathcal{A}}(\overline{a}) = \mathrm{tp}^{\mathcal{A}}(\overline{a}')$.

If a structure $\mathcal{A}$ is saturated then, among other things, it realizes all types $p \in S^{\mathcal{A}}_{\kappa}(\emptyset)$ such that $\kappa \leq |\mathcal{A}|$ (see [18].)

We repeat the following definition of the Ramsey Property for substructures from [8, Intro part (D)],[14],[12]:

Definition 2.1. We say that an age $\mathcal{K}$ of structures has the **Ramsey property (RP)** if for all $A, B \in \mathcal{K}$ and finite integers $k \geq 1$ there exists $C \in \mathcal{K}$ such that for any k-coloring c of $\binom{C}{A}$, there is $B' \in \binom{C}{B}$ such that for any $A', A'' \in \binom{B'}{A}$, $c(A') = c(A'')$.

We say that B' is a copy of B **homogeneous for** c **(on copies of** A**)**.

Definition 2.2. Given a structure $\mathcal{A}$, we say that A **has RP** if $\mathrm{age}(\mathcal{A})$ has RP.

We also list examples of classes, mostly listed in [8, 12], as they fall within this characterization.

Example 2.3. The following classes have RP:

(1) All simple graphs with no loops with an ordering on the vertices in $L = \{R, <\}$.
(2) Convexly ordered finite equivalence relations in $L = \{E, <\}$.
(3) All finite sets in $L = \emptyset$.
(4) All finite linear orders in $L = \{<\}$.

Example 2.4. The following classes do not have RP:

(1) Finite equivalence relations with any ordering on points in $L = \{E, <\}$.
(2) Partial orders with any linear ordering on points in $L = \{<, \prec\}$ [**?**]

2.1. **Semi-retractions.** The following two definitions were given in [17].

Definition 2.5. Given any structures $\mathcal{A}, \mathcal{B}$, we say that an injection $h : \mathcal{A} \to \mathcal{B}$ is

(i) **quantifier-free type-respecting (qftp-respecting)** if for all finite, same-length tuples $\bar{\imath}, \bar{\jmath}$ from $\mathcal{A}$,
$$\bar{\imath} \sim_{\mathcal{A}} \bar{\jmath} \Rightarrow h(\bar{\imath}) \sim_{\mathcal{B}} h(\bar{\jmath}).$$
(ii) **quantifier-free type-preserving (qftp-preserving)** if $\mathcal{A}, \mathcal{B}$ are structures in the same signature and $\mathrm{qftp}^{\mathcal{A}}(\bar{\imath}) = \mathrm{qftp}^{\mathcal{B}}(h(\bar{\imath}))$ (thus, it is also qftp-respecting.)

Definition 2.6. Let $\mathcal{A}$, $\mathcal{B}$ be any structures. We say that $\mathcal{A}$ is a **semi-retract of** $\mathcal{B}$ **(via** (g, f)**)** if

(1) there exist qftp-respecting injections: $\mathcal{A} \xrightarrow{g} \mathcal{B} \xrightarrow{f} \mathcal{A}$
(2) such that: $\mathcal{A} \xrightarrow{fg} \mathcal{A}$ is an embedding (equivalently, is qftp-preserving.)

We refer to the pair (g, f) as the **semi-retraction between** $\mathcal{A}$ **and** $\mathcal{B}$. We will refer to property (1) in this Definition as the *qftp-respecting property of semi-retractions* and property (2) as the *composition property of semi-retractions*.

Observation 2.7. If $\mathcal{A}$ is a semi-retract of $\mathcal{B}$, then $||\mathcal{A}|| = ||\mathcal{B}||$, by the Schröder-Bernstein theorem.

We make some additional observations below.

Proposition 2.8. *If the theory of $\mathcal{A}$ eliminates quantifiers, age$(\mathcal{B})$ has AP and $\mathcal{A}$ is a semi-retract of $\mathcal{B}$, then age$(\mathcal{A})$ has AP.*

The following Proposition can be generalized to show that the existence of a semi-retraction is a property of the ages of $\mathcal{A}$ and $\mathcal{B}$, as these structures may be replaced with any $\mathcal{A}', \mathcal{B}'$ of the same cardinality with the same respective ages.

Proposition 2.9. *If $\mathcal{K}' :=$ age$(\mathcal{B})$ has AP and $\mathcal{B}'$ is the Fraïssé-limit of $\mathcal{K}'$ and $\mathcal{A}$ is a semi-retract of $\mathcal{B}$, then $\mathcal{A}$ is a semi-retract of $\mathcal{B}'$*

Proof. We can define a type in constants $a \in \mathcal{A}$ and new function symbols f, g:

$$\Gamma = \{(\overline{a}, g(\overline{a}), fg(\overline{a})) \mid n < \omega, \overline{a} \in A^n, P\}$$

where the conditions P (using additional relation symbols to pick out quantifier-free types) are that

(1) $\overline{a} \sim_{\mathcal{A}} fg(\overline{a})$
(2) $\overline{a}_1 \sim_{\mathcal{A}} \overline{a}_2 \Rightarrow g(\overline{a}_1) \sim_{\mathcal{B}} g(\overline{a}_2)$
(3) $g(\overline{a}_1) \sim_{\mathcal{B}} g(\overline{a}_2) \Rightarrow fg(\overline{a}_1) \sim_{\mathcal{A}} fg(\overline{a}_2)$
(4) f, g give injections

Now let $\overline{a}$ be an infinite tuple enumerating all elements of $\mathcal{A}$. We can take a union of Γ together with $\mathrm{Diag}_{\mathcal{A}}(\overline{a})$ and $\mathrm{Diag}_{\mathcal{B}'}(g(\overline{a}))$. This type is now finitely satisfiable in $(\mathcal{A}, \mathcal{B}')$ and in the realization, the constants $g(\overline{a})$ will be in the copy of $\mathcal{B}'$. □

2.2. **The modeling property.** The following is mostly from [17]. In our applications below, the infinite structure $\mathcal{I}$ is typically replaced by $\mathcal{A}$.

Definition 2.10. Fix a structure $\mathcal{I}$, an integer $l \geq 1$, and l-tuples $\overline{a}_i$ from some structure $\mathcal{M}$, for all $i \in \mathcal{I}$. We say that $(\overline{a}_i \mid i \in \mathcal{I})$ is an **$\mathcal{I}$-indexed indiscernible set** if for any integer $n \geq 1$, for all n-tuples $\overline{\imath}, \overline{\jmath}$ from $\mathcal{I}$,

$$\overline{\imath} \sim_{\mathcal{I}} \overline{\jmath} \Rightarrow \overline{a}_{\overline{\imath}} \equiv_{\mathcal{M}} \overline{a}_{\overline{\jmath}}.$$

We say that $(\overline{a}_i \mid i \in \mathcal{I})$ is an **$\mathcal{I}$-indexed indiscernible sequence** if $\mathcal{I}$ is an ordered structure, or a **generalized indiscernible sequence** if $\mathcal{I}$ is an ordered structure that is clear from context.

We repeat definitions from [16] as Definition 2.11 and Definition 2.15.

Definition 2.11. Given an integer $l \geq 1$, an L'-structure $\mathcal{I}$, an L-structure $\mathcal{M}$ and an $\mathcal{I}$-indexed set of l-tuples from $\mathcal{M}$, $X = (\overline{a}_i \mid i \in \mathcal{I})$, we define the **EM-type of X (EMtp(X))** to be a syntactic type in variables $(\overline{x}_i \mid i \in \mathcal{I})$, where $\ell(\overline{x}_i) = l$ for each $i \in \mathcal{I}$, as follows:

$$\mathrm{EMtp}(X) = \{\psi(\overline{x}_{i_0}, \ldots, \overline{x}_{i_{n-1}}) \mid \psi \in L, \overline{\imath} \in {}^n\mathcal{I} \text{ and } (\forall \overline{\jmath} \in {}^n\mathcal{I})(\overline{\jmath} \sim_{\mathcal{I}} \overline{\imath} \Rightarrow \mathcal{M} \vDash \psi(\overline{a}_{j_0}, \ldots, \overline{a}_{j_{n-1}}))\}$$

Proposition 2.12 is a useful equivalence which follows directly from Definition 2.11 (see Proposition 2 of [16] for more details):

Proposition 2.12. *Given an L'-structure $\mathcal{I}$ and an L-structure $\mathcal{M}$, fix sets of l-tuples from $\mathcal{M}$ indexed by $\mathcal{I}$, $X = (\overline{a}_i \mid i \in \mathcal{I})$ and $Y = (\overline{b}_i \mid i \in \mathcal{I})$. $Y \vDash EMtp(X)$ if and only if for any integer $n \geq 1$, for all complete quantifier-free n-types η in $\mathcal{I}$ and all $n \cdot l$-ary formulas $\varphi \in L$, if we have the* rule

$$(\forall \overline{\jmath})(\mathcal{I} \vDash \eta(\overline{\jmath}) \Rightarrow \mathcal{M} \vDash \varphi(\overline{a}_{\overline{\jmath}}))$$

then we have the rule

$$(\forall \overline{\jmath})(\mathcal{I} \vDash \eta(\overline{\jmath}) \Rightarrow \mathcal{M} \vDash \varphi(\overline{b}_{\overline{\jmath}}))$$

Definition 2.13. Fix sequences of parameters $X = (\overline{a}_i \mid i \in \mathcal{A})$, $Y = (\overline{b}_i \mid i \in \mathcal{A})$, where $\overline{a}_i, \overline{b}_i$ are from some L-structure $\mathbb{U}$.

We say Y **is locally based on** X if for any finite set of L-formulas, Δ, and for any finite tuple $(j_0, \ldots, j_{n-1})$ from $\mathcal{A}$, there exists a tuple $(i_0, \ldots, i_{n-1})$ from $\mathcal{A}$ such that

$$\overline{\jmath} \sim_{\mathcal{A}} \overline{\imath}$$

and

$$\overline{b}_{\overline{\jmath}} \equiv_{\Delta} \overline{a}_{\overline{\imath}}$$

Where Y and X are understood from context, this property will be referred to as the *local basedness*.

The following is from Proposition 2 in [16]. A proof sketch is given here.

Proposition 2.14. *Fix sequences of parameters $X = (a_i \mid i \in \mathcal{I})$, $Y = (b_i \mid i \in \mathcal{I})$, where a_i, b_i are from some L-structure $\mathbb{U}$. Y is locally based on X if and only if $Y \vDash EMtp(X)$.*

Proof. Suppose $Y \vDash \mathrm{EMtp}(X)$. To show local basedness, fix a finite set of L-formulas Δ and a finite tuple $\overline{\jmath}$ from $\mathcal{I}$ with complete quantifier-free type η in $\mathcal{I}$. Let φ be a conjunction of all the formulas in the finite Δ-type of $\overline{b}_{\overline{\jmath}}$. Suppose, for contradiction, there is no $\overline{\imath} \sim_{\mathcal{I}} \overline{\jmath}$ such that $\overline{a}_{\overline{\imath}} \equiv_{\Delta} \overline{b}_{\jmath}$. Then:

$$(\forall \overline{\imath})(\mathcal{I} \vDash \eta(\overline{\imath}) \Rightarrow \mathbb{U} \vDash \neg\varphi(\overline{a}_{\overline{\imath}}))$$

Since $Y \vDash \mathrm{EMtp}(X)$, we must have that

$$(\forall \overline{\jmath})(\mathcal{I} \vDash \eta(\overline{\jmath}) \Rightarrow \mathbb{U} \vDash \neg\varphi(\overline{b}_{\overline{\jmath}}))$$

which contradicts the Δ-type of $\overline{b}_{\overline{\jmath}}$.

Suppose Y is locally based on X. To show $Y \vDash \mathrm{EMtp}(X)$, consider a rule from $\mathrm{EMtp}(X)$:

$$(\forall \overline{\imath})(\mathcal{I} \vDash \eta(\overline{\imath}) \Rightarrow \mathbb{U} \vDash \varphi(\overline{a}_{\overline{\imath}}))$$

Fix any $\overline{\jmath}$ from $\mathcal{I}$ such that $\mathcal{I} \vDash \eta(\overline{\jmath})$. By local basedness, there is $\overline{\imath} \sim_{\mathcal{I}} \overline{\jmath}$ such that $\overline{b}_{\overline{\jmath}} \equiv_{\{\varphi\}} \overline{a}_{\overline{\imath}}$. By the rule for η, $\mathbb{U} \vDash \varphi(\overline{a}_{\overline{\imath}})$. Thus we have that $\mathbb{U} \vDash \varphi(\overline{b}_{\overline{\jmath}})$, as well. And so we have proved the rule:

$$(\forall \overline{\jmath})(\mathcal{I} \vDash \eta(\overline{\jmath}) \Rightarrow \mathbb{U} \vDash \varphi(\overline{b}_{\overline{\jmath}}))$$

Since this is true for any rule, $Y \vDash \mathrm{EMtp}(X)$. □

Definition 2.15. Given a structure $\mathcal{I}$, we say that $\mathcal{I}$-indexed indiscernible sets have the **modeling property** if for any integer $l \geq 1$, any $|\mathcal{I}|^+$-saturated structure $\mathbb{U}$, and any $\mathcal{I}$-indexed set of l-tuples from $\mathbb{U}$

$$X = (\overline{a}_i \mid i \in \mathcal{I}),$$

there exists an $\mathcal{I}$-indexed indiscernible set of l-tuples from $\mathbb{U}$

$$Y = (\overline{b}_i \mid i \in \mathcal{I})$$

such that $Y \models \mathrm{EMtp}(X)$ *(equivalently, Y is locally based on X, by Proposition 2.14, above.)*

Remark 2.16. In fact, it suffices to require that $\mathbb{U}$ as above be $|\mathcal{I}|$-saturated, since the type describing Y has $|\mathcal{I}|$-many variables and no parameters from $\mathbb{U}$. See [18].

Theorem 2.17 (dictionary theorem)**.** *Suppose that $\mathcal{I}$ is a locally finite ordered structure. $\mathcal{I}$-indexed indiscernible sequences have the modeling property if and only if age$(\mathcal{I})$ has RP.*

The dictionary theorem fails when we drop order:

Example 2.18. Let $\mathcal{I} = (\mathbb{N}, =)$ and note that age$(\mathcal{I})$ has RP by Example 2.3. If we take an $\mathcal{I}$-indexed set in $\mathcal{M} := (\mathbb{N}, <)$, $X = (i \mid i \in \mathcal{I})$, then there is no $\mathcal{I}$-indexed indiscernible set in any extension $\mathcal{M}' \succeq \mathcal{M}$ locally based on X. Such a set would need to have $\mathrm{tp}^{\mathcal{M}'}(i,j) = \mathrm{tp}^{\mathcal{M}'}(j,i)$ for $i \neq j \in \mathbb{N}$, which is not possible.

The dictionary theorem fails when we drop local finiteness:

Example 2.19. Let $\mathcal{I} = (\mathbb{Z}, p, s, <)$ be the structure on $\mathbb{Z}$ with the usual order $<$ and where p, s are unary function symbols interpreted as "predecessor" and "successor", respectively. The only possible finitely generated substructure of $\mathbb{Z}$ is the whole structure. Since $||\mathrm{age}(\mathcal{I})|| = 1$, the class trivially has the RP. However, we will show that $\mathcal{I}$-indexed indiscernibles do not have the modeling property, showing the essentialness of the assumption that $\mathcal{I}$ be locally finite in Theorem 2.17. Let $\mathcal{M}$ be the Fraïssé limit of finite convexly ordered equivalence classes in language $\{E, \prec\}$. Let $X = (a_i \mid i \in \mathbb{Z})$ such that all a_i for i odd are in one E-class that we call `Odd` and all a_j for j even are in a separate E-class that we call `Even`. Moreover, let $i < j \Rightarrow a_i \prec a_j$, and `Odd` $<$ `Even` in $\mathcal{M}$. Within this example, we use $\sim$ to denote E-equivalence in the following picture, where elements are listed in $\prec$-increasing order in $\mathcal{M}$:

$$\ldots a_1 \sim a_3 \sim a_5 \ldots \nsim \ldots a_2 \sim a_4 \sim a_6 \ldots$$

The type EMtp(X) declares that whenever $j = s(s(i))$, $a_i \prec a_j$ are E-equivalent. However a decision is not made about when $j = s(i)$, since sometimes $a_i \prec a_j$ and sometimes $a_j \prec a_i$, though a_i, a_j are always E-inequivalent in this case. Suppose there is an $\mathcal{I}$-indexed indiscernible set locally based on X. If this indiscernible set chooses the case $a_i \prec a_j$, whenever $j = s(i)$, then $\mathcal{M}$ would admit equivalence classes that are not convexly ordered, e.g. with $b_1 \sim b_3$ but

$$\ldots b_1 \nsim b_2 \nsim b_3 \ldots$$

a contradiction. The alternative, choosing $a_i \succ a_{s(i)}$, is incompatible with $a_i \prec a_{s(s(i))}$.

Theorem 2.20. *Let $\mathcal{A}$ and $\mathcal{B}$ be any structures. Suppose that $\mathcal{A}$ is a semi-retract of $\mathcal{B}$. Furthermore, suppose that $\mathcal{B}$-indexed indiscernible sets have the modeling property. Then $\mathcal{A}$-indexed indiscernible sets have the modeling property.*

Corollary 2.21. *Let $\mathcal{A}$ and $\mathcal{B}$ be locally finite ordered structures. Suppose that $\mathcal{A}$ is a semi-retract of $\mathcal{B}$ and $\mathcal{B}$ has RP. Then $\mathcal{A}$ has RP.*

3. Semi-retractions and categorical retractions

Definition 3.1. Say that a structure $\mathcal{A}$ has property `rqe` (realized, quantifier-eliminable types) if

$$S_n^{\mathcal{A}}(\emptyset) = \{\mathrm{qftp}(\overline{a}) : \overline{a} \in \mathcal{A}^n\}$$

Observation 3.2. If $\mathcal{A}$ has property `rqe` then every n-type over the empty set in $\mathcal{A}$ is realized (in $\mathcal{A}$).

Remark 3.3. If $\mathcal{A}$ is in a finite relational signature and $\mathrm{Th}(\mathcal{A})$ eliminates quantifiers, then complete quantifier-free types are complete types that are also finite types, so any type realized in an elementary extension is realized in $\mathcal{A}$.

If $\mathcal{A}$ is an ultrahomogeneous structure, then it suffices that $\mathcal{A}$ is uniformly locally finite for $\mathrm{Th}(\mathcal{A})$ to be $\aleph_0$-categorical and have quantifier elimination, thus, for it to be `rqe`. (see [5])

The following is from [3].

Definition 3.4. Given a category $\mathbf{C}$ and morphisms $f \in \mathbf{C}(X, Y)$ and $g \in \mathbf{C}(Y, X)$, if $fg = \mathrm{id}_Y$, i.e.

$$Y \xrightarrow{g} X \xrightarrow{f} Y$$

and

$$Y \xrightarrow{fg=\mathrm{id}} Y$$

we say that the pair of maps (g, f) [my choice of order] is a **retraction** of X onto Y and that Y is the **retract** of X (via f and g).

Observation 3.5. It is observed in [3] that in the category $\mathbf{C} = \mathbf{Set}$

- g as above must be 1-1
- $gf : X \xrightarrow{\text{onto}} B$ where $\mathrm{Img} = B \subseteq X$
- $B \xrightarrow{gf=\mathrm{id}} B$

Definition 3.6. Given $\mathcal{A}, \mathcal{B}$ that have `rqe`, let $g : \mathcal{A} \to \mathcal{B}$ be a qftp-respecting injection on the underlying sets. We define $\theta_g : S_n^{\mathcal{A}}(\emptyset) \to S_n^{\mathcal{B}}(\emptyset)$ to take $p \in S_n^{\mathcal{A}}(\emptyset)$ to $q \in S_n^{\mathcal{B}}(\emptyset)$ such that there exists $\bar{\imath} \in \mathcal{A}^n$ satisfying $p(\bar{x})$ and $g(\bar{\imath}) \in \mathcal{B}^n$ satisfies $q(\bar{x})$ (i.e. $q = \mathrm{qftp}^{\mathcal{B}}(g(\bar{\imath}))$).

Remark 3.7. Note that the above Defintion is well-defined since for any $\bar{\imath}, \bar{\imath}'$ satisfying $p(\bar{x})$, $\bar{\imath} \sim_{\mathcal{A}} \bar{\imath}'$ and so $g(\bar{\imath}) \sim_{\mathcal{B}} g(\bar{\imath}')$ and thus $\mathrm{qftp}^{\mathcal{B}}(g(\bar{\imath})) = \mathrm{qftp}^{\mathcal{B}}(g(\bar{\imath}'))$.

Proposition 3.8. *Given $\mathcal{A}, \mathcal{B}$ that have `rqe`, let $g : \mathcal{A} \to \mathcal{B}$ be a qftp-respecting injection on the underlying sets. The map θ_g as in Definition 3.6 is continuous.*

Proof. Fix a nonempty $[\varphi] \subseteq S_n^{\mathcal{B}}(\emptyset)$ for some formula φ in the language of $\mathcal{B}$. By the `rqe` assumption, for all $\bar{\imath}, \bar{\jmath}$ from $\mathcal{A}$

$$\mathrm{qftp}^{\mathcal{A}}(\bar{\imath}) = \mathrm{qftp}^{\mathcal{A}}(\bar{\jmath}) \vdash \mathrm{qftp}^{\mathcal{B}}(g(\bar{\imath})) = \mathrm{qftp}^{\mathcal{B}}(g(\bar{\jmath})) \tag{1}$$

Thus, in particular

$$\mathrm{qftp}^{\mathcal{A}}(\bar{\imath}) = \mathrm{qftp}^{\mathcal{A}}(\bar{\jmath}) \vdash \varphi(g(\bar{\imath})) \leftrightarrow \varphi(g(\bar{\jmath})) \tag{2}$$

By compactness in the context of the 2-sorted structure with a named map $(\mathcal{A}, \mathcal{B}, g)$:

$$\bigwedge_s (\psi_s(\bar{\imath}) \leftrightarrow \psi_s(\bar{\jmath})) \vdash \varphi(g(\bar{\imath})) \leftrightarrow \varphi(g(\bar{\jmath})) \tag{3}$$

We may code this finite set of formulas into one formula ψ (as in [18])

$$\psi(\bar{\imath}) \leftrightarrow \psi(\bar{\jmath}) \vdash \varphi(g(\bar{\imath})) \leftrightarrow \varphi(g(\bar{\jmath})) \tag{4}$$

If $\theta_g^{-1}([\varphi]) = \emptyset$, then there is nothing to show, so fix any $p \in \theta_g^{-1}([\varphi])$. Thus $\varphi \in \theta_g(p) =: q$. Either ψ or $\neg\psi$ is in p, without loss of generality, $\psi \in p$. Let $\bar{\imath}$ realize p in $\mathcal{A}$ (so also

$\mathcal{A} \vDash \psi(\bar{\imath})$). Thus $g(\bar{\imath})$ realizes $\theta_g(p) = q$ in $\mathcal{B}$ (by the definition of θ_g) and so $\mathcal{B} \vDash \varphi(g(\bar{\imath}))$ By the universal condition (4):

$$\psi(\overline{x}) \vdash \varphi(g(\overline{x})) \tag{5}$$

Thus, for any type $p' \in [\psi]$, $\theta_g(p') \in [\varphi]$. This proves that the open subset $[\psi] \ni p$ is included in $\theta_g^{-1}([\varphi])$. □

Proposition 3.9. *Let $\mathcal{A}, \mathcal{B}$ satisfy property* `rqe` *from Definition 3.1. If $\mathcal{A}$ is a semi-retract of $\mathcal{B}$ via (g, f), then the pair of maps (θ_g, θ_f) is a retraction of $S_n^{\mathcal{B}}(\emptyset)$ onto $S_n^{\mathcal{A}}(\emptyset)$ in the category* **C** *of continuous maps between type spaces.*

Proof. By Proposition 3.8, we know that (θ_g, θ_f) are maps in the category **C**.

By assumption, there exist qftp-respecting injections on the underlying sets:

$$A \xrightarrow{g} \mathcal{B} \xrightarrow{f} \mathcal{A}$$

such that the composition is qftp-preserving:

$$\mathcal{A} \xrightarrow{fg} \mathcal{A}$$

Thus,

$$S_n^{\mathcal{A}}(\emptyset) \xrightarrow{\theta_g} S_n^{\mathcal{B}}(\emptyset) \xrightarrow{\theta_f} S_n^{\mathcal{A}}(\emptyset)$$

and

$$S_n^{\mathcal{A}}(\emptyset) \xrightarrow{\theta_f \cdot \theta_g = \mathrm{id}_{S_n^{\mathcal{A}}(\emptyset)}} S_n^{\mathcal{A}}(\emptyset)$$

which shows that the pair of maps is indeed a retraction in this category. □

It is interesting to compare a retraction in this category to another studied in model theory, albeit, ours concerns the type spaces (the orbits) whereas the following concerns the automorphism groups. In [2], Ahlbrandt and Ziegler, in the context of $\aleph_0$-categorical structures, give the following definition:

Definition 3.10. $\mathcal{A}$ is a **retraction** of $\mathcal{B}$ if there exist interpretations $f : \mathcal{A} \rightsquigarrow \mathcal{B}$ $g : \mathcal{B} \rightsquigarrow \mathcal{A}$ such that $g \circ f$ is homotopic to the identity on A.

An unpublished result attributed to T. Coquand in [2] states: $\mathcal{A}$ is a retraction of $\mathcal{B}$ iff there are continuous homomorphisms

$$\mathrm{Aut}(\mathcal{A}) \xrightarrow{\varphi} \mathrm{Aut}(\mathcal{B}) \xrightarrow{\psi} \mathrm{Aut}(\mathcal{A})$$

such that $\psi \circ \varphi = 1$.

4. Examples and Non-examples

To obtain a non-example, a case where $\mathcal{A}$ could not be a semi-retract of $\mathcal{B}$, we need a characterization of semi-retractions under certain assumptions.

Definition 4.1. We say that $\mathcal{A}$ is a **quantifier-free reduct** of $\mathcal{B}$ if $|\mathcal{A}| = |\mathcal{B}| = M$ and $\sim_{\mathcal{B}}$ refines $\sim_{\mathcal{A}}$ on M.

Proposition 4.2. *$\mathcal{A}$ is a quantifier-free reduct of $\mathcal{B}$ if and only if $|\mathcal{A}| = |\mathcal{B}|$ and the identity map $id : \mathcal{B} \to \mathcal{A}$ is qftp-respecting.*

Remark 4.3. For two structures $\mathcal{A}, \mathcal{B}$ such that $|\mathcal{A}| = |\mathcal{B}|$, $\mathcal{A}$ is a quantifier-free reduct of $\mathcal{B}$ if any of the following:

(1) $\sigma_A \subseteq \sigma_B$ (containment of signatures)
(2) Every basic relation in the signature of $\mathcal{A}$ is equivalent to a quantifier-free formula of $\mathcal{B}$.

Theorem 4.4. *Fix locally finite ordered structures $\mathcal{A}$ and $\mathcal{B}$ and suppose that $\mathcal{A}$ is a quantifier-free reduct of $\mathcal{B}$, i.e., for some κ, $|\mathcal{A}| = |\mathcal{B}| = \kappa$. Assume that $\mathcal{B}$ is saturated (i.e., κ-saturated.) Suppose that every quantifier-free type in $\mathcal{B}$ is equivalent to an $L_{\mathcal{B}}$-formula in $\mathcal{B}$. Suppose that $\mathcal{B}$ has RP. Then, $\mathcal{A}$ is a semi-retract of $\mathcal{B}$ if and only if $\mathcal{A}$ has RP.*

Proof. The $\Rightarrow$ is from Corollary 2.21. It remains to show $\Leftarrow$.

Suppose that $\mathcal{A}$ has RP. Let $X = (a_i \mid i \in \mathcal{A})$ enumerate $\mathcal{B}$ where $a_i = i$. Since $\mathcal{A}$ has RP, by Theorem 2.17, $\mathcal{A}$-indexed indiscernible sets have the modeling property (see Definitions 2.10, 2.13, 2.15 and Proposition 2.14.) Let $Y = (b_i \mid i \in \mathcal{A})$ be an $\mathcal{A}$-indexed indiscernible set locally based on X. Define $f : \mathcal{A} \to \mathcal{B}$ to take $i \mapsto b_i$. This is an injective map, since $a_i \neq a_j$ for all $i \neq j$ and Y is $\mathcal{A}$-indexed indiscernible locally based on X. It remains to show that

$$\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{id} \mathcal{A}$$

is a semi-retraction.

By Proposition 4.2, we already know that the identity map is qftp-respecting. We must show the remaining properties from Definition 2.5: (i) $\bar{\imath}_1 \sim_{\mathcal{A}} \bar{\imath}_2 \Rightarrow f(\bar{\imath}_1) \sim_{\mathcal{B}} f(\bar{\imath}_2)$ and in fact that (ii) $\bar{\jmath} \sim_{\mathcal{A}} id(f(\bar{\jmath})) = f(\bar{\jmath})$ for all n-tuples $\bar{\jmath}, \bar{\imath}_1, \bar{\imath}_2$ from $\mathcal{A}$, for all $n < \omega$. We have (i) as a direct consequence of $\mathcal{A}$-indexed indiscernibility. We have (ii) from the local basedness: for every finite subset of formulas in the language of $\mathcal{B}$, Δ, for every length-n $\bar{\jmath}$ from $\mathcal{A}$, there is $\sim_{\mathcal{A}}$-equivalent $\bar{\imath}$ from $\mathcal{A}$ such that $b_{\bar{\jmath}} \equiv_{\Delta} a_{\bar{\imath}}$. In other words, $f(\bar{\jmath}) = b_{\bar{\jmath}} \equiv_{\Delta} a_{\bar{\imath}} = \bar{\imath} \sim_{\mathcal{A}} \bar{\jmath}$, and that is for every finite Δ. Let θ be the formula in $L_{\mathcal{B}}$ equivalent to $\mathrm{qftp}^{\mathcal{B}}(\bar{\imath})$. Since we may let $\Delta = \{\theta(\overline{x})\}$ this shows that $f(\bar{\jmath}) \sim_{\mathcal{B}} \bar{\imath} \sim_{\mathcal{A}} \bar{\jmath}$, which by the assumption that $\mathcal{A}$ is a quantifier-free reduct of $\mathcal{B}$ shows that $f(\bar{\jmath}) \sim_{\mathcal{A}} \bar{\jmath}$, showing (ii). ☐

Example 4.5. Let $\mathcal{B} := \mathcal{R}^<$ be the random ordered graph (the Fraïssé limit of ordered graphs) and $\mathcal{A} := (\mathbb{Q}, <)$ the rational order, both on underlying set $\mathbb{Q}$. $\mathcal{A}$ is a quantifier-free reduct of $\mathcal{B}$, $\mathcal{B}$ is countably saturated, and both are locally finite and ordered. By Theorem 4.4, $\mathcal{A}$ is a semi-retract of $\mathcal{R}$B

In order to find the semi-retraction, one strategy is to take an indiscernible sequence $(f(i) \mid i \in \mathcal{A})$ in $\mathcal{R}^<$, such as when $f(\mathbb{Q})$ is densely ordered copy of K_ω:

$$\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{id} \mathcal{A}$$

witnesses that $\mathcal{A}$ is a semi-retract of $\mathcal{R}$.

Both are known to have RP, and the RP can be thought of as transferring by Theorem 5.9.

We have several examples of semi-retractions, not all witnessing transfer of the RP:

Example 4.6. Let $\mathcal{A} = (\mathbb{Q}, <)$ and $\mathcal{B} = (\mathbb{Q}, <, Q_1, \ldots, Q_n)$ where each Q_i is a dense piece of the partition (see [7].) Then $\mathcal{A}$ is a semi-retract of $\mathcal{B}$, since we can map $\mathcal{A}$ into one piece of the partition in $\mathcal{B}$ and back out again.

Both are known to have finite (small) Ramsey degrees, and the finite Ramsey degrees can be thought of as transferring by Theorem 5.13.

However $\mathcal{B}$ is not a semi-retract of $\mathcal{A}$ because $\mathcal{A}$ has RP, $\mathcal{B}$ fails RP, and we are satisfying the conditions of Theorem 4.4.

Next we look at three locally finite ordered structures known to have RP: the *Shelah tree* $\mathcal{I}_{\text{stree}}$, the *strong tree* $\mathcal{I}_{\text{strtree}}$ and the *convexly ordered equivalence relation* $\mathcal{I}_{\text{eq}}$. A proof that $\mathcal{I}_{\text{eq}}$ has RP is given in [8, Theorem 6.6]. The trees can be seen to have RP by way of Theorem 2.17 and methods of generalized indiscernible sequences, see [9].

Definition 4.7.

- Define $\mathcal{I}_{\text{stree}}$ to be the structure on ${}^{<\omega}\omega$ (finite sequences from ω) in the signature $\{\trianglelefteq, \wedge, <_{\text{lex}}, \{P_n\}_{n\in\omega}\}$ where for all $\eta, \nu \in {}^{<\omega}\omega$, $\eta \trianglelefteq \nu$ if and only if η is an initial segment of ν, $\wedge$ is the meet in the partial order $\trianglelefteq$, $<_{\text{lex}}$ is the lexicographic order on finite sequences, i.e. $\eta <_{\text{lex}} \nu$ if and only if
$$\eta \trianglelefteq \nu \vee \eta(\ell(\eta \wedge \nu)) < \nu(\ell(\eta \wedge \nu)),$$
and $\eta \in P_n \Leftrightarrow \ell(\eta) = n$, for all $n \in \omega$.
- Define $\mathcal{I}_{\text{strtree}}$ to be the structure on ${}^{<\omega}\omega$ in the signature $\{\trianglelefteq, \wedge, <_{\text{lex}}, <_{\text{len}}\}$ where $\trianglelefteq, \wedge, <_{\text{lex}}$ are interpreted as in $\mathcal{I}_{\text{stree}}$ and $<_{\text{len}}$ is the preorder on $\mu, \nu \in {}^{<\omega}\omega$ defined by the lengths of the sequences:
$$\mu <_{\text{len}} \nu \Leftrightarrow \ell(\mu) < \ell(\nu)$$
- Define $\mathcal{I}_{\text{eq}}$ to be the structure on $\omega \times \omega$ in the signature $\{E, \prec\}$ where for all $(i,j), (s,t) \in \omega \times \omega$, $(i,j)E(s,t) \Leftrightarrow i = s$ and $(i,j) \prec (s,t) \Leftrightarrow i < s \vee (i = s \wedge j < t)$.

We recap the following example from [17], which suffices to transfer the Ramsey property from $\mathcal{I}_{\text{strtree}}$ to $\mathcal{I}_{\text{eq}}$ by way of more saturated models.

Proposition 4.8. *Let $\mathcal{A}$ be the structure on the underlying set $\omega \times \mathbb{Q}$ such that age$(\mathcal{A})$ = age$(\mathcal{I}_{eq})$ and each equivalence class in $\mathcal{A}$ is densely ordered by $\prec$. Let $\mathcal{B}$ be the structure on the underlying set ${}^{<\omega}\mathbb{Q}$ such that age$(\mathcal{B})$ = age$(\mathcal{I}_{strtree})$ and the $\trianglelefteq$-successors of any fixed node in $\mathcal{B}$ are densely ordered by $<_{lex}$. $\mathcal{A}$ is a semi-retract of $\mathcal{B}$.*

Proof. Given $i \in \omega$, by the ith level in $\mathcal{B}$, we mean all sequences in ${}^{<\omega}\mathbb{Q}$ of length i, and by the ith equivalence class in $\mathcal{A}$, we mean $\{(i,x) \mid x \in \mathbb{Q}\}$.

Let $\eta_i = \underbrace{\langle 0, \ldots, 0 \rangle}_{2i}$. Let g take the ith equivalence class in $\mathcal{A}$ into $\{\eta_i^\frown \langle j \rangle \mid j \in \mathbb{Q}_{>0}\}$ in a way that preserves the order. Let $f : \mathcal{B} \to \mathcal{A}$ be the map that takes the ith level in $\mathcal{B}$ into the ith equivalence class in $\mathcal{A}$ in a way that preserves the order. $\mathcal{A}$ is a semi-retract of $\mathcal{B}$ via (g, f). $\square$

In Theorem 5.9 below, it helps to have the following trees example in mind.

Example 4.9. Fix a qftp $p(x_0, x_1) = \{E(x_0, x_1, x_2), x_0 \prec x_1 \prec x_2\}$ in $\mathcal{A} := \mathcal{I}_{\text{eq}}$ and let $\overline{a}$ be a realization of this type in $\mathcal{A}$. Let $\overline{b}$ be a finite tuple from $\mathcal{A}$ containing the elements of $\overline{a}$. Let (g, f) be the semiretraction described in Proposition 4.8 above. Define $\overline{a}_0 := g(\overline{a})$, $\overline{b}_0 := g(\overline{b})$.

We can think of the various types of $\{b_0 < b_1 < b_2\}$ in the tree $\mathcal{B} := \mathcal{I}_{\text{strtree}}$ that map to copies of $\overline{a}$ in $\mathcal{A}$ under the map $f : \mathcal{B} \to \mathcal{A}$:

(1) $\neg(b_0 \wedge b_1 \trianglelefteq b_2)$
(2) $\neg(b_1 \wedge b_2 \trianglelefteq b_0)$
(3) $b_0 \wedge b_1 = b_1 \wedge b_2$

The last type is the only one that is realized within $\overline{b}_0$ in $\mathcal{B}$. So it corresponds to the special generators $\overline{a}_0$ described below.

5. Formula-free approach to transfer the RP

Definition 5.1. Fix finite tuples $\overline{a}, \overline{b}$ from $\mathcal{A}$ and an injection $f : \mathcal{B} \to \mathcal{A}$. Fix finite tuples $\overline{a}_0, \overline{b}_0$ from $\mathcal{B}$ such that $\ell(\overline{a}) = \ell(\overline{a}_0)$. We say that $\overline{b}_0$ has the *restricted inverse images under f property for $\overline{a}$ witnessed by $\overline{a}_0$* if for any $\overline{c}_1$ in $\langle f(\overline{b}_0)\rangle_{\mathcal{A}}$ such that $\overline{c}_1 \sim_{\mathcal{A}} \overline{a}$, $\overline{c}_0 := f^{-1}(\overline{c}_1) \subseteq \overline{b}_0$ and $\overline{c}_0 \sim_{\mathcal{B}} \overline{a}_0$.

Proposition 5.2. *For any structures $\mathcal{A}, \mathcal{B}$, for any finite tuples $\overline{a}, \overline{b}$ enumerating substructures of $\mathcal{A}$, for any $\overline{b}_0' \sim_{\mathcal{B}} g(\overline{b})$, $\overline{b}_0'$ has the restricted inverse images under f property for $\overline{a}$ witnessed by $g(\overline{a})$.*

Proof. Fix structures $\mathcal{A}, \mathcal{B}$ and a pair of maps $g : \mathcal{A} \to \mathcal{B}$ and $f : \mathcal{B} \to \mathcal{A}$ witnessing that $\mathcal{A}$ is a semi-retract of $\mathcal{B}$. Fix finite tuples $\overline{a}, \overline{b}$ enumerating substructures of $\mathcal{A}$ and let $\overline{a}_0 := g(\overline{a}), \overline{b}_0 := g(\overline{b})$, $\overline{a}_1 := f(\overline{a}_0)$, and $\overline{b}_1 := f(\overline{b}_0)$. Let $n := \ell(\overline{a})$.

Fix a tuple $\overline{c}_1$ in $\langle \overline{b}_1\rangle_{\mathcal{A}}$ such that:

$$\overline{c}_1 \sim_{\mathcal{A}} \overline{a}. \tag{1}$$

We will argue that $\overline{c}_0 := f^{-1}(\overline{c}_1)$ is in $\langle \overline{b}_0\rangle_{\mathcal{B}}$: since fg is an $L_{\mathcal{A}}$-isomorphism and we assumed $\overline{b} = \langle \overline{b}\rangle_{\mathcal{A}}$, it must be that $\overline{b}_1 = \langle \overline{b}_1\rangle_{\mathcal{A}}$. Thus $\overline{c}_1 \subseteq \overline{b}_1$ so there is some $\overline{c} \subseteq \overline{b}$ such that $g(\overline{c}) = \overline{c}_0$ and $f(\overline{c}_0) = \overline{c}_1$, and moreover $f^{-1}(\overline{c}_1) = g(\overline{c}) \subseteq \overline{b}_0$.

By the embedding property of semi-retractions,

$$\overline{c} \sim_{\mathcal{A}} \overline{c}_1. \tag{2}$$

The equations (1) and (2) imply that:

$$\overline{c} \sim_{\mathcal{A}} \overline{a}. \tag{3}$$

Thus, by the qftp-respecting property of semi-retractions:

$$\overline{c}_0 = g(\overline{c}) \sim_{\mathcal{B}} g(\overline{a}) = \overline{a}_0 \tag{4}$$

By (4), we may conclude that $\overline{c}_0 = f^{-1}(\overline{c}_1) \sim_{\mathcal{B}} \overline{a}_0$, as desired. We summarize these facts in the diagram below.

$$\begin{tikzcd}
\overline{a} \arrow[r, mapsto, "g"] \arrow[d, dashed, no head, "(3)"] \arrow[rrd, dashed, no head, bend left=60, "(1)"] & \overline{a}_0 \arrow[r, mapsto, "f"] \arrow[d, dashed, no head, "(4)"] & \overline{a}_1 \arrow[d, mapsto] \\
\overline{c} \arrow[r, mapsto, "g"] \arrow[rr, dashed, no head, bend right, "(2)"'] & \overline{c}_0 \arrow[r, mapsto, "f"] & \overline{c}_1
\end{tikzcd}$$

To complete the proof, fix any $\overline{b}_0'$ from $\mathcal{B}$ such that

$$\overline{b}_0' \sim_{\mathcal{B}} \overline{b}_0. \tag{5}$$

Since $\overline{b}_0' \sim_{\mathcal{B}} \overline{b}_0$, $f(\overline{b}_0') \sim_{\mathcal{A}} f(\overline{b}_0)$ and so $f(\overline{b}_0')$ inherits $\overline{b}_1$'s property of enumerating a substructure of $\mathcal{A}$, i.e. $\left\langle f(\overline{b}_0')\right\rangle_{\mathcal{A}} = f(\overline{b}_0')$.

Fix any $\bar{e}_1 \subseteq \bar{b}_1' = \left\langle f(\bar{b}_0') \right\rangle_{\mathcal{A}} = f(\bar{b}_0')$ such that $\bar{e}_1 \sim_{\mathcal{A}} \bar{a}$. Then there exists $\bar{e}_0 := f^{-1}(\bar{e}_1) \subseteq \bar{b}_0'$. The similarity (5) guarantees the existence of some $\bar{e}_0' \subseteq \bar{b}_0$ such that $\bar{e}_0' \sim_{\mathcal{B}} \bar{e}_0$. By the qftp-respecting property for semi-retractions, $f(\bar{e}_0') \sim_{\mathcal{A}} f(\bar{e}_0) = \bar{e}_1 \sim_{\mathcal{A}} \bar{a}$, and we just argued that $\bar{b}_0$ has the restricted inverse images under f property for $\bar{a}$ witnessed by $\bar{a}_0$, so $\bar{e}_0' \sim_{\mathcal{B}} \bar{a}_0$, thus $\bar{e}_0 \sim_{\mathcal{B}} \bar{a}_0$, as desired. □

Definition 5.3. Fix structures $\mathcal{A}, \mathcal{B}$ in relational signatures, finite substructures $A, B \subseteq \mathcal{A}$ and an injection $f : \mathcal{B} \to \mathcal{A}$. Fix substructures A_0, B_0 from $\mathcal{B}$ such that $|A_0| = |A|$. We say that B_0 has the *relational restricted inverse images under f property for A witnessed by A_0* if for any $C_1 \subseteq f(B_0)$ such that $C_1 \cong_{\mathcal{A}} A$, $f^{-1}(C_1) \cong_{\mathcal{B}} A_0$.

Proposition 5.4. *Fix structures $\mathcal{A}, \mathcal{B}$ in relational signatures. Fix finite substructures $A, B \subseteq \mathcal{A}$, and an injection $f : \mathcal{B} \to \mathcal{A}$. Let $A_0 = g(A), B_0 = g(B), A_1 = f(A_0)$. Then, for any $B_0' \cong_{\mathcal{B}} B_0$, B_0' has the relational restricted inverse images under f property for A witnessed by A_0.*

Proof. Let $\mathcal{A}, \mathcal{B}, f, A, B, A_0, B_0, A_1$ be as in the statement above. Let $\bar{a}, \bar{b}$ be enumerations of A, B, respectively, and let $\bar{a}_0 := g(\bar{a}), \bar{b}_0 := g(\bar{b}), \bar{a}_1 := f(\bar{a}_0)$. Clearly $\bar{a}_0, \bar{b}_0, \bar{a}_1$ enumerate A_0, B_0, A_1, respectively, and $\ell(\bar{a}) = \ell(\bar{a}_0)$. Fix any $B_0' \cong_{\mathcal{B}} \mathcal{B}_0$ and fix an isomorphism $\sigma : B_0 \to B_0'$ and let $\bar{b}_0' := \sigma(\bar{b}_0)$. By Proposition 7.2, $\bar{b}_0'$ has the restricted inverse images under f property for $\bar{a}$ witnessed by $\bar{a}_0$. Clearly, this implies that B_0' has the relational restricted inverse images under f property for A witnessed by A_0. □

We warm up with a shorter argument for relational signatures.

Theorem 5.5. *Let $\mathcal{A}, \mathcal{B}$ be structures each in (possibly distinct) relational signatures and suppose that $\mathcal{A}$ is a semi-retract of $\mathcal{B}$. If $\mathcal{B}$ has RP, then $\mathcal{A}$ has RP.*

Proof. Fix structures $\mathcal{A}, \mathcal{B}$ in relational signatures. Fix the pair of maps $g : \mathcal{A} \to \mathcal{B}$ and $f : \mathcal{B} \to \mathcal{A}$ witnessing that $\mathcal{A}$ is a semi-retract of $\mathcal{B}$. Assume $\mathcal{B}$ has RP.

Fix finite structures $A, B \subseteq \mathcal{A}$ and a coloring $c : \binom{\mathcal{A}}{A} \to 2$. Define the structures $A_0 := g(A), B_0 := g(B), A_1 := f(A_0), B_1 := f(B_0)$. Since the signatures are relational, A_0, B_0 are finite substructures of $\mathcal{B}$ and A, B, A_1, B_1 are finite substructures of $\mathcal{A}$. Define an induced coloring $c_0 : \binom{\mathcal{B}}{A_0} \to 2$ by: $c_0(A_0') := c(f(A_0'))$. This coloring is well-defined by the qftp-preserving property of semi-retractions.

By RP for $\mathcal{B}$, there exists a copy B_0' of B_0 that is homogeneous for c_0. Thus, there exists $d < 2$ such that $c_0(A_0') = d$ for all $A_0' \cong_{\mathcal{B}} A_0$ in B_0'. We will argue that $B_1' := f(B_0') \subseteq \mathcal{A}$ is a copy of B homogeneous for the coloring c.

First, note that since $B_0' \cong_{\mathcal{B}} B_0$, we must have that $B_1' \cong_{\mathcal{A}} B_1 (\cong_{\mathcal{A}} B)$ by the qftp-respecting property of semi-retractions. Now fix any copy A_1' of A in B_1', and let $A_0' := f^{-1}(A_1') \subseteq B_0'$. By Proposition 5.4, B_0' has the relational restricted inverse images under f property for A witnessed by A_0, thus, $A_0' \cong_{\mathcal{B}} A_0$. But then A_0' is a copy of A_0 in B_0', so $d = c_0(A_0') = c(f(A_0')) = c(A_1')$. This proves the claim. □

Definition 5.6. We say that a structure A is **rigid** if the only automorphism of A is the identity map.

Proposition 5.7. *If age$(\mathcal{B})$ consists of rigid elements, then for any $C, C' \in$ age$(\mathcal{B})$, if $C \cong_{\mathcal{B}} C'$, then this is witnessed by a unique isomorphism $\tau : C \to C'$.*

Proof. If $\tau_1, \tau_2 : C \to C'$ are two isomorphisms, then $\tau_2^{-1}\tau_1 : C \to C$ is an automorphism of C, but since $\mathrm{Aut}(C)$ is trivial, $\tau_2^{-1}\tau_1 = \mathrm{id}$, i.e. $\tau_1 = \tau_2$. □

Remark 5.8. In the following argument, we make essential use of the local finiteness of $\mathcal{A}$. If $\mathcal{A}$ is not locally finite, then some finite tuple $\overline{b}$ from $\mathcal{A}$ generates an infinite substructure $B \subseteq \mathcal{A}$. Then $g(B) \subseteq |\mathcal{B}|$ is an infinite set that may or may not be generated by a finite set of generators from $\mathcal{B}$, in which case it may not be contained in an element of age($\mathcal{B}$). This is a problem, because any copy of $A \subseteq B$ in $fg(B) \subseteq \mathcal{A}$ has a preimage that is somewhere in $g(B)$, and may be outside the image of the original generators $g(\overline{b})$. But the RP for $\mathcal{B}$ does not cover infinite sets such as $g(B)$ that are not contained in elements of age($\mathcal{B}$).

Theorem 5.9. *Let $\mathcal{A}, \mathcal{B}$ be structures in any signatures and suppose that $\mathcal{A}$ is a semi-retract of $\mathcal{B}$. Suppose age($\mathcal{B}$) consists of rigid elements and $\mathcal{A}$ is locally finite. If $\mathcal{B}$ has RP, then $\mathcal{A}$ has RP.*

Proof. Fix structures $\mathcal{A}, \mathcal{B}$ as in the statement. Fix the pair of maps $g : \mathcal{A} \to \mathcal{B}$ and $f : \mathcal{B} \to \mathcal{A}$ witnessing that $\mathcal{A}$ is a semi-retract of $\mathcal{B}$. Assume $\mathcal{B}$ has RP.

Fix finitely generated structures $A, B \subseteq \mathcal{A}$ and fix generators, $\overline{a}$ and $\overline{b}$ for these structures, respectively, in some fixed enumeration. Since $\mathcal{A}$ is assumed locally finite, we may assume that $\overline{a}, \overline{b}$ enumerate A, B, respectively.

Fix a coloring $c : \binom{\mathcal{A}}{A} \to 2$. We define the tuples $\overline{a}_0 := g(\overline{a}), \overline{b}_0 := g(\overline{b}), \overline{a}_1 := f(\overline{a}_0), \overline{b}_1 := f(\overline{b}_0)$. Moreover, let $A := \langle \overline{a} \rangle_{\mathcal{A}}, B := \langle \overline{b} \rangle_{\mathcal{A}}$, $A_0 := \langle \overline{a}_0 \rangle_{\mathcal{B}}$, $B_0 = \langle \overline{b}_0 \rangle_{\mathcal{B}}$, and $A_1 := \langle \overline{a}_1 \rangle_{\mathcal{A}}$, $B_1 = \langle \overline{b}_1 \rangle_{\mathcal{A}}$. We summarize our notation in a diagram:

$$\overline{a} \xmapsto{g} \overline{a}_0 \xmapsto{f} \overline{a}_1$$

$$\overline{b} \xmapsto{g} \overline{b}_0 \xmapsto{f} \overline{b}_1$$

Define an induced coloring $c_0 : \binom{\mathcal{B}}{A_0} \to 2$ by: if there exists an L_B-isomorphism $\rho : A_0 \to A'_0$, for some substructure $A'_0 \subseteq \mathcal{B}$, let $\overline{a}'_0 := \rho(\overline{a}_0)$ and define

$$c_0(A'_0) := c(\langle f(\overline{a}'_0) \rangle_{\mathcal{A}})$$

Since ρ is an $L_{\mathcal{B}}$-isomorphism, we know that $\overline{a}'_0 \sim_{\mathcal{B}} \overline{a}_0$, and since f is qftp-respecting, we know that $f(\overline{a}'_0) \sim_{\mathcal{A}} f(\overline{a}_0)$ which implies that $\langle f(\overline{a}'_0) \rangle_{\mathcal{A}} \cong \langle f(\overline{a}_0) \rangle_{\mathcal{A}} \cong \mathcal{A}$. This, and the fact that the isomorphism ρ, if it exists, is unique, guarantee that this coloring c_0 is well-defined and total on the domain $\binom{\mathcal{B}}{A_0}$.

By the assumption of RP for $\mathcal{B}$, there is a copy B'_0 of B_0 in $\mathcal{B}$ **homogeneous** for copies of A_0 under the coloring c_0. Thus, there is $d < 2$ such that $c_0(A'_0) = d$ for all $A'_0 \cong_{\mathcal{B}} A_0$ in B'_0. Since $B_0 \cong_{\mathcal{B}} B'_0$, Proposition 5.7 guarantees the existence of a unique isomorphism $\sigma : B_0 \to B'_0$. Define

$$\begin{aligned} \overline{b}'_0 &:= \sigma(\overline{b}_0) \\ \overline{b}'_1 &:= f(\overline{b}'_0) \end{aligned} \tag{6}$$

and observe that

$$\overline{b}'_0 \sim_{\mathcal{B}} \overline{b}_0 \tag{7}$$

By (7) and the qftp-respecting property of semi-retractions,

$$\overline{b}'_1 = f(\overline{b}'_0) \sim_{\mathcal{A}} f(\overline{b}_0) = \overline{b}_1 \tag{8}$$

and we will denote the map $\bar{b}_1 \mapsto \bar{b}_1'$ by τ. By the composition property of semi-retractions,

$$\bar{b}_1 \sim_{\mathcal{A}} \bar{b} \tag{9}$$

Thus, by (8) and (9),

$$\bar{b}_1' \sim_{\mathcal{A}} \bar{b} \tag{10}$$

This is summarized in the diagram below:

$$\begin{tikzcd}
\bar{b} \arrow[r, mapsto, "g"] \arrow[rr, bend left, dashed, "(9)"] \arrow[drr, bend left=60, dashed, no head, "(10)"] & \bar{b}_0 \arrow[r, mapsto, "f"] \arrow[d, "\sigma{:}(6)"] & \bar{b}_1 \arrow[d, "\tau{:}(8)"] \\
& \bar{b}_0' \arrow[r, mapsto, "f"] & \bar{b}_1'
\end{tikzcd}$$

Let $B_1' := \left\langle \bar{b}_1' \right\rangle_{\mathcal{A}}$. We have just shown that B_1' is a copy of B in $\mathcal{A}$.

We claim that B_1' is the desired homogeneous copy of B in $\mathcal{A}$ for the coloring c. To verify, let A_1' be a copy of A in B_1'. Fix any isomorphism $\xi : A \to A_1'$ and let $\bar{a}_1' := \xi(\bar{a})$. Thus,

$$\bar{a}_1' \sim_{\mathcal{A}} \bar{a} \tag{11}$$

and $A_1' = \langle \bar{a}_1' \rangle_{\mathcal{A}} \subseteq B_1'$.

Since $\bar{b}_0 \sim_{\mathcal{B}} \bar{b}_0'$ by (7) and we have assumed that $\bar{a}, \bar{b}$ enumerate finite substructures of $\mathcal{A}$, by Proposition 5.2, $\bar{b}_0'$ has the restricted inverse images under f property for $\bar{a}$ witnessed by $\bar{a}_0$. That is, for $\bar{a}_1'$ in $B_1' = \left\langle \bar{b}_1' \right\rangle_{\mathcal{A}} = \bar{b}_1'$, since $\bar{a}_1' \sim_{\mathcal{A}} \bar{a}$, we have $\bar{a}_0' := f^{-1}(\bar{a}_1') \subseteq \bar{b}_0'$ and

$$\bar{a}_0 \sim_{\mathcal{B}} \bar{a}_0'. \tag{12}$$

By homogeneity of B_0' under the coloring c_0, $c_0(\langle \bar{a}_0' \rangle_{\mathcal{B}}) = d$, and since $\bar{a}_0' \sim_{\mathcal{B}} \bar{a}_0$, $c_0(\langle \bar{a}_0' \rangle_{\mathcal{B}}) := c(\langle f(\bar{a}_0') \rangle_{\mathcal{A}})$ thus $d = c(\langle f(\bar{a}_0') \rangle_{\mathcal{A}}) = c(\langle \bar{a}_1' \rangle_{\mathcal{A}}) = c(A_1')$, as desired.

We illustrate the maps as follows:

$$\begin{tikzcd}
\bar{a} \subseteq \bar{b} \arrow[r, mapsto, "g"] \arrow[drr, bend left=60, dashed, no head, "(11)"] & \bar{a}_0 \subseteq \bar{b}_0 \arrow[r, mapsto, "f"] \arrow[d, dashed, no head, "(12)"] & \bar{a}_1 \subseteq \bar{b}_1 \\
& \bar{a}_0' \subseteq \bar{b}_0' \arrow[r, mapsto, "f"] & \bar{a}_1' \subseteq \bar{b}_1'
\end{tikzcd}$$

$\square$

The argument just given does not work without local finiteness.

Example 5.10. Let $\mathcal{A} = (\mathbb{Z}, s)$ be the structure where s is interpreted as the successor function on $\mathbb{Z}$. Let $A = B = \langle 0 \rangle_{\mathcal{A}}$. For any copy of A, there is a unique element a that generates A (no other single element will do so.) Color copies of A in $\mathcal{A}$ red if that unique element is odd, blue, if it is even. There is no copy of B in $\mathcal{A}$ that is homogeneous for this coloring. Now let $\mathcal{B} = (\mathbb{Z}, \{R_{(n,\overline{k})}\})$ where $R_{(n,\overline{k})}(a_0, \ldots, a_{n-1})$ holds exactly of increasing n-tuples from $\mathbb{Z}$ such that $d(a_i, a_{i+1}) = (\overline{k})_i$. $\mathcal{B}$ is interdefinable with $\mathcal{A}$ but it has a (canonical) relation language. Thus, the identity maps give a semi-retraction (by way of the interdefinability, they are in a sense quantifier-free preserving.) If we let $a := x, b := y$ and $a_0 := g(x), b_0 := g(y)$, notice that a_0, b_0 each represent 1-point substructures of $\mathcal{B}$. Then for any coloring of copies of a_0 in $\mathcal{B}$, we can trivially find a copy of b_0 that is homogeneous for this coloring. However it does not mean that we can find a copy of B homogeneous for copies of A in $\mathcal{A}$.

A similar proof transfers finite small Ramsey degrees.

Definition 5.11. We say that a class of finite L-structures $\mathcal{K}$ has **finite (small) Ramsey degrees** if for every $A \in \mathcal{K}$ there exists $d = d(A, \mathcal{K}) < \omega$ such that for all $k < \omega$, for all desired $B \in \mathcal{K}$ there exists $C \in \mathcal{K}$ such that $C \to (B)^A_{k,d}$, i.e. for all k-colorings $c : \binom{C}{A} \to k$ there exists $B' \subseteq C$, $B' \cong B$, such that $|c''\binom{B'}{A}| \leq d$.

We say that the structure B' above is **$\leq d$-chromatic (for the coloring c on copies of A)**.

Proposition 5.12. *Let $\mathcal{A}$ be any infinite, locally finite structure and let age$(\mathcal{A}) = \mathcal{K}$. A has finite Ramsey degree $\leq d$ in $\mathcal{K}$ if and only if for all $B \in \mathcal{K}$ and $k < \omega$, for any coloring $c : \binom{\mathcal{A}}{A} \to k$ there is a copy of B, $B' \subseteq \mathcal{A}$, such that $|c''\binom{B'}{A}| \leq d$*

Proof. The $\Rightarrow$ direction is immediate. Let $n := |A|$. To see the $\Leftarrow$ direction, we prove the contrapositive: assume A does not have finite Ramsey degree $\leq d$ ($d < k$) in $\mathcal{K}$. Thus, for some $B \in \mathcal{K}$ and $k < \omega$, no $C \in \mathcal{K}$ works for $C \to (B)^A_{k,\leq d}$. We will find a coloring $c : \binom{\mathcal{A}}{A} \to k$ such that for all $B' \cong B$ in $\mathcal{A}$, $|c''\binom{B'}{A}| > d$. Expand the language of $\mathcal{A}$ to include predicates for all complete quantifier-free types of finite tuples. Expand the language further to contain predicates $\{R_i\}_{i \in k}$ the k-coloring on n-tuples. Using local finiteness, let $p_B(\overline{y}), p_A(\overline{x})$ be the predicates for B, A (in some enumeration). Consider the type Γ in variables $\{x_c : c \in \mathcal{A}\}$ consisting of the diagram of $\mathcal{A}$ in the expanded language as well as the statement of failure to not exceed d colors:

$$\forall \overline{y}(p_B(\overline{y}) \to \bigvee_{\substack{\overline{x}_0, \ldots, \overline{x}_d \subseteq \overline{y};\\ \ell(\overline{x}_i) = n, i < d+1}} \left[\bigwedge_{i<d+1} p_A(\overline{x}_i) \wedge \bigvee_{i_0<i_1<\ldots<i_d<k} \left(\bigwedge_{j<d+1} R_{i_j}(\overline{x}_i) \right) \right])$$

$\square$

Theorem 5.13. *Let $\mathcal{A}, \mathcal{B}$ be structures and let $\mathcal{K} :=$ age$(\mathcal{A}), \mathcal{K}' =$ age$(\mathcal{B})$. Assume $\mathcal{A}$ is locally finite and $\mathcal{K}'$ consists of rigid elements. Suppose that $\mathcal{A}$ is a semi-retract of $\mathcal{B}$ by the maps (g, f). For any $A \in \mathcal{K}$, if $g(A)$ has finite Ramsey degree in $\mathcal{K}'$ and we let $d := d(g(A), \mathcal{K}') < \omega$, then A has finite Ramsey degree in $\mathcal{K}$, and in fact, $d(A, \mathcal{K}) \leq d$.*

Proof. Fix the semi-retraction maps:

$$\mathcal{A} \xrightarrow{g} \mathcal{B} \xrightarrow{f} \mathcal{A}$$

Fix $A \in \mathcal{K}$ such that $A_0 := g(A)$ has finite Ramsey degree d in $\mathcal{K}'$. Since $\mathcal{A}$ is assumed to be locally finite, we can use Proposition 5.12. It suffices to show that for all $B \in \mathcal{K}$ and $k < \omega$, for any coloring $c : \binom{\mathcal{A}}{A} \to k$ there is a copy of B, $B' \subseteq \mathcal{A}$, such that $|c''\binom{B'}{A}| \leq d$. As in the argument for Theorem 5.9, the structure $\langle \overline{b}_0 \rangle_{\mathcal{B}}$ contains only copies of the generators $\overline{a}_0$, out of all $\overline{a}_i$ that possibly map to copies of $\overline{a}$ in $\mathcal{A}$ by f. We induce the coloring on $\binom{\mathcal{B}}{\langle \overline{a}_0 \rangle_{\mathcal{B}}}$ in the same way as in Theorem 5.9 and obtain $\leq d$ colors on some copy of $\overline{b}_0$. Then the image of this (under f) in $\mathcal{A}$ can have only $\leq d$ colors. □

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