Construction of Directed Strongly Regular Graphs Using Block Matrices

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Notation

- $\Gamma$ represents a directed graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$ where $|V(\Gamma)| = v$
- $\Gamma$ does not have loops nor multiple edges
- $a \rightarrow b$ illustrates a directed edge from $a$ to $b$ and $a \leftrightarrow b$ illustrates an undirected edge from $a$ to $b$
- The adjacency matrix $A$ of $\Gamma$ is a $v \times v \{0,1\}$-matrix such that $A_{ij} = 1$ iff $i \rightarrow j$.
- $I$ is the $v \times v$ identity matrix
- $J$ is the $v \times v$ the all-ones matrix
Regular Graphs

Definition

Regular Graph: a graph where every vertex has the same degree
Regular Graphs: An Example

Complete graph on 4 vertices
Strongly Regular Graphs

Definition

**Strongly Regular Graph**: an undirected graph with parameters \((v, k, \lambda, \mu)\) where

- \(v = \text{number of vertices of } \Gamma\)
- \(k = \text{valency of each vertex}\)
- \(\lambda = \text{number of common neighbors if } a \text{ and } b \text{ are adjacent}\)
- \(\mu = \text{number of common neighbors if } a \text{ and } b \text{ are not adjacent}\)

\[
A^2 = kl + \lambda A + \mu(J - I - A)
\]

\[
AJ = JA = kJ
\]
Strongly Regular Graphs: An Example

Petersen Graph: parameters (10, 3, 0, 1)
Directed Strongly Regular Graphs

Definition

Directed Strongly Regular Graph: a directed graph with parameters \((v, k, t, \lambda, \mu)\) where

- \(k = \text{total in or out degree of each vertex}\)
- \(t = \text{number of undirected edges at each vertex}\)
- \(\lambda = \text{number of paths of length 2 from } a \rightarrow b \text{ or } a \leftrightarrow b\)
- \(\mu = \text{number of paths of length 2 from } a \not\rightarrow b\)
- \(A^2 = tl + \lambda A + \mu(J - I - A)\)
- \(AJ = JA = kJ\)
Directed Strongly Regular Graphs: An Example

DSRG(8, 3, 2, 1, 1)
Duval’s Conditions

- If $\Gamma$ is a DSRG, then complement of $\Gamma$ is a DSRG
  - $A' = J - I - A$
  - $k' = (n - 2k) + (k - 1)$
  - $\lambda' = (n - 2k) + (\mu - 2)$
  - $t' = (n - 2k) + (t - 1)$
  - $\mu' = (n - 2k) + \lambda$

- We will exclude graphs that are totally undirected, totally directed, or complete
Duval’s Conditions

For all the DSRGs we will consider the following equations hold

- $k(k + (\mu - \lambda)) = t + (v - 1)\mu$
- $(\mu - \lambda)^2 + 4(t - \mu) = d^2$
- $d \mid 2k - (\mu - \lambda)(v - 1)$
- $\frac{2k - (\mu - \lambda)(v - 1)}{d} \equiv v - 1 \pmod{2}$
- $\left| \frac{2k - (\mu - \lambda)(v - 1)}{d} \right| \leq v - 1$
- $0 \leq \lambda < t < k$
- $0 < \mu \leq t < k$
- $-2(k - t - 1) \leq \mu - \lambda \leq 2(k - t)$
Construction Using Quadratic Residues

- DSRGs with parameters $(2q, q - 1, \frac{1}{2}(q - 1), \frac{1}{2}(q - 1) - 1, \frac{1}{2}(q - 1))$, where $q = 4m + 1$

- $A = \begin{bmatrix} Q & C_1 \\ C_2 & Q \end{bmatrix}$

- $C_1$ and $C_2$ are $\sigma_1$ and $\sigma_2$ circulant matrices and $Q$ is a quadratic residue matrix.

- $Q_{ij} = \begin{cases} 1 & \text{if } i - j \in R \\ 0 & \text{if } i - j \in N \end{cases}$
Using Quadratic Residues: DSRG-(10, 4, 2, 1, 2)

- \{1, 4\} are quadratic residues and \{2, 3\} are non-residues in \(\mathbb{Z}_5\).
- \(\sigma_1 = 2, \sigma_2 = 3\) and \(2 \times 3 \equiv 1 \mod 5\).
- Quadratic Residue Matrix (Q)
  
  \[
  \begin{bmatrix}
  0 & 1 & 0 & 0 & 1 \\
  1 & 0 & 1 & 0 & 0 \\
  0 & 1 & 0 & 1 & 0 \\
  0 & 0 & 1 & 0 & 1 \\
  1 & 0 & 0 & 1 & 0 
  \end{bmatrix}
  \]

- Circulant Matrix (first line)
  
  \[
  (0 1 0 0 1)
  \]
Using Quadratic Residues: DSRG-(10, 4, 2, 1, 2) (con.)

\[ A = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
\end{bmatrix} \]
Graph of DSRG-(10, 4, 2, 1, 2)
Block Construction Using Permutation Matrices

- DSRGs with parameters $(2(2\mu + 1), 2\mu, \mu, \mu - 1, \mu)$
  and adjacency matrix
  \[
  \begin{bmatrix}
  Q & PQ \\
  (PQ)^T & Q
  \end{bmatrix}
  \]

- $Q + Q^T = J - I$
- $QJ = JQ = \mu J$
- $P$ is a permutation matrix with rank 2
- $PJ = JP = J$
- $P = P^T = P^{-1}$
- This construction works iff $PQ = (PQ)^T$
Block Construction with Permutation Matrices: DSRG-(14, 6, 3, 2, 3)

\[ Q = \begin{bmatrix}
0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \]

\[ P = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \]
Block Construction with Permutation Matrices: DSRG-(14, 6, 3, 2, 3) (con.)

\[ A = \begin{bmatrix}
0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \]
Graph of DSRG-(14, 6, 3, 2, 3)
Construction using the Kronecker Product

- An example of the Kronecker Product

\[ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \]

\[ A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix} \]

- \( J_m \) is the \( m \times m \) all-ones matrix and \( A \) is the adjacency matrix of a DSRG with \( t = \mu \)

- \( A \otimes J_m \) is an adjacency matrix of a DSRG with parameters \((nm, km, tm, \lambda m, \mu m)\)
DSRG with parameters \((6, 2, 1, 0, 1)\) with adjacency matrix

\[
A = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
\end{bmatrix}
\]

and

\[
J_2 = \begin{bmatrix}
1 & 1 \\
1 & 1 \\
\end{bmatrix}
\]
Kronecker Product: DSRG-(12, 4, 2, 0, 2) (con.)

\[ A \otimes J_2 = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \]
Graph of DSRG-(12, 4, 2, 0, 2)
Using Cayley Graphs to Construct of DSRGs

Definition

Let $G$ be a finite group and $S \subseteq G - \{e\}$. The Cayley Graph, $\text{Cay}(G; S)$, is the digraph $\Gamma$ such that $V(\Gamma) = G$ and $E(\Gamma) = \{(x, y) : \exists s \in S \text{ such that } xs = y\}$.

- In the multiplication table of $S$:
  - The identity of the group, $e$ appears $t$ times. An easier way to check this is that $|S \cap S^{-1}| = t$ where $S^{-1} = \{x \in \Gamma : x^{-1} \in S\}$.
  - Each element of $S$ appears $\lambda$ times.
  - Each element of $G - S - \{e\}$ appears $\mu$ times.
- A result by Jørgensen shows that if $\Gamma$ is Abelian, than $\text{Cay}(G; S)$ is not a DSRG for any $S \subseteq \Gamma$. 
Examples of Constructing Graphs using Cayley Graphs

\[ \text{Cay}(Z_6, \{1\}) \quad \text{Cay}(S_3, \{(12), (123)\}) \]
Constructing DSRGs using the Dihedral Group

- $D_{2 \nu} = \langle \alpha, \beta : \beta^2 = \alpha^\nu = e \text{ and } \beta \alpha \beta = \alpha^{-1} \rangle$

  where

  - if $\nu = 2 \lambda$ and $S$ is the subset
    \[
    \{ \alpha, \alpha^2, \ldots, \alpha^{\lambda-1}, \beta, \beta \alpha, \ldots, \beta^{\lambda-1} \}
    \]
    will construct the DSRG($4 \lambda, 2 \lambda - 1, \lambda, \lambda - 1, \lambda - 1$) from $\text{Cay}(D_{2 \nu}, S)$

  - if $\nu = 2 \lambda + 1$ and $S$ is the subset
    \[
    \{ \alpha, \alpha^2, \ldots, \alpha^{\lambda}, \beta, \beta \alpha, \ldots, \beta^{\lambda} \}
    \]
    will construct the DSRG($4 \lambda + 2, 2 \lambda + 1, \lambda + 1, \lambda, \lambda + 1$)

- The adjacency matrices of these graphs can be deconstructed into block matrices $\begin{bmatrix} A & A^T \\ A & A^T \end{bmatrix}$

- $A$ is the adjacency matrix of a highly structured graph called a regular tournament.
A tournament is a directed graph $\Gamma$ such that for any $x, y \in V(\Gamma)$ exactly one of $x \to y$ or $y \to x$ holds. A tournament $\Gamma$ is said to be regular if every vertex in $V(\Gamma)$ has the same out-degree. Thus a regular tournament has $n = 2k + 1$ if $n$ and $k$ denote the number of vertices and the valency of the graph, respectively.

If $A$ is the adjacency matrix of a tournament $\Gamma$, this means $A + A^T = J - I$. If $A$ is the adjacency matrix of a regular tournament, then $JA = AJ = kJ$. 
Example of a Regular Tournament

\[ A = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
\end{bmatrix} \quad A^T = \begin{bmatrix}
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
\end{bmatrix} \]
Constructing DSRGs using Tournament Matrices Part I

- $A$ is an adjacency matrix of a regular tournament with valency $k$
- $B = \begin{bmatrix} A & A^T \\ A & A^T \end{bmatrix}$ or $C = \begin{bmatrix} A & A \\ A^T & A^T \end{bmatrix}$ is the adjacency matrix of a DSRG with parameters $(4k + 2, 2k, k, k - 1, k)$
- $M(A) = \begin{bmatrix} A & A^T + I \\ A + I & A^T \end{bmatrix}$ is also an adjacency matrix of a DSRG with parameters $(4k + 2, 2k + 1, k + 1, k, k + 1)$
- $M(A)$ is isomorphic to the complement of $B$
A regular tournament $T$ is said to be doubly regular if for every vertex $x \in V(T)$, the out-neighbors of $x$ span a regular tournament. If $T$ is a doubly regular tournament of order $n$, with regular valency $k$ and the degree of the induced subgraph on the out-neighbors $\lambda$, then $n = 2k + 1 = 4\lambda + 3$

An $(m, r)$-team tournament is a digraph obtained as an orientation of $m \circ K_r$, the complement of the $m$ copies of the complete graph $K_r$
Tournaments (con.)

**Definition**

\( \Gamma \) is an \((m, r)\) team tournament \( \Gamma \) is doubly regular iff

- Every vertex of \( \Gamma \) has in-degree and out-degree
  \( k = \frac{1}{2}(m - 1)r \)

- For every pair of distinct vertices \( x \) and \( y \), the number of directed paths of length 2 from \( x \) to \( y \) is
  \[
  \begin{cases} 
  \alpha & \text{if } x \rightarrow y \\
  \beta & \text{if } x \leftarrow y \\
  \gamma & \text{otherwise}
  \end{cases}
  \]

- \( A^2 = \alpha A + \beta A^T + \gamma(J - I - A - A^T) \)
Constructing DSRGs using Tournament Matrices Part II

Block Construction: \((m, 2)\)-team tournament

\[
D = \begin{bmatrix}
0 & 1^T & 0 & 0^T \\
0 & A & 1 & A^T \\
0 & 0^T & 0 & 1^T \\
1 & A^T & 0 & A
\end{bmatrix}
\]

\[
M = M(D) = \begin{bmatrix}
D & D^T + I \\
D + I & D^T
\end{bmatrix}
\]

\[
M^2 = \begin{bmatrix}
mJ + I & mJ \\
mJ & mJ + I
\end{bmatrix}
\]

DSRG with parameters
\((4m, 2m - 1, m, m - 1, m - 1)\) where
\(m \equiv 0 \pmod{4}\)
Constructing a DSRG-(16, 7, 4, 3, 3)

\[ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \]

is the adjacency matrix of a doubly regular tournament

\[ D = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \]

is the adjacency matrix of a doubly regular \((m, 2)\) team tournament
Constructing a DSRG-(16, 7, 4, 3, 3)

Graph of a doubly regular (4, 2)-team tournament
Constructing a DSRG-(16, 7, 4, 3, 3)
Constructing DSRGs using Tournament Matrices Part III

- **Block Construction: Regular Tournaments A**
  \[
  D = \begin{bmatrix}
  0 & 1^T & 0 & 0^T \\
  0 & A & 1 & A^T \\
  0 & 0^T & 0 & 1^T \\
  1 & A^T & 0 & A
  \end{bmatrix}
  \]

- \[ M = M(D) = \begin{bmatrix}
  D & D^T + I \\
  D + I & D^T
  \end{bmatrix} \]

- \[ M^2 = \begin{bmatrix}
  nJ + I & nJ \\
  nJ & nJ + I
  \end{bmatrix} \]

- DSRG with parameters \((4(n + 1), 2n + 1, n + 1, n, n)\) where \(n\) is an odd positive number
Constructing DSRGs using Circulant Matrices

Block Construction: $L$ is an $(2s + 2) \times (2s + 2)$ matrix equal to $\Pi + \Pi^2 + \cdots + \Pi^s$ where

$$
\Pi = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & 0 \\
1 & 0 & \cdots & \cdots & 0 \\
\end{bmatrix}
$$

$$
L = \begin{bmatrix}
0 & 1 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 1 & \cdots & 1 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & 1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
\end{bmatrix}
$$
Constructing DSRGs using Circulant Matrices (con.)

\[ M = M(L) = \begin{bmatrix} L & L^T + I \\ L + I & L^T \end{bmatrix} \]

\[ M^2 = \begin{bmatrix} sJ + I & sJ \\ sJ & sJ + I \end{bmatrix} \]

This adjacency matrix creates a DSRG with parameters \((4(s + 1), 2s + 1, s + 1, s, s)\).
First we will show that $MJ = JM = kJ = (2s + 1)J$

$$MJ = \begin{bmatrix} L\bar{J} + L^T \bar{J} + \bar{J} & L\bar{J} + L^T \bar{J} + \bar{J} \\ L\bar{J} + L^T \bar{J} + \bar{J} & L\bar{J} + L^T \bar{J} + \bar{J} \end{bmatrix} = JM$$

Since we know that the matrix $L$ has $s$ ones per row we can transform

$$MJ = JM = \begin{bmatrix} \bar{J}(s + s + 1) & \bar{J}(s + s + 1) \\ \bar{J}(s + s + 1) & \bar{J}(s + s + 1) \end{bmatrix} = (2s + 1)J$$

Therefore the equation $MJ = JM = kJ$ holds.
Next we will show that the following equation holds true

\[ M^2 = tI + \lambda M + \mu(J - I - M) \]

\[ = (t - \mu)I + (\lambda - \mu)M + \mu J \]
Constructing DSRGs using Circulant Matrices brief Proof (con.)

\[ L = \begin{bmatrix} B & B^T \\ B^T & B \end{bmatrix} \text{ with the } (s + 1) \times (s + 1)-\text{matrix} \]

\[ B = \begin{bmatrix} 0 & 1 & \ldots & \ldots & 1 \\ 0 & 0 & 1 & \ldots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 \\ 0 & \ldots & \ldots & \ldots & 0 \end{bmatrix} \]

It is easy to see that

\[ L + L^T = \begin{bmatrix} B + B^T & B + B^T \\ B + B^T & B + B^T \end{bmatrix} = \begin{bmatrix} J - I & J - I \\ J - I & J - I \end{bmatrix} \]
Constructing DSRGs using Circulant Matrices brief Proof (con.)

\[
L^2 = (L^T)^2 = \begin{bmatrix}
B^2 + B^T^2 & BB^T + B^T B \\
BB^T + B^T B & B^2 + B^T^2
\end{bmatrix}
\]

\[
LL^T = L^T L = \begin{bmatrix}
BB^T + B^T B & B^2 + B^T^2 \\
B^2 + B^T^2 & BB^T + B^T B
\end{bmatrix}
\]

\[
L^2 + LL^T + L + L^T = sJ_{2s+2}
\]

\[
M^2 = \begin{bmatrix}
\frac{sJ_{2s+2} + l_{2s+2}}{sJ_{2s+2}} & \frac{sJ_{2s+2}}{sJ_{2s+2} + l_{2s+2}} \\
\frac{sJ_{2s+2}}{sJ_{2s+2} + l_{2s+2}} & \frac{sJ_{2s+2} + l_{2s+2}}{sJ_{2s+2}}
\end{bmatrix} = sJ_{4s+4} + l_{4s+4}.
\]

As you can see, \( M \) is the adjacency matrix of a DSRG with parameters \((4(s + 1), 2s + 1, s + 1, s, s)\).
Isomorphisms between DSRGs Constructed using Tournament Matrices

- When does a single construction produce isomorphic graphs?
- For construction method $C$ where $C(A)$ is the graph constructed using matrix $A$ in $C$, if $A \cong B$, then $C(A) \cong C(B)$
- This is done by finding a permutation matrix $P$ such that $PC(A)P^{-1} = C(B)$
When do different construction methods produce isomorphic graphs?

\[
\begin{bmatrix}
A & A^T \\
A & A^T
\end{bmatrix} \begin{bmatrix}
A & A^T \\
A^T & A^T
\end{bmatrix} \cong
\begin{bmatrix}
A & A \\
A^T & A^T
\end{bmatrix}
\]

if there is a permutation matrix \( P \) such that \( PA = A^T = AP \)

The generalization of this construction has this same property.
Isomorphisms between DSRGs Constructed using Tournament Matrices

- Why do we care if they produce non-isomorphic graphs?
- None of the parameters satisfied by these graphs are new.
- We have provided several new ways for these parameters to be realized.
Isomorphisms between DSRGs Constructed using Tournament Matrices

- What do we get from non-isomorphic tournaments?
- It’s reasonable to expect that non-isomorphic tournaments produce non-isomorphic graphs.

- Order 7: 3 regular tournaments
- Order 9: 15 regular tournaments
- Order 11: 1223 regular tournaments
- Order 13: 1495297 regular tournaments
When does our construction using $M(A)$ create isomorphic graphs?

What matrices do we need for $M(A)$ to be a DSRG?
### Summary

#### Table: Constructing DSRGs

<table>
<thead>
<tr>
<th>Source</th>
<th>DSRG((v, k, \mu, \mu - 1, \mu))</th>
</tr>
</thead>
<tbody>
<tr>
<td>A regular tournament with adjacency matrix (A) and a DSRG with adjacency matrix ([A \ A^T \ A \ A^T] \text{ or } [A \ A \ A^T \ A^T])</td>
<td>((4k + 2, 2k, k, k - 1, k))</td>
</tr>
<tr>
<td>A regular tournament with adjacency matrix (A) and a DSRG with adjacency matrix (M(A) = \begin{bmatrix} A &amp; A^T + I \ A + I &amp; A^T \end{bmatrix})</td>
<td>((4k + 2, 2k + 1, k + 1, k, k + 1))</td>
</tr>
</tbody>
</table>
| A regular tournament with adjacency matrix \(A\) and a DSRG with adjacency matrix \[
\begin{bmatrix}
A & A^T & \ldots & A^T \\
A & A^T & \ldots & A^T \\
\vdots & \vdots & \ddots & \vdots \\
A & A^T & \ldots & A^T \\
\end{bmatrix}
\text{ or }
\begin{bmatrix}
A & A & \ldots & A \\
A^T & A^T & \ldots & A^T \\
\vdots & \vdots & \ddots & \vdots \\
A^T & A^T & \ldots & A^T \\
\end{bmatrix}
\]
| \((w(4k + 2), 2wk, wk, w(k - 1), wk)\) |
### Table: Our Constructions of DSRGs

<table>
<thead>
<tr>
<th>Source</th>
<th>DSRG($v, k, \mu + 1, \mu, \mu$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D = D(T)$ the adjacency matrix of a doubly regular $(m, 2)$-team</td>
<td>$(4m, 2m-1, m, m-1, m-1)$ if $m \equiv 0 \pmod{4}$</td>
</tr>
<tr>
<td>tournament and a DSRG with adjacency matrix $M(D) = \begin{bmatrix} D &amp; DT + I \ D + I &amp; DT \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>A regular tournament with adjacency matrix $A$, $D = D(A)$, DSRG with</td>
<td>$(4n + 4, 2n + 1, n + 1, n, n)$ if $n$ is odd</td>
</tr>
<tr>
<td>adjacency matrix $M(D) = \begin{bmatrix} D &amp; DT + I \ D + I &amp; DT \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$L$ is an $(2s + 2) \times (2s + 2)$ matrix equal to $\Pi + \Pi^2 + \ldots + \Pi^s$, DSRG with adjacency matrix</td>
<td>$(4(s + 1), 2s + 1, s + 1, s, s)$</td>
</tr>
<tr>
<td>$M(L) = \begin{bmatrix} L &amp; L^T + I \ L + I &amp; L^T \end{bmatrix}$</td>
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</tr>
</tbody>
</table>
Examples

Table: DSRGs with parameters \((v, k, \mu, \mu - 1, \mu)\)

<table>
<thead>
<tr>
<th>(v)</th>
<th>(k)</th>
<th>(t)</th>
<th>(\lambda)</th>
<th>(\mu)</th>
<th>(v')</th>
<th>(k')</th>
<th>(t')</th>
<th>(\lambda')</th>
<th>(\mu')</th>
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</table>
Table: DSRGs with parameters \((v, k, \mu + 1, \mu, \mu)\)

<table>
<thead>
<tr>
<th>v</th>
<th>k</th>
<th>t</th>
<th>\lambda</th>
<th>\mu</th>
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Thank you to Dr. Sung Yell Song and Oktay Olmez for aiding us in our research and instructing us in the background information.