ARITHMETIC OF ADDITIVELY REDUCED MONOID SEMIDOMAINS

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ABSTRACT. A subset S of an integral domain R is called a semidomain if the pairs $(S, +)$ and $(S, \cdot)$ are semigroups with identities; additionally, we say that S is additively reduced provided that S contains no additive inverses. Given an additively reduced semidomain S and a torsion-free monoid M, we denote by $S[M]$ the semidomain consisting of polynomial expressions with coefficients in S and exponents in M; we refer to these objects as additively reduced monoid semidomains. We study the factorization properties of additively reduced monoid semidomains. Specifically, we determine necessary and sufficient conditions for an additively reduced monoid semidomain to be a bounded factorization semidomain, a finite factorization semidomain, and a unique factorization semidomain. We also provide large classes of semidomains with full and infinity elasticity. Throughout the paper we provide examples aiming to shed some light upon the arithmetic of additively reduced semidomains.

1. INTRODUCTION

It is well known that the polynomial extension of a unique factorization domain (UFD) is also a unique factorization domain. In particular, the integral domain $\mathbb{Z}[x]$ is a UFD. In contrast, it is not hard to find nonzero nonunit elements in the multiplicative monoid $\mathbb{N}_0[x]^*$, the monoid of polynomials with nonnegative coefficients, having multiple factorizations (see, for instance, [19]), even though the multiplicative monoid $\mathbb{N}$ is a unique factorization monoid. In other words, the property of having unique factorization does not ascend from $\mathbb{N}_0$ to its polynomial extension $\mathbb{N}_0[x]$. This makes the factorization properties of $\mathbb{N}_0[x]$ much more interesting than those of $\mathbb{Z}[x]$. As a matter of fact, the arithmetic of $\mathbb{N}_0[x]$ and its valuations has been the subject of various articles. In [6], Campanini and Facchini investigated the arithmetic and ideal structure of $\mathbb{N}_0[x]$, while Brunotte [5] studied the factors with positive coefficients of a given polynomial with no positive roots. In addition, Baeth and Gotti [3] briefly explored the factorization properties of the multiplicative monoid $\mathbb{N}_0[r]^*$ with $r \in \mathbb{Q}_{>0}$.

A subset S of an integral domain R is called a *semidomain* if the pairs $(S, +)$ and $(S, \cdot)$ are semigroups with identities; additionally, we say that S is *additively reduced* provided that S contains no additive inverses. Given an additively reduced semidomain S and a cancellative, commutative, and torsion-free monoid M, we denote by $S[M]$ the additively reduced semidomain consisting of polynomial expressions with coefficients in S and exponents in M; we refer to these objects as *additively reduced monoid semidomains*. Clearly, $\mathbb{N}_0[x]$ is an additively reduced monoid semidomain, perhaps the simplest one. But $\mathbb{N}_0[x]$ is not the only additively reduced monoid semidomain that has been investigated before. Motivated by potential applications in control theory, Barnard et al. [4] analyzed the relationship between the pairs of conjugate roots of a polynomial $f \in \mathbb{R}_{\geq 0}[x]$ and the divisors of f with positive coefficients, while Cesarz et al. [7] investigated the elasticity and delta set of $\mathbb{R}_{\geq 0}[x]$. Moreover, Ponomarenko [22] studied the factorization properties of semigroup semirings, a class of semirings strictly containing that of additively reduced semidomains.

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The purpose of the present paper is to investigate the arithmetic of additively reduced monoid semidomains, and our work is structured as follows. In Section 2, we introduce the necessary background to follow our exposition. Our first results are presented in Section 3, where we provide necessary and sufficient conditions for an additively reduced monoid semidomain to be atomic and to satisfy the ACCP. Then, in Section 4, we focus on the bounded and finite factorization properties. Specifically, we show that an additively reduced monoid semidomain $S[M]$ is a BFS (resp., an FFS) if and only if S is a BFS (resp., an FFS) and M is a BFM (resp., an FFM). Section 5 is devoted to the study of the factoriality properties of additively reduced monoid semidomains. Here we prove that an additively reduced monoid semidomain $S[M]$ is a UFS (resp., an LFS, an HFS) if and only if M is the trivial group and S is a UFS (resp., an LFS, an HFS). We conclude providing large classes of semidomains with full and infinite elasticity.

2. Background

We now review some of the standard notation and terminology we shall be using later. Reference material on factorization theory and semiring theory can be found in the monographs [13] and [15], respectively. We let $\mathbb{N}$, $\mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ denote the sets of positive integers, integers, rational numbers, and real numbers, respectively, and we set $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. In addition, given $r \in \mathbb{R}$ and $S \subseteq \mathbb{R}$, we set $S_{>r} := \{s \in S \mid s > r\}$. We define $S_{\ge r}$ in a similar way. For $m, n \in \mathbb{N}_0$, we set $[\![m, n]\!] := \{k \in \mathbb{Z} \mid m \le k \le n\}$. For $q \in \mathbb{Q}_{>0}$, there exist unique $n, d \in \mathbb{N}$ such that $q = d^{-1}n$ and $\gcd(n, d) = 1$; we refer to n and d as the *numerator* and *denominator* of q and denote them by $\mathsf{n}(q)$ and $\mathsf{d}(q)$, respectively.

2.1. **Monoids and Factorizations.** Throughout this paper, a *monoid* is defined to be a semigroup with identity that is cancellative and commutative and, unless we specify otherwise, we will use multiplicative notation for monoids. For the rest of the section, let M be a monoid. We let $\mathscr{U}(M)$ denote the group of units (i.e., invertible elements) of M. Additionally, we let M_{red} denote the quotient monoid $M/\mathscr{U}(M)$. We say that M is *reduced* provided that the group of units of M is trivial; in this case, we identify M_{red} with M. On the other hand, the monoid M is *torsion-free* if for all $x, y \in M$ and $n \in \mathbb{N}$, we have that $x^n = y^n$ implies that $x = y$. Given a subset S of M, we let $\langle S \rangle$ denote the smallest submonoid of M containing S. Also, we denote by $\mathscr{G}(M)$ the unique (up to isomorphism) abelian group satisfying that if an abelian group contains a homomorphic image of M then it has to contain a homomorphic image of $\mathscr{G}(M)$; this group is called the *Grothendieck group* of M.

For elements $b, c \in M$, we say that b *divides* c *in* M if there is $b' \in M$ such that $c = bb'$; in this case, we write $b \mid_M c$, dropping the subscript whenever M is the multiplicative monoid of the natural numbers. On the other hand, two elements $b, c \in M$ are *associates*, which we denote by $b \simeq_M c$, provided that $b = u \cdot c$ for some $u \in \mathscr{U}(M)$. A submonoid N of M is *divisor-closed* in M if for every $b \in N$ and $c \in M$ the relation $c \mid_M b$ implies that $c \in N$. Let S be a nonempty subset of M. While we say that $d \in M$ is a *common divisor* of S given that d divides all elements of S, we say that a common divisor d of S is a *greatest common divisor* of S provided that d is divisible by all common divisors of S. Moreover, a common divisor of S is a *maximal common divisor* if every greatest common divisor of S/d is a unit of M. We denote by $\gcd_M(S)$ (resp., $\text{mcd}_M(S)$) the set consisting of all greatest common divisors (resp., maximal common divisors) of S. We say that M is a *GCD-monoid* (resp., an *MCD-monoid*) on the condition that every nonempty and finite subset of M has a greatest common divisor (resp., maximal common divisor).

An element $a \in M$ is called an *atom* if for every $b, c \in M$ the equality $a = bc$ implies that either $b \in \mathscr{U}(M)$ or $c \in \mathscr{U}(M)$; we denote by $\mathscr{A}(M)$ the set of atoms of M. We say that M is *atomic* provided that every element in $M \setminus \mathscr{U}(M)$ can be written as a finite product of atoms. It is easy to verify that M is atomic if and only if M_{red} is atomic. On the other hand, a subset I of M is an *ideal* of M provided that $IM \subseteq I$. An ideal I of M is *principal* if $I = xM$ for some $x \in M$. We say that M satisfies the *ascending chain condition on principal ideals* (*ACCP*) if every increasing sequence of principal ideals of M (under inclusion) eventually terminates. It is not hard to see that if a monoid satisfies the ACCP then it is atomic.

Suppose now that the monoid M is atomic. We denote by $\mathsf{Z}(M)$ the free (commutative) monoid on $\mathscr{A}(M_{\text{red}})$ whose elements we call *factorizations*. Given a factorization $z = a_1 \cdots a_\ell \in \mathsf{Z}(M)$, where $a_1, \dots, a_\ell \in \mathscr{A}(M_{\text{red}})$, it is said that ℓ is the *length* of z. We let $|z|$ denote the length of a factorization $z \in \mathsf{Z}(M)$. Let $\pi \colon \mathsf{Z}(M) \to M_{\text{red}}$ be the unique (monoid) homomorphism fixing the set $\mathscr{A}(M_{\text{red}})$. For every element $x \in M$, the following sets associated to x play a crucial role in the study of factorizations:

$$\mathsf{Z}_M(x) := \pi^{-1}(x\mathscr{U}(M)) \subseteq \mathsf{Z}(M) \qquad \text{and} \qquad \mathsf{L}_M(x) := \{|z| : z \in \mathsf{Z}_M(x)\} \subseteq \mathbb{N}_0. \tag{2.1}$$

The subscript in (2.1) is dropped when there seems to be no risk of confusion. Following [18], the monoid M is called a *finite factorization monoid* (*FFM*) if $\mathsf{Z}(x)$ is finite for every $x \in M$, and M is called a *bounded factorization monoid* (*BFM*) if $\mathsf{L}(x)$ is finite for all $x \in M$. It is evident that every FFM is a BFM and, by [13, Corollary 1.3.3], every BFM satisfies the ACCP. Following [26], we say that M is a *half-factorial monoid* (*HFM*) provided that $|\mathsf{L}(x)| = 1$ for every $x \in M$. Moreover, a monoid M is called *factorial* or a *unique factorization monoid* (*UFM*) if $|\mathsf{Z}(x)| = 1$ for all $x \in M$. It is clear that a UFM is an HFM and also that an HFM is a BFM. Finally, we follow the terminology in [8] and say that M is a *length-factorial monoid* (*LFM*) if for every $x \in M$ and $z, z' \in \mathsf{Z}(x)$, the equality $|z| = |z'|$ implies that $z = z'$. It is obvious that if a monoid is a UFM then it is an LFM.

2.2. **Semirings and Semidomains.** A *semiring* S is a (nonempty) set endowed with two binary operations denoted by '$\cdot$' and '$+$' and called *multiplication* and *addition*, respectively, such that the following conditions hold:

- $(S, \cdot)$ is a commutative semigroup with an identity element denoted by 1;
- $(S, +)$ is a monoid with its identity element denoted by 0;
- $b \cdot (c + d) = b \cdot c + b \cdot d$ for all $b, c, d \in S$;
- $0 \cdot b = 0$ for all $b \in S$.

We sometimes write bc instead of $b \cdot c$ for elements b, c in a semiring S. We would like to emphasize that a more general notion of a 'semiring' does not usually assume the commutativity of the underlying multiplicative semigroup, but these algebraic objects are not of interest in the scope of this article.

If R and S are semirings then a function $\sigma \colon R \to S$ is a *semiring homomorphism* if, for all $b, c \in R$, the following conditions hold:

- $\sigma(bc) = \sigma(b)\sigma(c)$;
- $\sigma(b + c) = \sigma(b) + \sigma(c)$;
- $\sigma(1) = 1$.
- $\sigma(0) = 0$;

We say that σ is a *semiring isomorphism* provided that σ is injective and surjective. On the other hand, a subset S' of a semiring S is a *subsemiring* of S if $(S', +)$ is a submonoid of $(S, +)$ that contains 1 and is closed under multiplication. Clearly, every subsemiring of S is a semiring.

Definition 2.1. A *semidomain* is a subsemiring of an integral domain.

Let S be a semidomain. We say that $(S \setminus \{0\}, \cdot)$ is the *multiplicative monoid* of S, and we denote it by S^*. Following standard notation from ring theory, we refer to the units of the multiplicative monoid S^* simply as *units*, and we refer to the units of $(S, +)$ as invertible elements without risk of ambiguity; we let $S^\times$ denote the group of units of S, while we let $\mathscr{U}(S)$ denote the additive group of invertible elements of S. We denote the set of atoms of the multiplicative monoid S^* as $\mathscr{A}(S)$ instead of $\mathscr{A}(S^*)$.

Also, for $b, c \in S$ such that b and c are associates in S^*, we write $b \simeq_S c$ (instead of $b \simeq_{S^*} c$). Similarly, for $b, c \in S$ such that b divides c in S^*, we write $b \mid_S c$ (instead of $b \mid_{S^*} c$).

Lemma 2.2. *[17, Lemma 2.2] For a semiring S, the following conditions are equivalent.*

(a) The multiplication of S extends to $\mathscr{G}(S)$ turning $\mathscr{G}(S)$ into an integral domain.

(b) S is a semidomain.

Given a semidomain S, we let $\mathscr{F}(S)$ denote the field of fractions of $\mathscr{G}(S)$. On the other hand, we say that a semidomain S is *atomic* (resp., satisfies the *ACCP*) if its multiplicative monoid S^* is atomic (resp., satisfies the ACCP). In addition, we say that S is a *BFS*, *FFS*, *HFS*, *LFS*, or *UFS* provided that S^* is a BFM, FFM, HFM, LFM, or UFM, respectively. Observe that when S is an integral domain, we recover the usual definitions of a UFD, a BFD, an FFD, and an HFD, which are now established notions in factorization theory.

Following [2], a subsemiring of the positive cone of $\mathbb{R}$ (under the standard multiplication and addition) is called a *positive semiring*. The fact that the underlying additive monoids of positive semirings are reduced makes them more tractable. In the recent paper [2], the reader can find several examples of positive semirings. Note that the class of additively reduced semidomains clearly contains that of positive semirings.

2.3. **Monoid Semirings.** Given a semiring S and a torsion-free monoid M (written additively), consider the set $S[M]$ consisting of all maps $f \colon M \to S$ satisfying that the set $\{m \in M \mid f(m) \neq 0\}$ is finite. We shall conveniently represent an element $f \in S[M]$ by

$$f = \sum_{m \in M} f(m)x^m = \sum_{i=1}^{n} f(m_i)x^{m_i},$$

where the exponents $m_1, \dots, m_n$ are the elements of M whose image under f is nonzero. Addition and multiplication in $S[M]$ are defined as for polynomials, and we call the elements of $S[M]$ *polynomial expressions*. Under these operations, $S[M]$ is a commutative semiring, which we call the *monoid semiring of M over S* or, simply, a monoid semiring.

Lemma 2.3. *Let S be a semidomain, and let M be a torsion-free monoid. Then $S[M]$ is a semidomain and*

$$S[M]^\times = \left\{ sx^m \mid s \in S^\times \text{ and } m \in \mathscr{U}(M) \right\}.$$

Proof. By virtue of [14, Theorem 8.1], we have that $\mathscr{G}(S)[M]$ is an integral domain and, clearly, $S[M]$ is a subsemiring of $\mathscr{G}(S)[M]$. Hence $S[M]$ is a semidomain. The last part of our lemma is easy to verify; we leave this task to the reader. □

If S is a semidomain then we say that $S[M]$ is a *monoid semidomain*. Observe that $S[M]$ is additively reduced provided that S is additively reduced. Since the monoid M is torsion-free (and cancellative), M admits a total order compatible with its monoid operation ([14, Corollary 3.4]). For $n \in \mathbb{N}$, we say that

$$f = s_1 x^{m_1} + \cdots + s_n x^{m_n} \in S[M]^*$$

is written in *canonical form* when $s_i \neq 0$ for every $i \in [\![1, n]\!]$ and $m_1 > \cdots > m_n$. Observe that there is only one way to write f in canonical form. As for polynomials, we call $\deg(f) := m_1$ the *degree* of f and $\mathsf{c}(f) := s_1$ the *leading coefficient* of f. Additionally, we say that $\mathsf{Supp}(f) := \{m_1, \dots, m_n\}$ is the *support* of f, and f is called a *monomial* (resp., *binomial*, *trinomial*) if $|\mathsf{Supp}(f)| = 1$ (resp., $|\mathsf{Supp}(f)| = 2$, $|\mathsf{Supp}(f)| = 3$).

3. Atomicity and the ACCP

In this section, we study under which circumstances an additively reduced monoid semidomain satisfies the properties of being atomic and the ACCP.

Given a semidomain S, we say that a nonzero polynomial in $S[x]$ is *indecomposable* if it cannot be written as a product of two non-constant polynomials in $S[x]$. In [17] and [23], it was shown that indecomposable polynomials play an important role in the ascent of atomicity from a semidomain S to its polynomial extension $S[x]$. Next we introduce a generalization of the notion of indecomposability to the context of monoid semidomains.

Definition 3.1. Given a monoid semidomain $S[M]$, we say that a nonzero polynomial expression $f \in S[M]$ is *monolithic* if $f = gh$ implies that either g or h is a monomial in $S[M]$.

While there are monolithic polynomials that are not indecomposable (e.g., $x^2 + x^3 \in \mathbb{N}_0[x]$), indecomposable polynomials are clearly monolithic. Our next lemma sheds some light upon the role that monolithic polynomial expressions will play in this section.

Lemma 3.2. *Let S be an additively reduced semidomain, and let M be a torsion-free monoid. Every nonzero nonunit polynomial expression in $S[M]$ factors into monolithic polynomial expressions.*

Proof. Let $f = \sum_{i=1}^{n} s_i x^{m_i}$ be a nonzero nonunit polynomial expression in $S[M]$ written in canonical form, so $s_i \neq 0$ for every $i \in [\![1, n]\!]$ and $m_1 > \cdots > m_n$. We proceed by induction on n. If $n = 1$ then f is monolithic. Suppose now that all nonzero nonunit polynomial expressions whose support have cardinality strictly less than n factors into monolithic polynomial expressions. If f is not monolithic then $f = gh$, where neither g nor h is a monomial. Observe that since S is additively reduced, we have that $\max(|\mathsf{Supp}(g)|, |\mathsf{Supp}(h)|) < |\mathsf{Supp}(f)|$, from which our argument follows inductively. $\square$

Now we are in a position to provide a necessary and sufficient condition for an additively reduced monoid semidomain to be atomic.

Theorem 3.3. *Let S be an additively reduced semidomain, and let M be a torsion-free monoid. Then $S[M]$ is atomic if and only if S and M are both atomic and*

$$\textit{mcd}\,(s_1, \dots, s_n) \times \textit{mcd}\,(m_1, \dots, m_n) \neq \emptyset$$

for any monolithic polynomial expression $f = s_1 x^{m_1} + \cdots + s_n x^{m_n} \in S[M]$ written in canonical form.

Proof. Suppose that $S[M]$ is atomic. Observe that the multiplicative monoid $N = \{s x^m \mid s \in S^* \text{ and } m \in \mathscr{U}(M)\}$ is a divisor-closed submonoid of $S[M]^*$, so it is atomic. Since $S^*_{\text{red}} \cong N_{\text{red}}$, we can conclude that S is atomic. Similarly, the multiplicative monoid $H = \{s x^m \mid s \in S^\times \text{ and } m \in M\}$ is a divisor-closed submonoid of $S[M]^*$, so it is atomic. This implies that M is also atomic as $M_{\text{red}} \cong H_{\text{red}}$. Now let $f = s_1 x^{m_1} + \cdots + s_n x^{m_n} \in S[M]$ be a monolithic polynomial expression written in canonical form. Without loss of generality, assume that f is not a monomial of $S[M]$ (so, in particular, $f \notin S[M]^\times$). Write $f = g_1 \cdots g_t$, where $g_j \in \mathscr{A}(S[M])$ for every $j \in [\![1, t]\!]$. Since f is monolithic, there is no loss in assuming that $g_1, \dots, g_{t-1}$ are all monomials. Let $s = \prod_{g_i \in N} g_i$ and $y = \prod_{g_i \notin N} g_i$, where the empty product is considered to be equal to 1. It is easy to see that $\mathsf{c}(s) \in \text{mcd}(s_1, \dots, s_n)$ and that we can write $y = s' x^m g_t$ for some $s' \in S^\times$ and $m \in \text{mcd}(m_1, \dots, m_n)$. Hence $\text{mcd}(s_1, \dots, s_n) \times \text{mcd}(m_1, \dots, m_n) \neq \emptyset$.

As for the reverse implication, let us start by noticing that if $a \in \mathscr{A}(S)$ (resp., $a \in \mathscr{A}(M)$) then $a \in \mathscr{A}(S[M])$ (resp., $x^a \in \mathscr{A}(S[M])$). Now let $f = \sum_{i=1}^{n} s_i x^{m_i} \in S[M]$ be a nonzero nonunit element written in canonical form. Since S and M are both atomic, there is no loss in assuming that $n > 1$. By Lemma 3.2, we can write $f = g_1 \cdots g_k$, where g_j is monolithic for each $j \in [\![1, k]\!]$. Now fix $j \in [\![1, k]\!]$. Thus,

$$g_j = \sum_{i=1}^{l} s'_i x^{m'_i} = \text{mcd}(s'_1, \dots, s'_l)\, x^{\text{mcd}(m'_1, \dots, m'_l)} h_j$$

for some $h_j \in \mathscr{A}(S[M])$. By our initial observation, we have that $g_j \in \langle \mathscr{A}(S[M]) \rangle$ for every $j \in [\![1, k]\!]$. Therefore $S[M]$ is atomic. $\square$

The next result follows readily from Theorem 3.3.

Corollary 3.4. *Let S be an additively reduced semidomain, and let M be a torsion-free monoid. Then $S[M]^*$ is an atomic MCD-monoid if and only if S^* and M are both atomic MCD-monoids.*

Theorem 3.3 and Corollary 3.4 depend on the fact that the semidomain S is additively reduced. We now discuss an example (introduced in [11]) in which generalizations of Theorem 3.3 and Corollary 3.4 to the context of general semidomains utterly fail.

Example 3.5. Fix a prime number p, and let $(p_n)_{n \in \mathbb{N}}$ be a sequence consisting of all prime numbers different from p ordered increasingly. Now set $M_p \coloneqq \langle p^{-n} p_n^{-1} \mid n \in \mathbb{N} \rangle$, which is an additive submonoid of $(\mathbb{Q}_{\geq 0}, +)$, and take $M = M_p \times M_p$. It is known that M is an atomic torsion-free monoid (see [11, page 9]). In fact, it is easy to see that $\mathscr{A}(M_p) = \{p^{-n} p_n^{-1} \mid n \in \mathbb{N}\}$, which implies that M_p is atomic; consequently, M is also atomic by [11, Proposition 3.1]. An elementary argument can be used to verify that each nonzero element $m \in M_p$ has a unique representation in the form

$$m = m' + \sum_{i=1}^{n} \frac{c_i}{p^i p_i},$$

where $m' \in \mathbb{Q}_{\geq 0}$ with $\mathsf{d}(m') = p^k$ for some $k \in \mathbb{N}_0$ and $0 \leq c_i \leq p_i - 1$ for each $i \in [\![1, n]\!]$. Consequently, M_p is an MCD-monoid which, in turn, implies that M is also an MCD-monoid. Now consider the monoid semidomain $F[M]$, where F is a finite field of characteristic p. Since there is a ring isomorphism $F[x; M_p \times M_p] \cong F[y; M_p] \times F[z; M_p]$ induced by the assignment $x^{(a,b)} \mapsto y^a z^b$, we can write the elements of $F[M]$ as polynomial expressions in two variables. It is known that every nonunit factor of $f = y + z + yz$ in $F[M]$ has the form

$$\left(y^{\frac{1}{p^k}} + z^{\frac{1}{p^k}} + y^{\frac{1}{p^k}} z^{\frac{1}{p^k}} \right)^t$$

for some $k \in \mathbb{N}_0$ and $t \in \mathbb{N}$ (see [11, page 9]). Consequently, not only is $F[M]$ non-atomic, but also no atom of $F[M]$ divides f.

We now turn our attention to the ACCP, a property closely related to that of being atomic.

Theorem 3.6. *Let S be an additively reduced semidomain, and let M be a torsion-free monoid. Then $S[M]$ satisfies the ACCP if and only if S and M satisfy both the ACCP.*

Proof. Suppose that $S[M]$ satisfies the ACCP. Since S^* is a submonoid of $S[M]^*$ such that $S^\times = S[M]^\times \cap S$, we have that S satisfies the ACCP. On the other hand, the multiplicative monoid $M_1 = \{x^m \mid m \in M\}$ (which is clearly isomorphic to M) is also a submonoid of $S[M]^*$ satisfying that $M_1^\times = S[M]^\times \cap M_1$. Consequently, M satisfies the ACCP. Conversely, suppose that S and M satisfy both the ACCP, and consider the multiplicative monoid $M_2 = \{sx^m \mid s \in S^* \text{ and } m \in M\}$. Clearly, M_2 is a divisor-closed submonoid of $S[M]^*$. Moreover, it is easy to see that

$$S^\times \subseteq M_2^\times = S[M]^\times = \left\{ sx^m \mid s \in S^\times \text{ and } m \in \mathscr{U}(M) \right\}.$$

Let $(s_k x^{m_k} M_2)_{k \in \mathbb{N}}$ be an ascending chain of principal ideals of M_2. Since the ascending chain $(s_k S)_{k \in \mathbb{N}}$ of principal ideals of S eventually stabilizes, there exists $n \in \mathbb{N}$ such that $s_i \simeq_S s_n$ for every $i \geq n$. Since $S^\times \subseteq M_2^\times$, there is no loss in assuming that $s_n = s_1$ for every $n \in \mathbb{N}$. Observe now that the ascending chain $(m_k M)_{k \in \mathbb{N}}$ of principal ideals of M stabilizes, which implies that $(s_k x^{m_k} M_2)_{k \in \mathbb{N}}$ also stabilizes by Lemma 2.3. Hence M_2 satisfies the ACCP. By way of contradiction, assume that there exists an ascending chain $\sigma = (f_k S[M])_{k \in \mathbb{N}}$ of principal ideals of $S[M]$ such that σ does not stabilize. If f_t is a monomial for some $t \in \mathbb{N}$ then σ would stabilize because M_2 is a divisor-closed submonoid of

$S[M]^*$ that satisfies the ACCP. As a consequence, we may assume that f_k is not a monomial for any $k \in \mathbb{N}$. Note that $(|\mathsf{Supp}(f_k)|)_{k\in\mathbb{N}}$ is a non-increasing sequence of natural numbers, which implies that we can also assume that $|\mathsf{Supp}(f_1)| = |\mathsf{Supp}(f_k)|$ for every $k \in \mathbb{N}$. Hence, for every $k \in \mathbb{N}$, we have $f_k = f_{k+1}(s_{k+1}x^{m_{k+1}})$ for some $s_{k+1} \in S^*$ and $m_{k+1} \in M$. Observe that $s_{k+1}x^{m_{k+1}} \notin S[M]^\times$ for any $k \in \mathbb{N}$. Since $s_{k+1}x^{m_{k+1}} \notin M_2^\times$ for any $k \in \mathbb{N}$, we have that $\sigma^* = (\mathsf{c}(f_k)x^{\deg(f_k)}M_2)_{k\in\mathbb{N}}$ is an ascending chain of ideals of M_2 that does not stabilize. This contradiction proves that our hypothesis is untenable. Therefore $S[M]$ satisfies the ACCP. □

In Theorem 3.6, the assumption that S is additively reduced is not superfluous as $F[\mathbb{Q}]$ does not satisfy the ACCP for any field F by [14, Theorem 14.17]. On the other hand, we can combine theorems 3.3 and 3.6 to yield atomic semidomains that do not satisfy the ACCP as the following example illustrates.

Example 3.7. Take $r \in \mathbb{Q} \cap (0, 1)$ with $\mathsf{n}(r) \geq 2$, and consider the additive monoid $S_r := \langle r^n \mid n \in \mathbb{N}_0\rangle$. By [9, Corollary 4.4], the monoid S_r is atomic and does not satisfy the ACCP. Moreover, it was argued in [17, Example 3.2] that S_r is an MCD-monoid. By theorems 3.3 and 3.6, the semidomain $\mathbb{N}_0[S_r]$ is atomic and does not satisfy the ACCP.

Remark 3.8. Given an additive submonoid M of $\mathbb{Q}_{\geq 0}$, consider the additive monoid $E(M) := \langle e^m \mid m \in M\rangle$, which is free on the set $S' = \{e^m \mid m \in M\}$ by the Lindemann-Weierstrass Theorem stating that, for distinct algebraic numbers $\alpha_1, \dots, \alpha_n$, the set $\{e^{\alpha_1}, \dots, e^{\alpha_n}\}$ is linearly independent over the algebraic numbers. Observe that $E(M)$ is closed under multiplication and, consequently, it is a positive semiring. This construction has been used in the literature to construct semidomains with prescribed factorization properties (see, for instance, [3, Example 4.15] and [2, Proposition 4.1]). Observe that the arithmetic of positive semirings of the form $E(M)$ can be better understood in the scope of the present paper since $E(M) \cong \mathbb{N}[M]$ (as semirings).

4. The Bounded and Finite Factorization Properties

This section is devoted to the study of the bounded and finite factorization properties in the context of additively reduced monoid semidomains. We start with a well-known and useful characterization of BFMs.

Definition 4.1. Given a monoid M, a function $\ell\colon M \to \mathbb{N}_0$ is a *length function* of M if it satisfies the following two properties:

(i) $\ell(u) = 0$ if and only if $u \in \mathscr{U}(M)$;

(ii) $\ell(bc) \geq \ell(b) + \ell(c)$ for every $b, c \in M$.

The following result is well known.

Proposition 4.2. *[18, Theorem 1] A monoid M is a BFM if and only if there is a length function $\ell\colon M \to \mathbb{N}_0$.*

We are now in a position to characterize the additively reduced monoid semidomains that are BFSs.

Theorem 4.3. *Let S be an additively reduced semidomain, and let M be a torsion-free monoid. Then $S[M]$ is a BFS if and only if S is a BFS and M is a BFM.*

Proof. Suppose that $S[M]$ is a BFS. As before, consider the multiplicative submonoid $N = \{sx^m \mid s \in S^* \text{ and } m \in \mathscr{U}(M)\}$ of $S[M]$ that satisfies that $N_{\mathrm{red}} \cong S^*_{\mathrm{red}}$. By Lemma 2.3, we have that $N^\times = S[M]^\times \cap N$. Consequently, the monoid N is a BFM by virtue of [13, Corollary 1.3.3] which, in turn, implies that S is a BFS. Similarly, the multiplicative submonoid $H = \{sx^m \mid s \in S^\times \text{ and } m \in M\}$ of $S[M]$ is a BFM as $H^\times = S[M]^\times \cap H$. Since $M_{\mathrm{red}} \cong H_{\mathrm{red}}$, we have that M is a BFM. Conversely, suppose that S is a BFS and M is a BFM. Then there exist length functions $\ell_c\colon S^* \to \mathbb{N}_0$ and $\ell_e\colon M \to \mathbb{N}_0$. Let us argue that the function $\ell\colon S[M]^* \to \mathbb{N}_0$ given by

$$\ell\,(f) = \ell_c\,(\mathsf{c}\,(f)) + \ell_e\,(\deg\,(f)) + |\,\mathsf{Supp}\,(f)\,| - 1$$

is also a length function. By Lemma 2.3, we have that $f \in S[M]$ is a unit if and only if $f = sx^m$, where $s \in S^\times$ and $m \in \mathscr{U}(M)$. Hence $\ell(f) = 0$ if and only if $f \in S[M]^\times$ as the reader can easily verify. For $f, g \in S[M]^*$ we see that

$$\begin{aligned} \ell(fg) &= \ell_c(\mathsf{c}(fg)) + \ell_e(\deg(fg)) + |\mathsf{Supp}\,(fg)| - 1 \\ &\geq \ell_c(\mathsf{c}(f)) + \ell_c(\mathsf{c}(g)) + \ell_e(\deg(f)) + \ell_e(\deg(g)) + |\mathsf{Supp}\,(f)| + |\mathsf{Supp}\,(g)| - 2 \\ &= \ell(f) + \ell(g), \end{aligned}$$

where the inequality follows from ℓ_c and ℓ_e being both length functions along with the fact that the inequality $|\mathsf{Supp}(fg)| \geq |\mathsf{Supp}(f)| + |\mathsf{Supp}(g)| - 1$ holds. Therefore the map ℓ is a length function of $S[M]^*$, which implies that $S[M]$ is a BFS by Proposition 4.2. □

From the corresponding definitions, we see that an additively reduced FFS is a BFS. However, there are numerous examples in the literature illustrating that the reverse implication fails (e.g. [2, Example 6.4] and [17, Example 4.5]). We now provide a new example of an additively reduced BFS that is not an FFS.

Example 4.4. The semidomain $S = \mathbb{N}_0 \cup \mathbb{Q}_{\geq 2}$ is a BFS that is not an FFS (see [2, Example 6.4]). Now let

$$R = \operatorname{Int}(\mathbb{N}, S) \coloneqq \{f \in \mathbb{Q}[x] \mid f(\mathbb{N}) \subseteq S\}\,.$$

It is easy to check that R is an additively reduced semidomain. Since $S \subseteq R$ and $R^\times = S^\times = \{1\}$, the semidomain R is not an FFS by [13, Theorem 1.5.6]. For every element $f \in R^* \setminus R^\times$, set

$$\mathfrak{L}(f) \coloneqq \left\{\ell \in \mathbb{N} \mid f = g_1 \cdots g_\ell, \text{ where } g_i \in R^* \setminus R^\times \text{ for each } i \in [\![1, \ell]\!]\right\}.$$

Suppose towards a contradiction that there exists $f \in R^* \setminus R^\times$ such that $|\mathfrak{L}(f)| = \infty$. Since S is a BFS, we have that $\deg(f) \geq 1$. There is no loss in assuming that f has no zeros in the interval $[1, \deg(f)+3]$; otherwise, we can pick another interval $I = [k+1, k+\deg(f)+3]$ for some $k \in \mathbb{N}$ such that f has no zeros in I. By a similar reasoning, we may assume that $f(n) \neq 1$ for any $n \in [\![1, \deg(f)+3]\!]$. Set $m \coloneqq \sum_{n=1}^{\deg(f)+3} \max \mathsf{L}(f(n))$, which is well defined since S is a BFS, and let $g_1 \cdots g_\ell \in \mathfrak{L}(f)$ such that $\ell > m$. Observe that, for each $n \in \mathbb{N}$, we have $f(n) = g_1(n) \cdots g_\ell(n)$, which implies that there exists $j \in [\![1, \ell]\!]$ such that

$$g_j(1) = g_j(2) = \cdots = g_j(\deg(f)+2) = 1.$$

Since $\deg(g_j) \leq \deg(f)$, we have that $g_j = 1$, a contradiction. Consequently, for every $f \in S^* \setminus S^\times$, there exists $n_f \in \mathbb{N}$ such that $|\mathfrak{L}(f)| \leq n_f$. This, in turn, implies that R is atomic. Since $\mathsf{L}(f) \subseteq \mathfrak{L}(f)$, it is clear that the semidomain R is a BFS.

For the rest of the section, we focus on the finite factorization property.

Definition 4.5. Let g be an element of a torsion-free abelian group G (which is additively written), and let N_g be the set of positive integers n such that the equation $nx = g$ has a solution in G. We say that $g \in G$ is *of type* $(0, 0, \ldots)$ provided that N_g is finite. On the other hand, we say that $g \in G$ is *of height* $(0, 0, \ldots)$ if N_g is a singleton (i.e., $N_g = \{1\}$).

Theorem 4.6. *Let S be an additively reduced semidomain, and let M be a torsion-free monoid. Then $S[M]$ is an FFS if and only if S is an FFS and M is an FFM.*

Proof. Suppose that $S[M]$ is an FFS. Again, consider the multiplicative submonoid $N = \{sx^m \mid s \in S^* \text{ and } m \in \mathscr{U}(M)\}$. Since $N^\times = S[M]^\times$, the monoid N is an FFM by [13, Theorem 1.5.6] which, in turn, implies that S^* is an FFM as $S^*_{\text{red}} \cong N_{\text{red}}$. Similarly, consider the multiplicative submonoid $H = \{sx^m \mid s \in S^\times \text{ and } m \in M\}$. Since $H^\times = S[M]^\times$, the monoid H is an FFM. From the fact that $M_{\text{red}} \cong H_{\text{red}}$, we conclude that M is also an FFM.

For the rest of the proof, we assume that a polynomial expression in $S[M]$ is always written in canonical form. Now to tackle the reverse implication, suppose that S^* and M are both FFMs. By way of contradiction, assume that $S[M]$ is not an FFS. By [13, Proposition 1.5.5], there exists $f := \sum_{i=0}^n s_i x^{m_i} \in S[M]$ such that f has infinitely many divisors in $S[M]$ that are pairwise non-associates. Let $g := \sum_{j=0}^t s'_j x^{m'_j}$ be an arbitrary divisor of f in $S[M]$. Observe that, for every $j \in [\![0,t]\!]$, there exists $i \in [\![0,n]\!]$ such that $m'_j \mid_M m_i$. Moreover, the inequality $t \le n$ holds. Since M is an FFM, for some $r \in [\![0,n]\!]$, there exists a sequence

$$\sigma = \left(g^{(k)} := \sum_{\ell=0}^{r} s_\ell^{(k)} x^{m_\ell^{(k)}} \right)_{k \in \mathbb{N}}$$

of pairwise non-associates divisors of f in $S[M]$ satisfying that $m_\ell^{(k)} \simeq_M m_\ell^{(1)}$ for every $k \in \mathbb{N}$ and every $\ell \in [\![0,r]\!]$. Since S^* is an FFM, we may assume that $s_0^{(k)} \simeq_S s_0^{(1)}$ for every $k \in \mathbb{N}$; because the elements of σ are pairwise non-associates, we may further assume that $s_0^{(k)} = s_0^{(1)}$ for every $k \in \mathbb{N}$. Now let

$$\gamma = \left(h^{(k)} := \sum_{\ell=0}^{t_k} c_\ell^{(k)} x^{o_\ell^{(k)}} \right)_{k \in \mathbb{N}}$$

such that $f = g^{(k)} h^{(k)}$ for every $k \in \mathbb{N}$. Since the underlying set of the sequence γ is an infinite subset of divisors of f in $S[M]$ that are pairwise non-associates, we can assume without loss of generality that $t_k = t_1 = t$ and $o_\ell^{(k)} \simeq_M o_\ell^{(1)}$ for every $k \in \mathbb{N}$ and every $\ell \in [\![0,t]\!]$. Clearly, we have $c_0^{(k)} = c_0^{(1)}$ for each $k \in \mathbb{N}$. Also, there is no loss in assuming that the equality $m_i^{(k)} + o_j^{(k)} = m_i^{(1)} + o_j^{(1)}$ holds for every $k \in \mathbb{N}$, each $i \in [\![0,r]\!]$, and each $j \in [\![0,t]\!]$. Hence, for each $k \in \mathbb{N}$, there exists $u_k \in \mathscr{U}(M)$ such that

$$\mathsf{Supp}\left(g^{(k)}\right) = \left\{ m + u_k \,\middle|\, m \in \mathsf{Supp}\left(g^{(1)}\right) \right\} \quad \text{and} \quad \mathsf{Supp}\left(h^{(k)}\right) = \left\{ m - u_k \,\middle|\, m \in \mathsf{Supp}\left(h^{(1)}\right) \right\}.$$

Thus $\sigma^* = \left(x^{-u_k} g^{(k)}\right)_{k \in \mathbb{N}}$ is a sequence of pairwise non-associates divisors of f in $S[M]$ with the same support and leading coefficient, namely $\mathsf{Supp}(g^{(1)})$ and $\mathsf{c}(g^{(1)})$, respectively. Let M' be the submonoid of M generated by the set $S' := \mathsf{Supp}(f) \cup \mathsf{Supp}(g^{(1)}) \cup \mathsf{Supp}(h^{(1)})$, and consider the monoid domain $R = F[M']$, where F is a field containing S. Since every nonzero element of the Grothendieck group of a finitely generated torsion-free (cancellative) monoid is of type $(0, 0, \ldots)$, we have that R is an FFD by virtue of [20, Proposition 3.24]. Observe that f, $x^{-u_k} g^{(k)}$, and $x^{u_k} h^{(k)}$ are elements of R for every $k \in \mathbb{N}$, which implies that there exist $i, j \in \mathbb{N}$ with $i \neq j$ such that $g^{(i)} \simeq_R g^{(j)}$. Since $\deg(g^{(i)}) = \deg(g^{(j)})$ and $\mathsf{c}(g^{(i)}) = \mathsf{c}(g^{(j)})$, we obtain that $g^{(i)} = g^{(j)}$, a contradiction. Therefore $S[M]$ is an FFS. □

A monoid M is called a *strong finite factorization monoid* (*SFFM*) if each nonzero element of M has only finitely many divisors (counting associates)[1]. It is easy to see that a monoid M is an SFFM if and only if it is an FFM and $|\mathscr{U}(M)| < \infty$. Following this definition, we say that a semidomain S is a *strong finite factorization semidomain* (*SFFS*) provided that S^* is an SFFM. In Lemma 2.3, we established that, for a semidomain S and a torsion-free monoid M, the inequality $|S[M]^\times| < \infty$ holds if and only if $|S^\times| < \infty$ and $|\mathscr{U}(M)| < \infty$. Consequently, we obtain the following result as an easy corollary of Theorem 4.6.

[1]The notion of strong finite factorization was introduced by Anderson and Mullins in [1].

Corollary 4.7. *Let S be an additively reduced semidomain, and let M be a torsion-free monoid. Then $S[M]$ is an SFFS if and only if S is an SFFS and M is an SFFM.*

In theorems 4.3 and 4.6, the assumption that S is additively reduced is not superfluous as, again, $F[\mathbb{Q}]$ does not satisfy the ACCP for any field F by [14, Theorem 14.17]. We conclude this section providing a large class of finite factorization semidomains, but first let us introduce a definition: a positive semiring P is *well-ordered* if P contains no decreasing sequence.

Proposition 4.8. *Let P be a well-ordered positive semiring. Then P is an FFS.*

Proof. Suppose towards a contradiction that there exists an element $b \in P^*$ such that b has infinitely many non-associates (multiplicative) divisors. Then there exists an increasing sequence $(b_n)_{n \in \mathbb{N}}$ consisting of non-associates divisors of b, which means that the underlying set of the decreasing sequence $(bb_n^{-1})_{n \in \mathbb{N}}$ is a subset of P. This contradicts the fact that P is well-ordered. By [13, Proposition 1.5.5], P is an FFS. □

5. Factoriality Properties

An additively reduced HFS is an FFS. However, the reverse implication does not hold in general. Consider the following example.

Example 5.1. The semidomain $S \coloneqq \mathbb{N}_0[\sqrt{6}]$ is not half-factorial by [7, Theorem 3.1]. On the other hand, it is not hard to verify that if $b' + c'\sqrt{6} \mid_S b + c\sqrt{6}$, where $b + c \neq 0$, then $b' + c' \leq b + c$, which implies that S is an FFS by virtue of [13, Proposition 1.5.5].

Recall that a monoid M is a *length-factorial monoid* (an *LFM*) if for all $b \in M$ and $z, z' \in \mathsf{Z}(b)$, the equality $|z| = |z'|$ implies $z = z'$. We say that a semidomain S is a *length-factorial semidomain* (an *LFS*) if its multiplicative monoid S^* is an LFM. It is evident that every UFS is an LFS. However, it is not clear whether the reverse implication holds (see [17, Question 5.7]). It is known that an integral domain is an LFS if and only if it is a UFS ([12, Corollary 2.11]).

In this section, we prove that an additively reduced monoid semidomain $S[M]$ is not factorial (resp., half-factorial, length-factorial), unless M is the trivial group and S is factorial (resp., half-factorial, length-factorial). We also provide large classes of semidomains with full and infinity elasticity. Throughout this section, we assume that, in an additively reduced monoid semidomain, a polynomial expression is always written in canonical form.

Theorem 5.2. *Let S be an additively reduced semidomain, and let M be a torsion-free monoid. The following statements hold.*

1. *$S[M]$ is a UFS if and only if S is a UFS and M is the trivial group.*
2. *$S[M]$ is an LFS if and only if S is an LFS and M is the trivial group.*
3. *$S[M]$ is an HFS if and only if S is an HFS and M is the trivial group.*

Proof. Let us start by proving statements (1) and (2). The reverse implications of both statements hold trivially. Now assume that $S[M]$ is an LFS. Suppose towards a contradiction that there exists a nonzero $m \in M$, and consider the polynomial expressions

$$f_1 = x^m + 1, \quad f_2 = x^{3m} + 1, \quad f_3 = x^{2m} + x^m + 1, \quad \text{and} \quad f_4 = x^{4m} + x^{2m} + 1$$

in $S[M]$. Since M is torsion-free, we have that f_i and f_j are not associates in $S[M]$ for $i \neq j$. Clearly, the equality $f_1 f_4 = f_2 f_3$ holds. Since f_1 and f_2 are binomials and $f_1(0) = f_2(0) = 1$, the polynomial expressions f_1 and f_2 are irreducibles in $S[M]$. On the other hand, observe that $|\mathsf{L}(f_3)| = 1$. Indeed, either f_3 is irreducible in $S[M]$ or f_3 is a product of two irreducible binomials. Similarly, we have $|\mathsf{L}(f_4)| = 1$. We argue that $\mathsf{L}(f_3) = \mathsf{L}(f_4)$. If we can write f_3 as a product of two irreducible binomials in $S[M]$ then none of the factors is an associate of x^m+1 in $S[M]$ as the reader can easily verify and, in this case, we can also write f_4 as a product of two irreducible binomials in $S[M]$ using a straightforward substitution. Conversely, suppose that

$$(s_1x^{m_1}+s_2x^{m_2})\,(s_3x^{m_3}+s_4x^{m_4}) \in \mathsf{Z}(f_4), \tag{5.1}$$

where $s_1, s_2, s_3, s_4 \in S^*$ and $m_1, m_2, m_3, m_4 \in M$. From Equation (5.1), we obtain that $s_2s_4 = 1$ and $m_2+m_4 = 0$. Consequently, there is no loss in assuming that $s_2 = s_4 = 1$ and $m_2 = m_4 = 0$. This, in turn, implies that $m_1 = m_3 = 2m$. Hence we have $f_4 = (s_1x^{2m}+1)(s_3x^{2m}+1)$, which implies that $f_3 = (s_1x^m+1)(s_3x^m+1)$ for $s_1, s_3 \in S^*$. Observe that neither s_1x^m+1 nor s_3x^m+1 is an associate of x^m+1 in $S[M]$. Thus $\mathsf{L}(f_3) = \mathsf{L}(f_4)$, and we can conclude that the polynomial expression

$$x^{5m}+x^{4m}+x^{3m}+x^{2m}+x^m+1 \in S[M]$$

has two different factorizations of the same length, which contradicts that $S[M]$ is an LFS. Therefore M is the trivial group which, in turn, implies that S is an LFS. If, additionally, the semidomain $S[M]$ is a UFS then S is also a UFS. We can conclude that statements (1) and (2) hold.

To tackle the nontrivial implication of statement (3), suppose that $S[M]$ is an HFS, and assume towards a contradiction that there exists a nonzero $m \in M$. Consider the polynomial expressions

$$f_1 = x^{4m}+x^{2m}+x^m+1, \qquad f_2 = x^{6m}+x^{5m}+x^{3m}+1,$$
$$f_3 = x^m+1, \qquad f_4 = x^{2m}+1, \qquad \text{and} \qquad f_5 = x^{7m}+2x^{4m}+1$$

in $S[M]$. Again, since M is torsion-free, we have that f_i and f_j are not associates in $S[M]$ for $i \neq j$. As the reader can easily check, the equality $f_1f_2 = f_3f_4f_5$ holds. On the other hand, we already established that polynomial expressions similar to f_3 and f_4 are irreducibles in $S[M]$. Next we argue that f_1 and f_2 are also irreducibles in $S[M]$.

Case 1: $f_1 = x^{4m}+x^{2m}+x^m+1$. By way of contradiction, suppose that f_1 reduces in $S[M]$. Since f_1 is not divisible in $S[M]$ by any nonunit monomial, f_1 factors in $S[M]$ either as a binomial times a trinomial, or into two binomials, yielding the following two subcases.

Case 1.1: $f_1 = (s_1x^{m_1}+s_2x^{m_2})(s_3x^{m_3}+s_4x^{m_4}+s_5x^{m_5})$ with coefficients $s_1, s_2, s_3, s_4, s_5 \in S^*$ and exponents $m_1, m_2, m_3, m_4, m_5 \in M$. From this decomposition, we obtain the following equations:

$$m_1+m_3 = 4m, \quad m_2+m_5 = 0, \quad m_2+m_3 = 2m, \quad \text{and} \quad m_1+m_5 = m,$$

which generate the contradiction $4m = 3m$.

Case 1.2: $f_1 = (s_1x^{m_1}+s_2x^{m_2})(s_3x^{m_3}+s_4x^{m_4})$ with coefficients $s_1, s_2, s_3, s_4 \in S^*$ and exponents $m_1, m_2, m_3, m_4 \in M$. From this decomposition, we obtain the following equations:

$$m_1+m_3 = 4m, \quad m_2+m_4 = 0, \quad \text{and} \quad m_1+m_4+m_2+m_3 = 3m,$$

which is evidently a contradiction.

As a consequence, we may conclude that the polynomial expression f_1 is irreducible in $S[M]$. To show that f_2 is irreducible in $S[M]$, we proceed similarly.

Case 2: $f_2 = x^{6m}+x^{5m}+x^{3m}+1$. By way of contradiction, suppose that f_2 reduces in $S[M]$. Since f_2 is not divisible in $S[M]$ by any nonunit monomial, f_2 factors in $S[M]$ either as a binomial times a trinomial, or into two binomials, yielding the following two subcases.

Case 2.1: $f_2 = (s_1x^{m_1}+s_2x^{m_2})(s_3x^{m_3}+s_4x^{m_4}+s_5x^{m_5})$ with coefficients $s_1, s_2, s_3, s_4, s_5 \in S^*$ and exponents $m_1, m_2, m_3, m_4, m_5 \in M$. From this decomposition, we obtain the following equations:

$$m_1+m_3 = 6m, \quad m_2+m_5 = 0, \quad m_2+m_3 = 5m, \quad \text{and} \quad m_1+m_5 = 3m,$$

which generate the contradiction $6m = 8m$.

CASE 2.2: $f_2 = (s_1x^{m_1} + s_2x^{m_2})(s_3x^{m_3} + s_4x^{m_4})$ with coefficients $s_1, s_2, s_3, s_4 \in S^*$ and exponents $m_1, m_2, m_3, m_4 \in M$. From this decomposition, we obtain the following equations:

$$m_1 + m_3 = 6m, \quad m_2 + m_4 = 0, \quad \text{and} \quad m_1 + m_4 + m_2 + m_3 = 8m,$$

which is evidently a contradiction.

Since the polynomial expressions f_1 and f_2 are irreducibles in $S[M]$, the element

$$x^{10m} + x^{9m} + x^{8m} + 3x^{7m} + 2x^{6m} + 2x^{5m} + 2x^{4m} + x^{3m} + x^{2m} + x^m + 1 \in S[M]$$

has a factorization of length 2 (i.e., f_1f_2) and a factorization of length at least 3 (i.e., $f_3f_4f_5$). Consequently, the semidomain $S[M]$ is not an HFS. This contradiction proves that M is the trivial group which, in turn, implies that S is an HFS. □

Based on Theorem 5.2, one might think that an additively reduced HFS is a UFS, but this is not the case.

Example 5.3. Let $D = \mathbb{Z}[M]$, where $M = \langle (1, n) \mid n \in \mathbb{N}\rangle \subseteq \mathbb{N}_0^2$. Clearly, the (cancellative and commutative) monoid M is torsion-free, which implies that D is an integral domain by [14, Theorem 8.1]. Let $S = \{f \in \mathbb{N}_0[M] \mid f(0) > 0\}$. Since S is a multiplicatively closed subset of D containing 1, we can consider the localization of D at S, which we denote by $S^{-1}D$. Set $R = (\mathbb{N}_0[M] \times S)/\sim$, where $\sim$ is the equivalence relation on $\mathbb{N}_0[M] \times S$ defined by $(f, g) \sim (f', g')$ if and only if $fg' = gf'$ for (f, g) and (f', g') in $\mathbb{N}_0[M] \times S$. We let f/g denote the equivalence class of (f, g). Define the following operations in R:

$$\frac{f}{g} \cdot \frac{f'}{g'} = \frac{ff'}{gg'} \quad \text{and} \quad \frac{f}{g} + \frac{f'}{g'} = \frac{fg' + gf'}{gg'}.$$

It is routine to verify that these operations are well defined and that $(R, +, \cdot)$ is an additively reduced semiring[2]. Let $\varphi\colon R \to S^{-1}D$ be a function given by $\varphi(f/g) = \overline{f/g}$, where $\overline{f/g}$ represents the equivalence class of (f, g) as an element of $S^{-1}D$. It is easy to see that φ is a well-defined semiring homomorphism. Since φ is injective, R is an additively reduced semidomain.

It is known that M is an HFM that is not a UFM (see [9, Example 4.23]). Consequently, the semidomain R is not a UFS. Next we show that R is an HFS. Since all the elements of M that are not atoms are divisible by $(1, 1)$ and $x^{(1,1)}/1$ is irreducible in R, the semidomain R is atomic. Now let f/g be a nonzero nonunit element of R. Since $f/g \simeq_R f$, there is no loss in assuming that $g = 1$. Write $f = c_kx^{m_k} + \cdots + c_1x^{m_1}$, where $m_k > \cdots > m_1 > (0, 0)$ in the lexicographic order. Let

$$z = \left(\frac{f_1}{g_1}\right) \cdots \left(\frac{f_n}{g_n}\right) \quad \text{and} \quad z' = \left(\frac{f_1'}{g_1'}\right) \cdots \left(\frac{f_m'}{g_m'}\right)$$

be two different factorizations of $f/1$ in R, and suppose towards a contradiction that $n \neq m$. Observe that if f'/g' is an atom of R then writing $f' = d_lx^{o_l} + \cdots + d_1x^{o_1}$ with $o_l > \cdots > o_1 > (0, 0)$ in the lexicographic order, we have that $o_1 \in \mathscr{A}(M)$ because all non-atoms of M are divisible by $(1, 1)$. Consequently, the element $m_1 \in M$ has two factorizations of lengths n and m, which contradicts that M is an HFM. Therefore R is an additively reduced HFS that is not a UFS.

For an atomic monoid M, the *elasticity* of a nonunit $x \in M$, denoted by $\rho(x)$, is defined as

$$\rho(x) = \frac{\sup \mathsf{L}(x)}{\inf \mathsf{L}(x)}.$$

By convention, we set $\rho(u) = 1$ for every $u \in \mathscr{U}(M)$. It is easy to see that, for all $x \in M$, we have that $\rho(x) \in \mathbb{Q}_{\geq 1} \cup \{\infty\}$. The *elasticity* of the monoid M is defined to be

$$\rho(M) := \sup\{\rho(x) \mid x \in M\}.$$

[2]The localization of semirings is presented in greater generality in [15, Chapter 11].

The *set of elasticities* of M is denoted by $R(M) := \{\rho(x) \mid x \in M\}$, and M is said to have *full elasticity* provided that $R(M) = (\mathbb{Q} \cup \{\infty\}) \cap [1, \rho(M)]$. Observe that a monoid M is an HFM if and only if $\rho(M) = 1$ (resp., $|R(M)| = 1$). So, we can think of monoids having full and infinite elasticity as being as far as they can possibly be from being an HFM. In fact, the elasticity was first studied by Steffan [24] and Valenza [25] with the purpose of measuring the deviation of an atomic monoid from being half-factorial.

Next we show that an atomic monoid semidomain $S[M]$ has full and infinite elasticity provided that $(S, +)$ is reduced, M is nontrivial and torsion-free, and $\mathscr{F}(S)[M]$ is a UFD, where $\mathscr{F}(S)$ denotes the quotient field of $\mathscr{G}(S)$. This generalizes [17, Proposition 5.7] in which the authors proved that a semidomain $S[x]$ has full and infinite elasticity if S is additively reduced. For the rest of the section, we identify a semidomain S with a subsemiring of the integral domain $\mathscr{G}(S)$ (resp., $\mathscr{F}(S)$) (see Lemma 2.2).

Proposition 5.4. *An atomic monoid semidomain $S[M]$ has full and infinite elasticity provided that $(S, +)$ is reduced, M is nontrivial and torsion-free, and $\mathscr{F}(S)[M]$ is a UFD.*

Proof. First we show that there exists a nonzero element $a \in M$ such that $x^a + b$ is irreducible in $\mathscr{F}(S)[M]$ for every nonzero $b \in \mathscr{F}(S)$. Since $\mathscr{F}(S)[M]$ is a UFD, the monoid M is factorial and each nonzero element of the group $\mathscr{U}(M)$ of invertible elements of M is of type $(0, 0, \ldots)$ by [14, Theorem 14.16]. If M is reduced then it is not hard to see that, for $a \in \mathscr{A}(M)$, the polynomial expression $x^a + b$ is irreducible in $\mathscr{F}(S)[M]$ for any nonzero $b \in \mathscr{F}(S)$. On the other hand, if M is not reduced then there is no loss in assuming that $M = \mathscr{U}(M)$. To see why our previous assumption is valid, observe that $S[\mathscr{U}(M)]^*$ is a divisor-closed submonoid of $S[M]^*$. Indeed, if there exist polynomial expressions $f = s_1 x^{h_1} + \cdots + s_n x^{h_n} \in S[\mathscr{U}(M)]^*$ and $g = s'_1 x^{m_1} + \cdots + s'_k x^{m_k} \in S[M]^*$ such that $g \mid_{S[M]} f$ then, since S is additively reduced, for each $i \in [\![1, k]\!]$ there exists $j \in [\![1, n]\!]$ such that $m_i \mid_M h_j$, but $\mathscr{U}(M)$ is a divisor-closed submonoid of M; consequently, we have that $g \in S[\mathscr{U}(M)]^*$. Now since $S[\mathscr{U}(M)]^*$ is a divisor-closed submonoid of $S[M]^*$, the semidomain $S[\mathscr{U}(M)]$ is atomic and if $S[\mathscr{U}(M)]$ has full and infinite elasticity then $S[M]$ has full and infinite elasticity too. Consequently, we may assume that M is a group satisfying that all of its nonzero elements are of type $(0, 0, \ldots)$. This, in turn, implies that there exists a nonzero $a \in M$ of height $(0, 0, \ldots)$. By virtue of [21, Lemma 4.1], the polynomial expression $x^a + b$ is irreducible in $\mathscr{F}(S)[M]$ for every nonzero $b \in \mathscr{F}(S)$.

For every field $\mathscr{F}$ containing $\mathscr{F}(S)$, the integral domain $\mathscr{F}[M]$ is a UFD by [14, Theorem 14.16]. Consequently, there is no loss in assuming that $\mathscr{F}(S)$ is algebraically closed. Consider now the polynomial expression $f = x^{2a} - x^a + 1$, where a is a nonzero element of M such that $x^a + b$ is irreducible in $\mathscr{F}(S)[M]$ for every nonzero $b \in \mathscr{F}(S)$. Observe that f reduces in $\mathscr{F}(S)[M]$. In fact, $f = (x^a + \alpha)(x^a + \beta)$ for some nonzero $\alpha, \beta \in \mathscr{F}(S)$ satisfying that $\alpha\beta = 1$ and $\alpha + \beta = -1$. We already established that $x^a + \alpha$ and $x^a + \beta$ are irreducibles (in fact, primes) in $\mathscr{F}(S)[M]$. Note that either $n\alpha \notin S$ for any $n \in \mathbb{N}$ or $n\beta \notin S$ for any $n \in \mathbb{N}$. Indeed, if $k\alpha$ and $t\beta$ are in S for some $k, t \in \mathbb{N}$ then we have

$$-(tk) = tk(\alpha + \beta) = t(k\alpha) + k(t\beta) \in S,$$

which contradicts that $(S, +)$ is reduced. Without loss of generality, assume that $n\alpha \notin S$ for any $n \in \mathbb{N}$. We now claim that the polynomial expression $(x^a+n)^n(x^{2a}-x^a+1)$ is irreducible in $S[M]$ for every $n \in \mathbb{N}$. It follows from [7, Lemma 2.1] that, for every $n, m \in \mathbb{N}$, the polynomial $(y+n)^m(y^2-y+1) \in \mathbb{N}_0[y]$ if and only if $m \geq n$. By a straightforward substitution, we obtain that, for every $n, m \in \mathbb{N}$, the polynomial expression $(x^a + n)^m(x^{2a} - x^a + 1) \in S[M]$ if and only if $m \geq n$. Since $S[M]$ is atomic, we can write

$$(x^a + n)^n(x^{2a} - x^a + 1) = (x^a + n)^n(x^a + \alpha)(x^a + \beta) = f_1 \cdots f_k,$$

where $k \in \mathbb{N}$ and $f_1, \ldots, f_k$ are irreducibles in $S[M]$. Since $n\alpha \notin S$ for any $n \in \mathbb{N}$, if $x^a + \alpha \mid_{\mathscr{F}(S)[M]} f_j$ for some $j \in [\![1, k]\!]$ then $x^a + \beta \mid_{\mathscr{F}(S)[M]} f_j$. Hence if $k \geq 2$ then, for some $j \in [\![1, k]\!]$, we have that $f_j = (x^a+n)^l(x^{2a}-x^a+1)$ for some $0 \leq l < n$, but we already showed that this is impossible. Therefore $(x^a+n)^n(x^{2a}-x^a+1)$ is irreducible in $S[M]$ for every $n \in \mathbb{N}$. Clearly, x^a+1 and $x^{3a}+1$ are irreducibles in $S[M]$. For $n, k \in \mathbb{N}$, consider the polynomial expression

$$g = (x^a + n)^n (x^{2a} - x^a + 1)(x^a + 1)^k \in S[M].$$

Since $\mathscr{F}(S)[M]$ is a UFD and $x^a + b$ is irreducible in $\mathscr{F}(S)[M]$ for every nonzero $b \in \mathscr{F}(S)$, we have that the only two factorizations of g in $S[M]$ are

$$[(x^a + n)^n (x^{2a} - x^a + 1)] \cdot [x^a + 1]^k \quad \text{and} \quad [x^a + n]^n \cdot [(x^{2a} - x^a + 1)(x^a + 1)] \cdot [x^a + 1]^{k-1}$$

with lengths $k + 1$ and $k + n$, respectively. Since $\{(k + n)/(k + 1) \mid k, n \in \mathbb{N}\} = \mathbb{Q}_{\geq 1}$, we conclude that $S[M]$ has full and infinite elasticity. $\square$

The reverse implication of Proposition 5.4 does not hold as the following example illustrates.

Example 5.5. Let $M = \langle (3/2)^n \mid n \in \mathbb{N}_0 \rangle \subseteq (\mathbb{Q}_{\geq 0}, +)$, and consider the monoid semidomain $\mathbb{N}_0[M]$. Since M is an FFM ([16, Theorem 5.6]), the semidomain $\mathbb{N}_0[M]$ is atomic (in fact, an FFS) by Theorem 4.6. It was proved in [10, Proposition 4.4] that M has full and infinite elasticity, which implies that $\mathbb{N}_0[M]$ has full and infinite elasticity too. However, since M is not a UFM, the domain $F[M]$ is not a UFD for any field F containing $\mathbb{N}_0$ ([14, Theorem 14.7]).

We conclude this section using Proposition 5.4 to provide two large classes of additively reduced monoid semidomains with full and infinite elasticity.

Corollary 5.6. *Let $S[M]$ be an atomic monoid semidomain such that S is additively reduced and M is nontrivial and torsion-free. The following statements hold.*

(1) If M is a reduced UFM then $S[M]$ has full and infinite elasticity.

(2) If M is not reduced and every nonzero element of $\mathscr{U}(M)$ is of type $(0, 0, \dots)$ then $S[M]$ has full and infinite elasticity.

Proof. Observe that if M is a reduced UFM then $\mathscr{F}(S)[M]$ is a UFD by [14, Theorem 14.16]. Consequently, the statement (1) follows from Proposition 5.4. Now suppose that M is not reduced and that every nonzero element of $\mathscr{U}(M)$ is of type $(0, 0, \dots)$. We already established that $S[\mathscr{U}(M)]^*$ is a divisor-closed submonoid of $S[M]^*$, which implies that $S[\mathscr{U}(M)]$ is atomic. By [14, Theorem 14.15], the integral domain $\mathscr{F}(S)[\mathscr{U}(M)]$ is a UFD. Then $S[\mathscr{U}(M)]$ has full and infinite elasticity by virtue of Proposition 5.4, which concludes our argument. $\square$

Corollary 5.7. *An atomic polynomial semidomain $S[x]$ (resp., $S[x, x^{-1}]$) has full and infinite elasticity provided that $(S, +)$ is reduced.*

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