Extremely Amenable Groups and Structural Ramsey Theory

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A topological group is a group $G$ with a topology that makes multiplication and inversion continuous.

An action of $G$ on a topological space $X$ is continuous if it is continuous as a map $G \times X \to X$.

We will assume throughout that $G, X$ are Hausdorff and $X$ is compact.
Continuous G-flows

Call a continuous action of $G$ on $X$ a $G$-flow.

**Definition**

A group $G$ is called extremely amenable if every $G$-flow $X$ has a fixed point; that is, some $x \in X$ such that for all $g \in G$ $g \cdot x = x$. 
Non-examples of Extremely Amenable Groups

Example

Let $G$ be a non-trivial, compact group and let $G$ act on itself by left translation. This is a $G$-flow with no fixed point. In fact it is a free $G$-flow, meaning the only thing that fixes anything is the identity. This means that no compact group is extremely amenable.
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Example

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Theorem

(Veech) Every locally compact group $G$ has a free $G$-flow.

Thus, no locally compact group is extremely amenable.
Examples of Extremely Amenable Groups

1. (Gromov-Milman) $U(H)$, the unitary group of the infinite dimensional, separable Hilbert Space (strong operator).

2. (Pestov) $\text{Aut}(\langle \mathbb{Q}, < \rangle)$ (pointwise convergence).

3. (Giordano-Pestov) $\text{Iso}(U)$, the isometries of the Urysohn space (pointwise convergence).

4. (Furstenberg-Weiss, Glasner) $L([0, 1], \mathbb{T})$, the measurable maps from the unit interval to the unit circle under pointwise multiplication (convergence in measure).
Structures

- Let $L$ be a countable language of relation symbols $\{R_i\}_{i \in I}$ with arity $n(i) \geq 1$ and function symbols $\{f_j\}_{j \in J}$ with arity $m(j) \geq 0$.

- An $L$-structure is $\mathcal{A} = \langle A, \{R_i^A\}_{i \in I}, \{f_j^A\}_{j \in J} \rangle$ where
  - $A \neq \emptyset$ is a set, the universe of $\mathcal{A}$.
  - For each $i$, $R_i^A \subseteq A^{n(i)}$ is a $n(i)$-ary relation on $A$.
  - For each $j$, $f_j^A : A^{m(j)} \to A$ is a $m(j)$-ary function on $A$.

- $B$ is a substructure of $\mathcal{A}$ if $B \subseteq A$, $B$ is closed under each $f_j^A$ and the functions and relations of $B$ are those of $\mathcal{A}$ restricted to $B$. 

Examples

References
Homomorphisms

Given two $L$-structures $\mathcal{A}, \mathcal{B}$, a map $\pi : A \to B$ is a homomorphism if

- $R^A_i(a_1, \ldots, a_{n(i)}) \iff R^B_i(\pi(a_i), \ldots, \pi(a_{n(i)}))$
- $\pi(f^A_j(a_1, \ldots, a_{m(j)})) = f^B_j(\pi(a_i), \ldots, \pi(a_{m(j)}))$

We will be particularly interested in the automorphism groups of structures, $\text{Aut}(\mathcal{A})$. 
More on Structures

- A structure is countable if its universe is countable, and similarly for any other cardinality.
- A structure is locally finite if every finitely generated substructure is actually finite.
- A structure is ultrahomogeneous if every isomorphism between two finitely generated substructures extends to an automorphism of the whole structure.

Example

For $L = \{<\}$, the structure $\langle \mathbb{Q}, < \rangle$ is countable, locally finite, and ultrahomogeneous.
Classes of Structures

Let $\mathbf{K}$ be a class of $L$-structures.

- $\mathbf{K}$ has the Hereditary Property (HP) if for $\mathcal{B} \in \mathbf{K}$ and any finitely generated $\mathcal{C} \leq \mathcal{B}$ ($\mathcal{C}$ embeds into $\mathcal{B}$), then $\mathcal{C} \in \mathbf{K}$.
- $\mathbf{K}$ has the Joint Embedding Property (JEP) if for $\mathcal{B}, \mathcal{C} \in \mathbf{K}$, there is a $\mathcal{D} \in \mathbf{K}$ such that $\mathcal{B}, \mathcal{C} \leq \mathcal{D}$.
- $\mathbf{K}$ has the Amalgamation Property (AP) if for $\mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathbf{K}$, and embeddings $f : \mathcal{B} \to \mathcal{C}$, $g : \mathcal{B} \to \mathcal{D}$, there is an $\mathcal{E} \in \mathbf{K}$ and embeddings $r : \mathcal{C} \to \mathcal{E}$, $s : \mathcal{D} \to \mathcal{E}$ such that $r \circ f = s \circ g$.

**Example**

The class of finite linear orders, $\mathbf{LO}$, satisfies HP, JEP, and AP.
The Age of a Structure

Definition

For an $L$-structure $\mathcal{A}$, the age of $\mathcal{A}$, denoted $\text{Age}(\mathcal{A})$, is the collection of finitely generated $L$-structures embeddable into $\mathcal{A}$.

- $\text{Age}(\mathcal{A})$ always satisfies HP and JEP.
- If $\mathcal{A}$ is ultrahomogeneous, then it also satisfies AP.
- If $\mathcal{A}$ is countable, then $\text{Age}(\mathcal{A})$ has only countably many isomorphism types.
- We will call a class of structures countable if it only has countably many isomorphism types.

Example

$\text{Age}(\langle \mathbb{Q}, < \rangle) = \text{LO}$. 
Fraïssé’s Theorem

**Theorem**

Fix a countable language $L$. Let $\mathbf{K}$ be a nonempty, countable class of finitely generated $L$-structures satisfying HP, JEP, and AP. Then there is a unique (up to isomorphism) countable, ultrahomogeneous structure $\mathcal{A}$, called the Fraïssé limit of $\mathbf{K}$, such that $\mathbf{K} = \text{Age}(\mathcal{A})$.

So a countable, ultrahomogeneous structure is the Fraïssé limit of its age.

**Example**

Since $\text{Age}(\langle \mathbb{Q}, < \rangle) = \mathbf{LO}$, we have $\text{Flim}(\mathbf{LO}) = \langle \mathbb{Q}, < \rangle$. 
Fraïssé classes and structures

Definition

For a language $L$, a Fraïssé class in $L$ is a class of finite $L$-structures which

- is countable
- contains structures of arbitrarily high finite cardinality
- satisfies HP, JEP, and AP

Definition

A Fraïssé structure in $L$ is a countably infinite, locally finite, ultrahomogeneous structure.
So the map $\mathbf{K} \rightarrow Flim(\mathbf{K})$ is a bijection between Fraïssé classes and Fraïssé structures (up to isomorphism type) and has as its inverse the map $\mathcal{A} \rightarrow Age(\mathcal{A})$. 
Finite Ramsey Theorem

Theorem

For $a, b, k \in \omega$ with $a \leq b$, there is a $c \in \omega$ such that if $|S| \geq c$, then for any $k$-coloring of $[S]^a$, there is a $T \subseteq S$ such that $|T| = b$ and $[T]^a$ is monochromatic.

This is denoted for all $a \leq b, k \in \omega$, there is a $c \in \omega$ such that $c \rightarrow (b)^a_k$. 
Since a finite subset of $\omega$ is a finite linear order, what does this theorem say about the class of finite linear orders, $\text{LO}$?

**Theorem**

For any $A, B \in \text{LO}$, $A \leq B$ and any $k \in \omega$, there is a $C \in \text{LO}$ such that any $k$-coloring of the substructures of $C$ isomorphic to $A$, there is a $B_0 \subseteq C$ isomorphic to $B$ such that $B_0$ is homogeneous.
Fix $L$ and let $A, B, C$ be $L$-structures.

- If $A \leq B$, let $\binom{B}{A} = \{A_0 : A_0 \subseteq B \text{ and } A_0 \cong A\}$
- For $A \leq B \leq C$, $k \in \omega$, say $C \rightarrow \binom{B}{A}_k^A$ if for any $k$-coloring of $\binom{C}{A}$, there is a homogeneous $B_0 \in \binom{C}{B}$, meaning $\binom{B_0}{A}$ is monochromatic.

**Definition**

A class of finite $L$-structures $\mathcal{K}$ has the Ramsey Property if $\mathcal{K}$ satisfies HP and for $A \leq B, k \in \omega$, there is a $C \in \mathcal{K}$ where $B \leq C$ and $C \rightarrow \binom{B}{A}_k^A$. 
Examples of the Ramsey Property

Example

For $L = \{<\}$, $\textbf{LO}$ has the Ramsey Property, by Ramsey’s Theorem.

Example

For $L = \{<, E\}$, let $\textbf{OG}$ be the class of finite ordered graphs $\mathcal{A} = \langle A, <^A, E^A \rangle$ where $<^A$ is a linear order and $E^A$ is irreflexive and symmetric. (Nešetřil-Rödl) $\textbf{OG}$ has the Ramsey Property.
Examples of the Ramsey Property

Example

- Let $F$ be a finite field, and $L = \{+, f_\alpha : \alpha \in F\}$. Then vector spaces over $F$ are $L$-structures.
- Let $\mathbf{V}_F$ be the class of all finite dimensional vector spaces over $F$.
- (Graham-Leeb-Rothschild) $\mathbf{V}_F$ has the Ramsey Property.
The topological group $S_\infty$ is the group of permutations of $\omega$ topologized as a Polish subgroup of $\omega^\omega$.

It has a left-invariant compatible metric:

$$d(x, y) = \begin{cases} 
0 & x = y \\
2^{-n} & \text{if } x \neq y \text{ and } n \text{ is least such that } x(n) \neq y(n) 
\end{cases}$$

The basic open sets are of the form $[\sigma|n]$, the permutations agreeing with $\sigma$ up to $n$. 

$S_\infty$
Closed Subgroups of $S_\infty$

- For a countable structure $\mathcal{A}$, $Aut(\mathcal{A})$ is isomorphic to a closed subgroup of $S_\infty$.
- And in fact, for any closed $G \leq S_\infty$, there is a countable structure $\mathcal{A}_G$ such that $Aut(\mathcal{A}_G) \cong G$.
- By the construction of $\mathcal{A}_G$, it is seen to be ultrahomogeneous.
Let $L$ be a language with a distinguished binary relation symbol $\prec$.

- An order structure for $L$ is a structure $\mathcal{A}$ where $\prec^\mathcal{A}$ is a linear order.
- An order class is a class $\mathbf{K}$ of $L$-structures where every $\mathcal{A} \in \mathbf{K}$ is an order structure.
Theorem

(Kechris-Pestov-Todorcevic) Let $G \leq S_\infty$ be closed. Then $G$ is extremely amenable iff $G = \text{Aut}(\mathcal{A})$ where $\mathcal{A}$ is the Fraïssé limit of a Fraïssé order class with the Ramsey Property.

Corollary

Let $\{<\} \subseteq L$ be a language and $\mathcal{K}$ a Fraïssé order class in $L$. Let $\mathcal{F} = \text{Flim}(\mathcal{K})$, so $\mathcal{F}$ is a Fraïssé order structure. Then $G = \text{Aut}(\mathcal{F})$ is extremely amenable iff $\mathcal{K}$ has the Ramsey Property.
Since $\mathbf{LO}$ is a Fraïssé order class with the Ramsey Property with $\langle \mathbb{Q}, < \rangle$ as its Fraïssé limit, we have that $\text{Aut}(\langle \mathbb{Q}, < \rangle)$ is extremely amenable.
Let $G$ be the class of all finite graphs. This is a Fraïssé class with Fraïssé limit $R$, the Rado graph, also known as the random graph.

Now look at $OG$, the class of all finite ordered graphs, a Fraïssé order class.

We’ll call $Flim(OG) = OR$ the random ordered graph.

$OR$ is the random graph along with a linear order $<OR$ isomorphic to $\mathbb{Q}$.

Since $OG$ has the Ramsey Property (by N-R), $Aut(OR)$ is extremely amenable.
Vector Spaces

- For a finite field $F$ we have $\mathcal{V}_F$, the class of finite vector spaces over $F$.
- This is a Fraïssé class over $L_0 = \{+, f_\alpha : \alpha \in F\}$.
- We can define an ordering on a vector space $\mathcal{V}$ by ordering $F = \{0 = a_0 < a_1 < \ldots < a_r\}$ and ordering a basis of $\mathcal{V}$, $B = \{b_0 < \ldots < b_s\}$ and ordering $\mathcal{V}$ antilexicographically.
- An ordering obtained this way is called a natural ordering.
Let $OV_F$ be the Fraïssé order class over $L = \{+, f_\alpha : \alpha \in F, <\}$ of finite vector spaces over $F$ with a natural ordering.

$Flim(OV_F) = OV_F = \langle V_F, +, f_\alpha, <_{OV_F} \rangle$, where $<_{OV_F}$ is an appropriate linear ordering.

We’ll call $OV_F$ the $\aleph_0$-dimensional vector space over $F$ with the canonical ordering.

$OV_F$ has the Ramsey Property, so $Aut(OV_F)$ is extremely amenable.
For a metric space \((X, d)\), we can define a structure over \(L_0 = \{R_q\}_{q \in \mathbb{Q}}\) by \(\mathcal{X} = \langle X, \{R^X_q\}_{q \in \mathbb{Q}} \rangle\) where \((x, y) \in R^X_q \iff d(x, y) < q\).

Let \(M_\mathbb{Q}\) be the class of finite metric spaces with rational distances.

This is a Fraïssé class with Fraïssé limit \(U_0\), the rational Urysohn space: the unique, universal, countable, ultrahomogeneous, rational metric space.

The completion of \(U_0\) is the Urysohn space \(U\), the unique, universal, ultrahomogeneous Polish space.
Let $\text{OM}_\mathbb{Q}$ be the class of ordered finite metric spaces with rational distances.

This is a Fraïssé order class with Fraïssé limit $\text{OU}_0$, the ordered rational Urysohn space.

Since $\text{OM}_\mathbb{Q}$ has the Ramsey Property (N), $\text{Aut}(\text{OM}_\mathbb{Q})$ is extremely amenable.
