ON THE ARITHMETIC OF POLYNOMIAL SEMIDOMAINS

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ABSTRACT. A subset S of an integral domain R is called a semidomain provided that the pairs $(S, +)$ and $(S, \cdot)$ are semigroups with identities. The study of factorizations in integral domains was initiated by D. D. Anderson, D. F. Anderson, and M. Zafrullah in 1990, and this area has been systematically investigated since then. In this paper, we study the divisibility and arithmetic of factorizations in the more general context of semidomains. We are specially concerned with the ascent of the most standard divisibility and factorization properties from a semidomain to its semidomain of (Laurent) polynomials. As in the case of integral domains, here we prove that the properties of satisfying ACCP, having bounded factorizations, and having finite factorizations ascend in the class of semidomains. We also consider the ascent of the property of being atomic, and we show that the property of having unique factorization ascends only when a semidomain is an integral domain. Throughout the paper we provide several examples aiming to shed some light upon the arithmetic of factorizations of semidomains.

1. INTRODUCTION

A subset of an integral domain containing 0 and 1 and closed under both addition and multiplication is called a semidomain. As for integral domains, we say that a semidomain is atomic if every nonzero element that is not a multiplicative unit factors into irreducibles. The first systematic study of factorizations in the context of integral domains was carried out by D. D. Anderson, D. F. Anderson, and M. Zafrullah in [2], where they not only introduced and studied the bounded and the finite factorization properties but also investigated further factorization properties, including being atomic, satisfying ACCP, and being factorial or half-factorial. With the same properties in mind, here we study the arithmetic of the more general class of semidomains, putting special emphasis on whether such properties ascend from a semidomain to its semidomains of (Laurent) polynomials.

A commutative semiring S is a nonempty set endowed with two compatible binary operations denoted by '$+$' and '$\cdot$' such that $(S, +)$ and $(S, \cdot)$ are commutative semigroups with identities. Commutative semirings consisting of nonnegative real numbers (under the standard addition and multiplication) are called positive semirings. Clearly, every positive semiring is a semidomain. The atomicity of positive semirings consisting of rational numbers was first considered in [1,14]. Positive semirings were also studied by N. R. Baeth and the first author [7] in connection with factorizations of matrices. Several examples of positive semirings were recently given in [6], where for the first time additive and multiplicative factorizations in positive semirings were considered simultaneously.

The algebraic structures of central interest in this paper are semidomains of polynomials and semidomains of Laurent polynomials. The arithmetic of certain semidomains of (Laurent) polynomials has been considered in the literature in the past few years. P. Cesarz et al. in [12] studied the elasticity of $\mathbb{R}_{\geq 0}[X]$, where $\mathbb{R}_{\geq 0}$ is the nonnegative cone of $\mathbb{R}$. In addition, methods to factorize polynomials in $\mathbb{N}_0[X]$ were investigated by H. Brunotte in [10]. More recently, F. Campanini and A. Facchini in [11]

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provided a systematic investigation of factorizations in $\mathbb{N}_0[X]$. More generally, semigroup semirings from the factorization point of view were studied by Ponomarenko in [37].

The positive semirings we obtain as homomorphic images of semidomains of polynomials and semidomains of Laurent polynomials have also been investigated recently. The additive structure of $\mathbb{N}_0[\alpha]$, where α is a positive algebraic number, was studied in [14] by Chapman et al. and then in [16] by J. Correa-Morris and the first author. More recently, the elasticity and the omega-primality of $\mathbb{N}_0[\alpha]$ have been considered in [36]. Also, the atomicity of the multiplicative structure of $\mathbb{N}_0[\alpha]$ was studied in [12], where it is shown that $\mathbb{N}_0[\alpha]$ has full infinite elasticity for reasonable quadratic algebraic integers α. On the other hand, S. Zhu has recently studied several factorization aspects of the homomorphic image $\mathbb{N}_0[\alpha^{\pm 1}]$ of the semidomain of Laurent polynomials $\mathbb{N}_0[X^{\pm 1}]$, where α is a positive algebraic number.

We revise the definitions and terminology relevant to this paper in Section 2. Our first results are presented in Section 3, where we study atomicity in connection with ACCP. It was proved by M. Roitman [39, Proposition 1.1] that atomicity ascends from any integral domain R to the ring of polynomials $R[x]$ provided that every finite subset of R has a maximal common divisor. In Section 3 we extend and generalize this result, proving that if an atomic semidomain has maximal common divisors for finite subsets, then its semidomain of (Laurent) polynomial is also atomic. In [32], A. Grams constructed a celebrated example of an atomic domain that does not satisfy ACCP. Further (nontrivial) examples have been given in [39,43] and more recently in [9,30]. Here we construct a positive semiring (which cannot be an integral domain) that is atomic but does not satisfy ACCP. On the other hand, we prove that ACCP ascends from any semidomain to its semidomain of (Laurent) polynomials.

In Section 4, we consider the bounded and the finite factorization properties. Let S be a semidomain. We say that S is a bounded factorization semidomain (BFS) if there is a function $\ell \colon S \setminus \{0\} \to \mathbb{N}_0$ that is zero on units and satisfies $\ell(rs) \geq \ell(r)\ell(s)$ for all $r, s \in S \setminus \{0\}$. On the other hand, we say that S is a finite factorization semidomain (FFS) if every nonzero element has finitely many divisors up to associates. Both notions are extensions of the corresponding notions introduced in [2] for integral domains. It is well known that both the bounded and the finite factorization properties ascend from an integral domain to its ring of (Laurent) polynomials (see [2, Propositions 2.5 and 5.3] and [3, Corollary 2.2]). We extend these facts by proving that the same properties ascend from any semidomain to its semidomain of (Laurent) polynomials.

In Section 5, we study factorial and length-factorial semidomains. As for integral domains, we say that a semidomain S is a factorial semidomain or a unique factorization semidomain (UFS) if every nonzero element has a unique factorization into irreducibles. On the other hand, following the terminology in [13], we say that S is a length-factorial semidomain (LFS) if S is atomic and any two distinct factorizations of the same element of S have distinct numbers of irreducible factors (counting repetitions). Length-factoriality was first studied in [18] and more recently in [13,24,29]. We prove that the only semidomains where the unique factorization and the length-factorial properties ascend to their corresponding (Laurent) polynomial semidomains are integral domains. We conclude the paper considering the elasticity of semidomains of polynomials.

2. Preliminaries

In this section, we introduce the notation and terminology we shall be using later. For a comprehensive background on factorization theory and semiring theory, the reader can consult [22] and [28], respectively. Following standard notation, we let $\mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ denote the sets of integers, rational numbers, and real numbers, respectively. Additionally, we let $\mathbb{N}$ denote the set of positive integers, and we set $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Given $r \in \mathbb{R}$ and $S \subseteq \mathbb{R}$, we set $S_{<r} \coloneqq \{s \in S \mid s < r\}$; we define $S_{>r}$ and $S_{\geq r}$ in a similar way. For $m, n \in \mathbb{N}_0$, we denote by $[\![m, n]\!]$ the discrete interval from m to n; that is, $[\![m, n]\!] \coloneqq \{k \in \mathbb{Z} \mid m \leq k \leq n\}$.

2.1. **Monoids and Factorizations.** Throughout this paper, a *monoid*[1] is defined to be a semigroup with identity that is cancellative and commutative. As we are primarily interested in the multiplicative structure of certain semirings, unless otherwise specified we will use multiplicative notation for monoids. Let M be a monoid with identity 1. We set $M^\bullet := M \setminus \{1\}$, and we let $\mathscr{U}(M)$ denote the group of units (i.e., invertible elements) of M. In addition, we let M_{red} denote the quotient $M/\mathscr{U}(M)$, which is also a monoid. The monoid M is *reduced* provided that $\mathscr{U}(M)$ is the trivial group, in which case we naturally identify M_{red} with M. The *Grothendieck group* of M, denoted here by $\mathscr{G}(M)$, is the abelian group (unique up to isomorphism) satisfying that any abelian group containing a homomorphic image of M also contains a homomorphic image of $\mathscr{G}(M)$. For a subset S of M, we let $\langle S \rangle$ denote the smallest submonoid of M containing S, and S is a *generating set* of M provided that $M = \langle S \rangle$.

For $b, c \in M$, we say that b *divides* c *in* M if there exists $b' \in M$ such that $c = bb'$, in which case we write $b \mid_M c$, dropping the subscript precisely when $M = (\mathbb{N}, \times)$. Two elements $b, c \in M$ are *associates* if $b \mid_M c$ and $c \mid_M b$. A submonoid N of M is *divisor-closed* if for each $b \in N$ and $d \in M$ the relation $d \mid_M b$ implies that $d \in N$. Let S be a nonempty subset of M. An element $d \in M$ is a *common divisor* of S provided that $d \mid_M s$ for all $s \in S$. A common divisor d of S is a *greatest common divisor* of S if d is divisible by all common divisors of S. Also, a common divisor of S is a *maximal common divisor* if every greatest common divisor of S/d belongs to $\mathscr{U}(M)$. We let $\gcd_M(S)$ (resp., $\text{mcd}_M(S)$) denote the set consisting of all greatest common divisors (resp., maximal common divisors) of S. The monoid M is a *GCD-monoid* (resp., an *MCD-monoid*) provided that every finite nonempty set of elements in M has a greatest common divisor (resp., maximal common divisor).

An element $a \in M \setminus \mathscr{U}(M)$ is an *atom* if for all $b, c \in M$ the equality $a = bc$ implies that either $b \in \mathscr{U}(M)$ or $c \in \mathscr{U}(M)$. We let $\mathscr{A}(M)$ denote the set of all atoms of M. The monoid M is *atomic* if each element in $M \setminus \mathscr{U}(M)$ can be written as a (finite) product of atoms. One can readily check that M is atomic if and only if M_{red} is atomic. A subset I of M is an *ideal* of M provided that $IM \subseteq I$ or, equivalently, $IM = I$. An ideal I of M is *principal* if $I = bM$ for some $b \in M$. The monoid M satisfies the *ascending chain condition on principal ideals* (*ACCP*) if every increasing sequence of principal ideals of M (under inclusion) eventually stabilizes. It is well known and not hard to verify that monoids satisfying ACCP are atomic.

Assume now that M is atomic. We let $\mathsf{Z}(M)$ denote the free (commutative) monoid on $\mathscr{A}(M_{\text{red}})$. The elements of $\mathsf{Z}(M)$ are *factorizations*, and if $z = a_1 \cdots a_\ell \in \mathsf{Z}(M)$ for $a_1, \dots, a_\ell \in \mathscr{A}(M_{\text{red}})$, then ℓ is the *length* of z, which is denoted by $|z|$. Let $\pi \colon \mathsf{Z}(M) \to M_{\text{red}}$ be the unique monoid homomorphism satisfying that $\pi(a) = a$ for all $a \in \mathscr{A}(M_{\text{red}})$. For each $b \in M$, the following sets associated to b are fundamental in the study of factorization theory:

$$\mathsf{Z}_M(b) := \pi^{-1}(b\mathscr{U}(M)) \subseteq \mathsf{Z}(M) \qquad \text{and} \qquad \mathsf{L}_M(b) := \{|z| : z \in \mathsf{Z}_M(b)\} \subseteq \mathbb{N}_0. \tag{2.1}$$

We drop the subscript M in (2.1) whenever the monoid is clear from the context. Following [2] and [33], we say that M is a *finite factorization monoid* (*FFM*) if $\mathsf{Z}(b)$ is finite for all $b \in M$, and we say that M is a *bounded factorization monoid* (*BFM*) if $\mathsf{L}(b)$ is finite for all $b \in M$. It is clear that every FFM is a BFM and, by virtue of [22, Corollary 1.3.3], every BFM satisfies ACCP. Following [44], we say that M is a *half-factorial monoid* (*HFM*) if $|\mathsf{L}(b)| = 1$ for all $b \in M$. Finally, M is a *unique factorization monoid* (*UFM*) if $|\mathsf{Z}(b)| = 1$ for all $b \in M$. It is clear that every UFM is an HFM and also that every HFM is a BFM. The atomic classes defined in this paragraph can be fitted in the following diagram, introduced by Anderson, Anderson, and Zafrullah in [2] and since then used as a methodology to study atomicity and the phenomenon of multiple factorizations. Finally, we follow the terminology in [13] and say that M is a *length-factorial monoid* (*LFM*) if for all $b \in M$ and $z, z' \in \mathsf{Z}(b)$, the equality $|z| = |z'|$ implies $z = z'$. It is clear that every UFM is an LFM.

[1] The standard definition of a monoid does not assume the cancellative and the commutative conditions.

$$\begin{array}{ccccccc}
\textbf{UFM} & \overset{\not\Longleftarrow}{\Longrightarrow} & \textbf{HFM} & & & & \\
\Downarrow \not\Uparrow & & \Downarrow \not\Uparrow & & & & \\
\textbf{FFM} & \overset{\not\Longleftarrow}{\Longrightarrow} & \textbf{BFM} & \overset{\not\Longleftarrow}{\Longrightarrow} & \textbf{ACCP monoid} & \overset{\not\Longleftarrow}{\Longrightarrow} & \textbf{atomic monoid}
\end{array}$$

FIGURE 1. The implications in the diagram show the general inclusions among the subclasses of atomic monoids we have previously mentioned. The diagram also emphasizes (with red marked arrows) that no further inclusion between any two of these classes holds in general.

2.2. **Semirings.** A *commutative semiring* S is a nonempty set endowed with two binary operations denoted by '$+$' and '$\cdot$' and called *addition* and *multiplication*, respectively, such that the following conditions hold:

- $(S, +)$ is a monoid with its identity element denoted by 0;
- $(S, \cdot)$ is a commutative semigroup with an identity element denoted by 1;
- $b \cdot (c + d) = b \cdot c + b \cdot d$ for all $b, c, d \in S$;
- $0 \cdot b = 0$ for all $b \in S$.

With notation as in the previous definition and for any $b, c \in S$, we write bc instead of $b \cdot c$ when there seems to be no risk of confusion. A more general notion of a 'semiring' S does not assume that the semigroup $(S, \cdot)$ is commutative. However, this more general type of algebraic objects are not of interest in the scope of this paper. Accordingly, from now on we will use the single term *semiring*, tacitly assuming the commutativity of both operations. A subset S' of a semiring S is a *subsemiring* of S if $(S', +)$ is a submonoid of $(S, +)$ that contains 1 and is closed under multiplication. Observe that every subsemiring of S is a semiring.

Definition 2.1. We say that a semiring S is a *semidomain* provided that S is a subsemiring of an integral domain.

Let S be a semidomain. Then $(S \setminus \{0\}, \cdot)$ is a monoid, which we denote by S^* and call the *multiplicative monoid* of S. In Example 2.3, we show a semiring S that is not a semidomain but still $(S \setminus \{0\}, \cdot)$ is a monoid. In order to reuse notation from ring theory, we refer to the units of $(S, +)$ as invertible elements, so that we can refer to the units of the multiplicative monoid S^* simply as *units* of S without the risk of ambiguity. Also, following standard notation from ring theory, we let $S^\times$ denote the group of units of S, letting $\mathscr{U}(S)$ refer to the additive group of invertible elements of S. In addition, we write $\mathscr{A}(S)$ instead of $\mathscr{A}(S^*)$ for the set of atoms of the multiplicative monoid S^* (we do not consider in this paper the set of atoms of the additive monoid of a semidomain, except briefly in Proposition 3.6). Finally, for $b, c \in S$ such that b divides c in S^*, we write $b \mid_S c$ (instead of $b \mid_{S^*} c$).

For the next example, we need the following lemma.

Lemma 2.2. *For a semiring S, the following conditions are equivalent.*

(a) *S is a semidomain.*

(b) *The multiplication of S extends to $\mathscr{G}(S)$ turning $\mathscr{G}(S)$ into an integral domain.*

Proof. (b) $\Rightarrow$ (a): This is clear.

(a) $\Rightarrow$ (b): Let S be a semidomain, and suppose that S is embedded into an integral domain R. We can identify the Grothendieck group $\mathscr{G}(S)$ of $(S, +)$ with the subgroup $\{r - s \mid r, s \in S\}$ of the underlying additive group of R. It is easy to see then that $\mathscr{G}(S)$ is closed under the multiplication it inherits from R, and it contains the multiplicative identity because $0, 1 \in S$. Hence $\mathscr{G}(S)$ is an integral domain having S as a subsemiring. $\square$

Example 2.3. Notice that the set $S := \{(0,0)\} \cup (\mathbb{N} \times \mathbb{N})$ is a monoid with the usual component-wise addition, and it is closed under the usual component-wise multiplication with multiplication identity $(1,1)$. Hence S is a semiring. Observe, on the other hand, that any extension of the multiplication of S to $\mathscr{G}(S)$ making the latter a commutative ring must respect the identity $(1,0)(0,1) = (0,0)$ and, therefore, will not turn $\mathscr{G}(S)$ into an integral domain. Hence it follows from Lemma 2.2 that S is not a semidomain.

We say that a semidomain S is *atomic* (resp., satisfies *ACCP*) if its multiplicative monoid S^* is atomic (resp., satisfies ACCP). In addition, we say that S is a *BFS*, *FFS*, *HFS*, or *UFS* provided that S^* is a BFM, FFM, HFM, or UFM, respectively. Note that when S is an integral domain, we recover the usual definition of a UFD as well as the definitions of a BFD, an FFD, and an HFD, which are now standard notions in atomicity and factorization theory. Although a semidomain S can be embedded into an integral domain R, the semiring S may not inherit any atomic property from R as, after all, the integral domain $\mathbb{Q}[x]$ is a UFD but it contains as a subring the integral domain $\mathbb{Z} + x\mathbb{Q}[x]$, which is not even atomic.

The set consisting of all polynomials with coefficients in a semiring S is also a semiring, which we denote by $S[x]$ and call the *semiring of polynomials over S*. In addition, if S is a semidomain embedded into an integral domain R, then it is clear that $S[x]$ is also a semidomain, and the elements of $S[x]$ are, in particular, polynomials in $R[x]$. As a result, when S is a semidomain all the standard terminology for polynomials can be applied to elements of $S[x]$, including *degree*, *order*, *leading coefficient*, etc. Observe that S^* is a divisor-closed submonoid of $S[x]^*$ and, therefore, $S[x]^\times = S^\times$ and $\mathscr{A}(S[x]) \cap S = \mathscr{A}(S)$. Following [39], we say that a nonzero polynomial in $S[x]$ is *indecomposable* if it cannot be written as a product of two non-constant polynomials in $S[x]$. Similarly, when S is a semidomain, the semiring of Laurent polynomials with coefficients in S is also a semidomain, which we denote by $S[x^{\pm 1}]$ and call the *semiring of Laurent polynomials*. Note that $\{sx^n \mid s \in S^* \text{ and } n \in \mathbb{Z}\}$ is a divisor-closed submonoid of $S[x^{\pm 1}]^*$, and so $S[x^{\pm 1}]^\times = \{sx^n \mid s \in S^\times \text{ and } n \in \mathbb{Z}\}$.

Following the terminology in [6], we call a subsemiring of $\mathbb{R}$ (under the standard addition and multiplication) consisting of nonnegative numbers a *positive semiring*. The fact that underlying additive monoids of positive semirings are reduced makes them more tractable. The reader can check the recent paper [6] for several examples of positive semirings. The class of semidomains clearly contains those of integral domains and positive semirings. As the following example illustrates, positive semirings and integral domains account for all semidomains that can be embedded into $\mathbb{Q}$.

Example 2.4. Suppose that S is a semidomain that is a subsemiring of $\mathbb{Q}$, and assume that S is not a positive semiring. Since S is not a positive semiring, it must contain a negative rational. By virtue of [25, Theorem 2.9] any additive submonoid of $\mathbb{Q}$ containing both a negative and a positive rationals must be a subgroup of $\mathbb{Q}$. As a result, the additive monoid of S is a subgroup of $\mathbb{Q}$. Thus, S is a subring of $\mathbb{Q}$, and so an integral domain.

In general, there are semidomains that are neither positive semirings nor integral domains.

Example 2.5. Consider the semidomain $\mathbb{N}_0[\alpha, \beta, -\beta]$, where $\alpha, \beta \in \mathbb{R}_{>0}$ are algebraically independent over $\mathbb{Q}$. Clearly, the additive monoid $\mathbb{N}_0[\alpha, \beta, -\beta]$ is not reduced. On the other hand, since α and β are algebraically independent over $\mathbb{Q}$, we see that $-\alpha \notin \mathbb{N}_0[\alpha, \beta, -\beta]$ and, therefore, $(\mathbb{N}_0[\alpha, \beta, -\beta], +)$ is not a group.

3. Atomicity and the ACCP

In this section, we investigate under which conditions the property of being atomic and that of satisfying ACCP ascend from a semidomain to its semidomain of (Laurent) polynomials. For an integral domain R, it is well known that the property of being ACCP ascends to both $R[x]$ and $R[x^{\pm 1}]$. However, this is not the case for the property of being atomic. Indeed, Roitman constructed in [39] examples of atomic domains whose polynomial rings are not atomic (in the same vein, examples of atomic monoids with non-atomic monoid algebras were constructed in [17]). The following result generalizes [39, Proposition 1.1] to the context of semidomains.

Theorem 3.1. *For a semidomain S, the following statements are equivalent.*

(a) S is atomic and $\operatorname{mcd}(s_1, \dots, s_n) \neq \emptyset$ for any coefficients $s_1, \dots, s_n$ of an indecomposable polynomial in $S[x]$.

(b) $S[x]$ is atomic.

(c) $S[x^{\pm 1}]$ is atomic.

Proof. (a) $\Rightarrow$ (b): Assume that $S[x]$ is not atomic. Let f be a minimum-degree nonunit polynomial in $S[x]^*$ that does not factor into irreducibles. Then f must be indecomposable and, as S is atomic, $\deg f \geq 1$. Write $f = cg$, where c is a maximal common divisor of the set of coefficients of f. As g is indecomposable, it is irreducible in $S[x]$. Hence c is a nonunit of S that cannot factor into irreducibles in S. Hence S is not atomic.

(b) $\Rightarrow$ (c): We claim that $\mathscr{A}(S[x]) \subseteq \mathscr{A}(S[x^{\pm 1}]) \cup S[x^{\pm 1}]^\times$. To argue this, take $f \in \mathscr{A}(S[x])$, and assume that $f \notin S[x^{\pm 1}]^\times = \{ux^n \mid u \in S^\times \text{ and } n \in \mathbb{Z}\}$. Now write $f = gh$ for some $g, h \in S[x^{\pm 1}]$. After replacing g and h by some of their associates in $S[x^{\pm 1}]$, we can assume that $g, h \in S[x]$. Since $f \in \mathscr{A}(S[x])$, either g or h belongs to $S[x]^\times \subseteq S[x^{\pm 1}]^\times$, and so $f \in \mathscr{A}(S[x^{\pm 1}])$.

(c) $\Rightarrow$ (a): The monoid $M := \{sx^n \mid s \in S^* \text{ and } n \in \mathbb{Z}\}$ is atomic because it is a divisor-closed submonoid of $S[x^{\pm 1}]^*$. Since $S^*_{\text{red}} \cong M_{\text{red}}$, we conclude that S is atomic. Now suppose, by way of contradiction, that there exists an indecomposable polynomial $f = \sum_{i=0}^n s_i x^i \in S[x]$ such that $\operatorname{mcd}(s_0, \dots, s_n) = \emptyset$. As f is indecomposable, $\operatorname{ord} f = 0$, and the fact that $\operatorname{mcd}(s_1, \dots, s_n) = \emptyset$ ensures that $\deg f \geq 1$. Hence f is a nonzero nonunit in $S[x^{\pm 1}]$, and so we can write $f = g_1 \cdots g_m$ for $g_1, \dots, g_m \in \mathscr{A}(S[x^{\pm 1}])$. After replacing $g_1, \dots, g_m$ for some of their associates in $S[x^{\pm 1}]$, we can assume that they all belong to $S[x]$. Because f is indecomposable, we can further assume that $\deg g_m = \deg f$ and $\deg g_i = 0$ for every $i \in [\![1, m-1]\!]$. Since $g_m \in \mathscr{A}(S[x^{\pm 1}]) \cap S[x]$ and $\operatorname{ord} g_m = 0$, we see that $g_m \in \mathscr{A}(S[x])$, which means that $S^\times$ is the set of greatest common divisors of the coefficients of g_m. Hence $g_1 \cdots g_{m-1}$ must belong to $\operatorname{mcd}(s_1, \dots, s_n)$, a contradiction. $\square$

As a consequence of Theorem 3.1, we obtain that if a semidomain S is atomic and MCD, then its polynomial extensions $S[x]$ and $S[x^{\pm 1}]$ are atomic. Next we show that in this case $S[x]$ and $S[x^{\pm 1}]$ are also MCDs.

Proposition 3.2. *For a semidomain S, the following statements are equivalent.*

(a) S is MCD.

(b) $S[x]$ is MCD.

(c) $S[x^{\pm 1}]$ is MCD.

Proof. (a) $\Rightarrow$ (b): Suppose that S is MCD. Let

$$T = \{(f_1, \dots, f_n) \mid n \in \mathbb{N}_{>1}, f_i \in S[x] \text{ for } i \in [\![1, n]\!], \text{ and } \operatorname{mcd}(f_1, \dots, f_n) = \emptyset\}.$$

Assume towards a contradiction that T is nonempty, and let $(g_1, \dots, g_m) \in T$ such that $\sum_{i=1}^m \deg(g_i)$ is minimal. For $f = c_n x^n + \cdots + c_0 \in S[x]$, set $\mathsf{c}(f) := \{c_n, \dots, c_0\}$. Let c be an arbitrary element of the set $\operatorname{mcd}(\cup_{i=1}^m \mathsf{c}(g_i))$. By the minimality of $\sum_{i=1}^m \deg(g_i)$, the set $\operatorname{mcd}(g_1/c, \dots, g_n/c) = S[x]^\times$, which contradicts that $(g_1, \dots, g_m)$ is an element of T. Hence $S[x]$ is MCD.

(b) $\Rightarrow$ (c): This implication follows readily from the fact that $S[x]^\times \subseteq S[x^{\pm 1}]^\times$.

(c) ⇒ (a): Suppose that $S[x^{\pm 1}]$ is MCD. Since $M = \{sx^n \mid s \in S^* \text{ and } n \in \mathbb{Z}\}$ is a divisor-closed submonoid of $S[x^{\pm 1}]^*$ satisfying that $M^\times = S[x^{\pm 1}]^\times$, the former is also MCD. This immediately implies that S is MCD because M and S^* have isomorphic reduced monoids. □

Now we switch gears from the property of being atomic to that of satisfying ACCP, keeping our setting of semidomains. Although it follows from the corresponding definitions that every semidomain satisfying ACCP is atomic, we have mentioned in the introduction that the converse does not hold in general. We proceed to provide an example of an atomic positive semiring that does not satisfy ACCP.

Example 3.3. Take $r \in \mathbb{Q} \cap (0,1)$ with $\mathsf{n}(r) \geq 2$, and consider the additive monoid $S_r := \langle r^n \mid n \in \mathbb{N}_0 \rangle$. By [14, Corollary 4.4], the monoid S_r is atomic and does not satisfy ACCP. We proceed to argue that S_r is an MCD-monoid. Take $s_1, \dots, s_k \in S_r$ for some $k \in \mathbb{N}$. For each $i \in [\![1,k]\!]$, we can write $s_i = \sum_{j=0}^{n_i} c_{i,j} r^j$, where $c_{i,j} \in \mathbb{N}_0$ for all $i \in [\![1,k]\!]$ and $j \in [\![0,n_i]\!]$. By virtue of the identity $\mathsf{n}(r) r^n = \mathsf{d}(r) r^{n+1}$, there is no loss of generality in assuming that $c_{i,j} < \mathsf{n}(r)$ for all $i \in [\![1,k]\!]$ and $j \in [\![0, n_i - 1]\!]$. In addition, the same identity allows us to assume that $n_1 = \cdots = n_k$. Set $y := (\min_{1 \leq i \leq k} c_{i,n_1}) r^{n_1}$. Clearly, y is a common divisor of $s_1, \dots, s_k$ in S_r. Observe that, for some $i \in [\![1,k]\!]$, the equality $s_i - y = \sum_{j=0}^{n_1 - 1} c_{i,j} r^j$ holds, where $c_{i,j} < \mathsf{n}(r)$ for each $j \in [\![0, n_1 - 1]\!]$. By [14, Lemma 3.1(4)], the element $s_i - y$ has finitely many nonzero divisors in S_r, which implies that $\operatorname{mcd}_{S_r}(s_1, \dots, s_k)$ is nonempty. Hence S_r is an MCD-monoid.

Let us now consider the additive monoid $E(S_r) := \langle e^s \mid s \in S_r \rangle$. It follows from Lindemann-Weierstrass Theorem that $E(S_r)$ is the free monoid on the set $M = \{e^s \mid s \in S_r\}$. Note that $E(S_r)$ is closed under multiplication and, consequently, it is a positive semiring (cf. [6, Example 4.15]). Since the multiplicative submonoid M is isomorphic to the additive monoid S_r, the monoid M does not satisfy ACCP which, in turn, implies that $E(S_r)$ does not satisfy ACCP as $E(S_r)^\times = \mathscr{U}(M) = \{1\}$. To argue that $E(S_r)$ is atomic, take a nonzero nonunit $f := c_1 e^{s_1} + \cdots + c_k e^{s_k} \in E(S_r)$ for some $c_1, \dots, c_k \in \mathbb{N}$ and $s_1, \dots, s_k \in S_r$. After taking $s \in \operatorname{mcd}(s_1, \dots, s_k)$ and setting $d := \operatorname{mcd}(c_1, \dots, c_k)$, we can write

$$f = de^s \left(\frac{c_1}{d} e^{s_1 - s} + \cdots + \frac{c_k}{d} e^{s_k - s} \right),$$

where $s_1 - s, \dots, s_k - s$ have no nonunit common divisor in S_r. Since both $\mathbb{P}$ and $\{e^a \mid a \in \mathscr{A}(S_r)\}$ are contained in $\mathscr{A}(E(S_r))$, the fact that $\mathbb{N}$ and S_r are atomic immediately implies that de^s factors into irreducibles in $E(S_r)$. Set $g := \frac{c_1}{d} e^{s_1 - s} + \cdots + \frac{c_k}{d} e^{s_k - s}$, and write $g = f_1 \cdots f_m$ for some nonunit elements $f_1, \dots, f_m \in E(S_r)$. As no element of the form e^t with $t \in S_r$ divides g in $E(S_r)$, it follows that $m \leq \log_2 \left(\frac{c_1}{d} + \cdots + \frac{c_k}{d} \right)$. Hence, after assuming that m is as large as it can possibly be, we obtain that $f_1 \cdots f_m$ is a factorization of g in $E(S_r)$, and so f factors into irreducibles. Thus, $E(S_r)$ is atomic.

Unlike the property of being atomic, it is well known that the property of satisfying ACCP ascends from every semidomain to its semidomain of (Laurent) polynomials (this is not the case in the more general context of commutative rings with identity, as shown by W. J. Heinzer and D. C. Lantz in [35]). In the next theorem, we generalize this result to the context of semidomains. If S is a semidomain and $f \in S[x]^*$, then we let $c(f)$ denote the leading coefficient of f.

Theorem 3.4. *For a semidomain S, the following statements are equivalent.*

(a) S satisfies ACCP.

(b) $S[x]$ satisfies ACCP.

(c) $S[x^{\pm 1}]$ satisfies ACCP.

Proof. (a) ⇒ (b): Let $(f_n S[x])_{n \in \mathbb{N}}$ be an ascending chain of principal ideals in $S[x]$. Since $\deg f_n \geq \deg f_{n+1}$ for every $n \in \mathbb{N}$, we can choose $N_1 \in \mathbb{N}$ with $\deg f_n = \deg f_{N_1}$ for every $n \geq N_1$. On the other hand, for each $n \in \mathbb{N}$, the fact that f_{n+1} divides f_n in $S[x]$ implies that $c(f_{n+1}) \mid_S c(f_n)$. Therefore $(c(f_n)S)_{n \in \mathbb{N}}$ is an ascending chain of principal ideals in S. Since S satisfies ACCP, there exists $N_2 \in \mathbb{N}$

such that $c(f_n)$ and $c(f_{N_2})$ are associates in S^* for every $n \ge N_2$. After setting $N := \max\{N_1, N_2\}$, we can take a sequence $(s_n)_{n\in\mathbb{N}}$ with terms in S^* and a sequence $(u_n)_{n\in\mathbb{N}}$ with terms in $S^\times$ such that $f_n = s_n f_N$ and $c(f_n) = u_n c(f_N)$ for every $n \ge N$. Thus, for each $n \in \mathbb{N}$ with $n \ge N$,

$$u_n c(f_N) = c(f_n) = c(s_n f_N) = s_n c(f_N),$$

which implies that $s_n \in S^\times$. As a result, $f_n S[x] = f_N S[x]$ for every $n \ge N$, and so the ascending chain of principal ideals $(f_n S[x])_{n\in\mathbb{N}}$ stabilizes. Hence $S[x]$ satisfies ACCP.

(b) $\Rightarrow$ (c): Now assume that $S[x]$ satisfies ACCP. Let $(x^{k_n} f_n S[x^{\pm 1}])_{n\in\mathbb{N}}$ be an ascending chain of principal ideals of $S[x^{\pm 1}]$, where $(k_n)_{n\in\mathbb{N}}$ is a sequence of integers and $f_n \in S[x]$ satisfies $\operatorname{ord} f_n = 0$ for every $n \in \mathbb{N}$. Take a sequence $(\ell_n)_{n\in\mathbb{N}}$ of integers and a sequence $(g_n)_{n\in\mathbb{N}}$ with terms in $S[x]$ satisfying $\operatorname{ord} g_n = 0$ and $x^{k_n} f_n = (x^{k_{n+1}} f_{n+1})(x^{\ell_{n+1}} g_{n+1})$ for every $n \in \mathbb{N}$. As $f_n = f_{n+1} g_{n+1}$ for each $n \in \mathbb{N}$, we see that $(f_n S[x])_{n\in\mathbb{N}}$ is an ascending chain of principal ideals in $S[x]$. Since $S[x]$ satisfies ACCP, $(f_n S[x])_{n\in\mathbb{N}}$ must stabilize, and so there is an $N \in \mathbb{N}$ such that f_n and f_{n+1} are associates in $S[x]$ for every $n \ge N$. This implies that $g_{n+1} = f_n / f_{n+1} \in S[x]^\times \subseteq S[x^{\pm 1}]^\times$ for every $n \ge N$. Thus, $x^{k_n} f_n$ and $x^{k_{n+1}} f_{n+1}$ are associates in $S[x^{\pm 1}]$ for every $n \ge N$, and so $(x^{k_n} f_n S[x^{\pm 1}])_{n\in\mathbb{N}}$ stabilizes. Hence $S[x^{\pm 1}]$ satisfies ACCP.

(c) $\Rightarrow$ (a): Suppose that $S[x^{\pm 1}]$ satisfies ACCP. Since $M = \{s x^n \mid s \in S^* \text{ and } n \in \mathbb{Z}\}$ is a divisor-closed submonoid of $S[x^{\pm 1}]^*$, the former also satisfies ACCP. This immediately implies that S satisfies ACCP because M and S^* have isomorphic reduced monoids. $\square$

A semidomain S is called *Noetherian* if every ideal of S is finitely generated. In contrast to Theorem 3.4, the polynomial extension of a Noetherian semidomain is not necessarily Noetherian as the next example illustrates.

Example 3.5. Consider the polynomial semidomain $\mathbb{Q}_{\ge 0}[x]$. Clearly, the semidomain $\mathbb{Q}_{\ge 0}$ has no nonzero proper ideals, which implies that $\mathbb{Q}_{\ge 0}$ is Noetherian. Let I be the set consisting of all polynomials in $\mathbb{Q}_{\ge 0}[x]$ whose support have cardinality strictly bigger than 2. As the reader can easily verify, the set I is, in fact, an ideal of $\mathbb{Q}_{\ge 0}[x]$. Suppose towards a contradiction that I is finitely generated, i.e., $I = (f_1, \dots, f_n)$ for some $n \in \mathbb{N}$ and $f_1, \dots, f_n \in \mathbb{Q}_{\ge 0}[x]$. For a polynomial $f = s_1 x^{n_1} + \dots + s_k x^{n_k} \in \mathbb{Q}_{\ge 0}[x]^*$ written in canonical form, we define $\Delta(f) \coloneqq \{n_i - n_{i+1} \mid i \in [\![1, k-1]\!]\}$. Set $\Delta \coloneqq \cup_{i\in[\![1,n]\!]} \Delta(f_i)$, and let $N \in \mathbb{N}$ such that $N > \max \Delta$. Consider the polynomial $g = x^{3N} + x^{2N} + x^N \in I$. Observe that, for polynomials $f', g' \in \mathbb{Q}_{\ge 0}[x]$, we have

$$\min \Delta(f' \cdot g') \le \min \Delta(f') \quad \text{and} \quad \min \Delta(f' + g') \le \min \Delta(f').$$

But this contradicts that $g \in (f_1, \dots, f_n)$. Therefore $\mathbb{Q}_{\ge 0}[x]$ is not Noetherian.

Let S be a semidomain, and suppose that it is embedded into an integral domain R. The subset of the ring of formal power series $R[\![x]\!]$ consisting of those power series with coefficients in S is a semidomain, which we call the *semidomain of formal power series over* S and denote by $S[\![x]\!]$. It is clear that $S[\![x]\!]$ does not depend on the embedding of S into R. In [40], Roitman showed that atomicity does not ascend from an integral domain to its ring of formal formal power series. Next we show that the semidomain of formal power series over S is not necessarily atomic, even when S is a UFS.

Proposition 3.6. *Let S be a semidomain such that $(S, +)$ is reduced. The following statements hold.*

(1) *$S[\![x]\!]$ does not satisfy the ACCP.*
(2) *If $\mathscr{A}(S, +) \ne \emptyset$, then $S[\![x]\!]$ is not atomic.*

Proof. For each $k \in \mathbb{N}_0$, set $f_k \coloneqq \sum_{n=0}^{\infty} x^{n\cdot 2^k}$. Observe that $f_k = (1 + x^{2^k}) f_{k+1}$ for every $k \in \mathbb{N}_0$, which implies that $(f_k S[\![x]\!])_{k\in\mathbb{N}_0}$ is an ascending chain of principal ideals of $S[\![x]\!]$. Since $S[\![x]\!]^\times = S^\times$, we have that $1 + x^{2^k} \notin S[\![x]\!]^\times$ for any $k \in \mathbb{N}_0$. Consequently, the ascending chain $(f_k S[\![x]\!])_{k\in\mathbb{N}_0}$ never stabilizes. Hence $S[\![x]\!]$ does not satisfy the ACCP.

To tackle the statement (2), we start by proving that $\mathbb{N}_0[\![x]\!]$ is not atomic. It is clear that $\mathbb{N}_0[\![x]\!]$ is reduced. Suppose towards a contradiction that $\mathbb{N}_0[\![x]\!]$ is atomic. Set $f := \sum_{i=0}^{\infty} x^i$. Observe that we can write $f = gh$, where either $g \coloneqq \sum_{i=0}^{\infty} c_i x^i$ or $h \coloneqq \sum_{i=0}^{\infty} d_i x^i$ is an irreducible of $\mathbb{N}_0[\![x]\!]$ that is not a polynomial. Clearly, the inequality $c_i + d_i \le 1$ holds for each $i \in \mathbb{N}$. We can assume, without loss of generality, that $c_0 = \cdots = c_t = c_{(k+1)(t+1)} = 1$ and $d_0 = d_{t+1} = \cdots = d_{k(t+1)} = 1$ for some $t, k \in \mathbb{N}$ (note that, implicitly, we are also assuming that certain coefficients of g and h are 0). Set

$$C = \Big\{ n \in \mathbb{N} : t+1 \nmid n \text{ and } d_n = 1 \Big\} \bigcup \Big\{ n \in \mathbb{N} : t+1 \mid n,\ c_n = 1, \text{ and } c_{n+j} = 0 \text{ for some } j \in [\![1, t]\!] \Big\}.$$

We now prove that C is empty. Let m be the minimal element of C. If $t+1 \nmid m$, then $m = k'(t+1)+j$ for some $j \in [\![1,t]\!]$ and $k' \in \mathbb{N}_{\ge k}$. It is easy to see that the term $x^{k'(t+1)}$ does not show up in h and, by the minimality of m, this term does not show up in g either. As a result, there exist $r, \ell \in (t+1)\mathbb{N}$ such that $r + \ell = k'(t+1)$, $c_\ell = 1$, $d_r = 1$, and $\max(r, \ell) < k'(t+1)$. The minimality of m now implies that $c_{\ell+j} = 1$, which contradicts that $d_m = 1$. Hence $m = k'(t+1)$ for some $k' \in \mathbb{N}_{\ge k+1}$, which implies the existence of $j \in [\![1,t]\!]$ with $c_{m+j} = 0$. Without loss of generality, we can assume that $c_{m+j-i} = 1$ for every $i \in [\![1, j]\!]$. It is easy to verify that $d_{m+j} = 0$. Consequently, there exist $r, \ell \in \mathbb{N}_{<m}$ with $r + l = m + j$, $c_\ell = 1$, and $d_r = 1$. The minimality of m ensures that $t+1 \mid r$ which, in turn, implies that $\ell = k''(t+1) + j$ for some $k'' \in \mathbb{N}$. Since $c_m = 1$, both equalities $c_{k''(t+1)} = 0$ and $d_{k''(t+1)} = 0$ hold. Again, there exist $r', \ell' \in \mathbb{N}_{<k''(t+1)}$ with $r' + \ell' = k''(t+1)$, $c_{\ell'} = 1$, and $d_{r'} = 1$. The minimality of m guarantees that $t+1 \mid r'$, which implies that $c_{\ell'+j} = 1$, but this contradicts that $c_\ell = 1$. Thus C is empty.

Since C is empty, it is not hard to see that if $c_{n(t+1)} = 0$ for some $n \in \mathbb{N}$, then $c_{n(t+1)+j} = 0$ for all $j \in [\![0, t]\!]$. Therefore $1 + x + \cdots + x^t$ divides g in $\mathbb{N}_0[\![x]\!]$. Assume now that, for some $m \in \mathbb{N}_0$, we can write

$$h = \sum_{i=0}^{m} \Big(x^{i(t+1)} + x^{(i+1)(t+1)} + \cdots + x^{(i+k)(t+1)} \Big) + \sum_{i=n}^{\infty} d_{i(t+1)} x^{i(t+1)},$$

where $n > m + k$. There is no loss in assuming that $d_{n(t+1)} = 1$. By way of contradiction, suppose that there exists $j \in [\![1, k]\!]$ such that $d_{(n+j)(t+1)} = 0$, and assume that j is minimal. Since $c_{(k+1)(t+1)} = d_{k(t+1)} = 1$ and $n > k$, we have $c_{(n+j)(t+1)} = 0$. Consequently, there exist $r, \ell \in \mathbb{N}$ such that $r + \ell = (n+j)(t+1)$, $c_r = 1$, and $d_\ell = 1$. Clearly, we can write $l = (u+v)(t+1)$, where $u \in [\![0, m]\!]$ and $v \in [\![0, k]\!]$. Observe that $v < j$; otherwise $d_{(u+v-j)(t+1)} = 1$, which implies that $d_{n(t+1)} = 0$. Thus,

$$x^{n(t+1)} \cdot x^{(k+1)(t+1)} = x^{(u+v+k-j+1)(t+1)} \cdot x^r, \tag{3.1}$$

where $1 \le v + k - j + 1 \le k$. Since $n > m + k \ge u + (v + k - j + 1)$, the equality (3.1) represents a contradiction. As a consequence, we have that $d_{(n+j)(t+1)} = 1$ for each $j \in [\![0, k]\!]$. By induction, we obtain that $1 + x^{t+1} + \cdots + x^{k(t+1)}$ divides h in $\mathbb{N}_0[\![x]\!]$. We can conclude that neither g nor h is an irreducible of $\mathbb{N}_0[\![x]\!]$ that is not a polynomial. Therefore $\mathbb{N}_0[\![x]\!]$ is not atomic.

Let us now take care of the general case. Since $\mathscr{A}(S, +) \neq \emptyset$, we have that $1 \in \mathscr{A}(S, +)$. Suppose towards a contradiction that $S[\![x]\!]$ is atomic. Set $f \coloneqq \sum_{i=0}^{\infty} x^i$. Again, since $S[\![x]\!]$ is atomic, we can write $f = gh$, where either $g \coloneqq \sum_{i=0}^{\infty} c_i x^i$ or $h \coloneqq \sum_{i=0}^{\infty} d_i x^i$ is an irreducible of $S[\![x]\!]$ that is not a polynomial. Clearly, we have $c_0 d_0 = 1$. On the other hand, if $t + s = t' + s'$ for indices $t, t', s, s' \in \mathbb{N}_0$ satisfying $t \neq t'$ and $s \neq s'$, then either $c_t d_s = 0$ or $c_{t'} d_{s'} = 0$ because $1 \in \mathscr{A}(S, +)$ and $(S, +)$ is reduced. As a consequence, we obtain that $c_i = c_0$ and $d_i = d_0$ for every $i \in \mathbb{N}_0$. This, in turn, implies that $f = g'h'$, where $g' = d_0 g$ and $h' = c_0 h$ are both elements of $\mathbb{N}_0[\![x]\!]$ and either g' or h' is an irreducible of $\mathbb{N}_0[\![x]\!]$ that is not a polynomial; however, we already established that this is a contradiction, which concludes our argument. □

Corollary 3.7. *The semidomain $\mathbb{N}_0[\![x]\!]$ is not atomic.*

Corollary 3.7 shows that, in the context of semidomains, no factorization property behaves well with respect to formal power series extensions. In particular, the statement of Theorem 3.4 is not longer true if we replace either $S[x]$ or $S[x^{\pm 1}]$ by $S[\![x]\!]$.

The main results we have established in this section are illustrated in the following diagram.

$$
\begin{tikzcd}
S \ \textbf{satisfies ACCP} \arrow[r, Rightarrow, shift right] \arrow[d, Leftrightarrow] & S \ \textbf{is atomic} \arrow[l, Rightarrow, shift right, "/" marking] \arrow[d, Rightarrow, shift left, "/" marking] \\
S[x] \ \textbf{satisfies ACCP} \arrow[r, Rightarrow, shift right] \arrow[d, Leftrightarrow] & S[x] \ \textbf{is atomic} \arrow[l, Rightarrow, shift right, "/" marking] \arrow[u, Rightarrow, shift left] \arrow[d, Leftrightarrow] \\
S[x^{\pm 1}] \ \textbf{satisfies ACCP} \arrow[r, Rightarrow, shift right] & S[x^{\pm 1}] \ \textbf{is atomic} \arrow[l, Rightarrow, shift right, "/" marking]
\end{tikzcd}
$$

FIGURE 2. As proved in Theorems 3.1 and 3.4 the implications in the above diagram hold for every semidomain S. Grams' construction in [32, Section 1] and Roitman's examples in [39, Section 5] confirm that neither the horizontal implications nor the top-right implication are reversible, which is illustrated by red marked arrows.

4. The Bounded and Finite Factorization Properties

In this section, we study the bounded and the finite factorization properties. Specifically, we prove that both properties ascend from a semidomain S to the semidomains $S[x]$ and $S[x^{\pm 1}]$. Let us start with a useful characterization of BFMs.

Definition 4.1. Given a monoid M, a function $\ell\colon M \to \mathbb{N}_0$ is a *length function* of M if it satisfies the following two properties:

(i) $\ell(u) = 0$ if and only if $u \in \mathscr{U}(M)$;

(ii) $\ell(bc) \ge \ell(b) + \ell(c)$ for every $b, c \in M$.

The following result is well known.

Proposition 4.2. *[33, Theorem 1] A monoid M is a BFM if and only if there is a length function $\ell\colon M \to \mathbb{N}_0$.*

We are now in a position to discuss the results of this section.

Theorem 4.3. *For a semidomain S, the following statements are equivalent.*

(a) S is a BFS.

(b) $S[x]$ is a BFS.

(c) $S[x^{\pm 1}]$ is a BFS.

Proof. (a) $\Rightarrow$ (b): Assume that S is a BFS. Then there exists a length function $\ell\colon S^* \to \mathbb{N}_0$. Let us argue now that the function $\ell_x\colon S[x]^* \to \mathbb{N}_0$ given by $\ell_x(f) = \ell(c(f)) + \deg f$ is also a length function. Since $f \in S[x]^*$ is a unit if and only if $f = c(f)$ and $c(f)$ is a unit of S, we see that for each $f \in S[x]^*$ the equality $\ell_x(f) = 0$ holds if and only if $f \in S[x]^\times$. Using now the fact that ℓ is a length function of S^*, for any $f, g \in S[x]^*$ we see that

$$\ell_x(fg) = \ell(c(fg)) + \deg fg \ge (\ell(c(f)) + \deg f) + (\ell(c(g)) + \deg g) = \ell_x(f) + \ell_x(g).$$

Therefore the map ℓ_x is a length function of $S[x]^*$, which implies that $S[x]$ is a BFS.

(b) ⇒ (c): Suppose now that $S[x]$ is a BFS, and let $\ell\colon S[x]^* \to \mathbb{N}_0$ be a length function of $S[x]^*$. Proving that $S[x^{\pm 1}]$ is a BFS amounts to showing that the map

$$\bar{\ell}\colon S[x^{\pm 1}]^* \to \mathbb{N}_0 \quad \text{defined by} \quad \bar{\ell}(f) = \ell\Big(\frac{f}{x^{\operatorname{ord} f}}\Big)$$

is a length function. For each $f \in S[x^{\pm 1}]^*$, we observe that $\bar{\ell}(f) = 0$ if and only if $f/x^{\operatorname{ord} f}$ is a unit of $S[x]$, which happens precisely when f is a unit in $S[x^{\pm 1}]$. In addition, for all $f, g \in S[x^{\pm 1}]^*$,

$$\bar{\ell}(fg) = \ell\Big(\frac{fg}{x^{\operatorname{ord} fg}}\Big) = \ell\Big(\frac{f}{x^{\operatorname{ord} f}} \cdot \frac{g}{x^{\operatorname{ord} g}}\Big) \ge \ell\Big(\frac{f}{x^{\operatorname{ord} f}}\Big) + \ell\Big(\frac{g}{x^{\operatorname{ord} g}}\Big) = \bar{\ell}(f) + \bar{\ell}(g).$$

As a consequence, $\bar{\ell}$ is a length function, and so $S[x^{\pm 1}]$ is a BFS.

(c) ⇒ (a): This follows from the fact that $\{sx^n \mid s \in S^* \text{ and } n \in \mathbb{Z}\}$ is a divisor-closed submonoid of $S[x^{\pm 1}]^*$ whose reduced monoid is isomorphic to that of S^*. □

It is known that the class of BFDs is strictly contained in that consisting of all integral domains satisfying ACCP. Moreover, it turns out that there are semidomains satisfying ACCP that are neither integral domains nor BFSs. Indeed, if $M := \langle \frac{1}{p} \mid p \in \mathbb{P} \rangle$, then the semidomain $E(M)$ satisfies ACCP but it is not a BFS [7, Example 4.15].

We now turn our attention to the finite factorization property.

Theorem 4.4. *For a semidomain S, the following statements are equivalent.*

(a) S is an FFS.

(b) $S[x]$ is an FFS.

(c) $S[x^{\pm 1}]$ is an FFS.

Proof. (a) ⇒ (b): Suppose that S is an FFS, and let K be a field containing S. Assume, by contradiction, that $S[x]$ is not an FFS. Take a nonunit $f_0 \in S[x]^*$ such that $\{gS[x]^\times \mid g \in S[x] \text{ and } g \mid_{S[x]} f_0\}$ is infinite (this element exists by [33, Corollary 2]). Now let $(f_n)_{n \in \mathbb{N}}$ be a sequence whose terms are non-associate divisors of f_0 in $S[x]$. For each $n \in \mathbb{N}_0$, let s_n be the leading coefficient of f_n. As $s_n \mid_S s_0$ for every $n \in \mathbb{N}$ and the set $\{tS^\times \in S^*_{\text{red}} \mid t \mid_{S^*} s_0\}$ is finite by [33, Corollary 2], after replacing $(f_n)_{n\in\mathbb{N}}$ by a subsequence we can assume that $s_nS^\times = s_1S^\times$ for every $n \in \mathbb{N}$. Then we can replace f_n by $s_1s_n^{-1}f_n$ for every $n \in \mathbb{N}_{\ge 2}$ and assume that each term of $(f_n)_{n\in\mathbb{N}}$ has leading coefficient s_1. Because $f_n \mid_{K[x]} f_0$ for every $n \in \mathbb{N}$ and $K[x]$ is an FFD (in fact, a UFD), we can take $i, j \in \mathbb{N}$ with $i \neq j$ and $f_iK^\times = f_jK^\times$. Since both f_i and f_j have leading coefficient s_1, we see that $f_i = f_j$, contradicting that they are not associates in $S[x]$. Hence $S[x]$ is an FFS.

(b) ⇒ (c): Suppose that $S[x]$ is an FFS. By [33, Corollary 2], it suffices to show that every nonzero $f \in S[x]$ with $\operatorname{ord} f = 0$ has only finitely many divisors in $S[x^{\pm 1}]$ up to associates. To do so, fix a nonzero $f \in S[x]$ with $\operatorname{ord} f = 0$. Now assume that $x^{d_1}g_1$ and $x^{d_2}g_2$ are divisors of f in $S[x^{\pm 1}]$ for some $d_1, d_2 \in \mathbb{Z}$ and $g_1, g_2 \in S[x]$ with $\operatorname{ord} g_1 = \operatorname{ord} g_2 = 0$. Observe that g_1 and g_2 divide f in $S[x]$ and also that $x^{d_1}g_1$ and $x^{d_2}g_2$ are associates in $S[x^{\pm 1}]$ if and only if g_1 and g_2 are associates in $S[x]$. As $S[x]$ is an FFS, it follows from [33, Corollary 2] that f has only finitely many divisors in $S[x]$ up to associates, and so our previous observation ensures that f has only finitely many divisors in $S[x^{\pm 1}]$ up to associates. Therefore $S[x^{\pm 1}]$ is an FFS.

(c) ⇒ (a): Suppose that $S[x^{\pm 1}]$ is an FFS. Then S is an FFS because S^* and the divisor-closed submonoid $\{sx^n \mid s \in S^* \text{ and } n \in \mathbb{Z}\}$ of $S[x^{\pm 1}]^*$ have isomorphic reduced monoids. □

In the class of integral domains, the bounded factorization property does not imply the finite factorization property (see, for instance, [5, Example 4.7]). Hence the same statement must hold in the class of semidomains. As the following example illustrates, there are positive semidomains that are BFS but not FFSs.

Example 4.5. Fix $k \in \mathbb{N}_{\ge 2}$ and observe that $S = \mathbb{N}_0 \cup \mathbb{R}_{\ge k}$ is a positive semiring. It follows from [6, Theorem 5.1] that S is a BFS with $\mathscr{A}(S) = \big(\mathbb{P}_{<k^2} \cup [k, k^2)\big) \setminus \mathbb{P} \cdot S_{>1}$. In addition, S is a reduced semiring because $\min S \setminus \{0\} = 1$. Showing that S is not an FFS amounts to observing that the equality $(k+1)^2 = (\mu(k+1))(\mu^{-1}(k+1))$ yields a factorization of $(k+1)^2$ for each $\mu \in \mathbb{R}_{>1}$ closed enough to 1 such that $k < \mu^{-1}(k+1) < \mu(k+1) < k+2 \le k^2$.

In light of Corollary 3.7, the statements of Theorems 4.3 and 4.4 are not longer true if one replaces either $S[x]$ or $S[x^{\pm 1}]$ by $S[\![x]\!]$.

We summarize the results we have established in this section in the following diagram.

$$\begin{array}{ccc}
S \textbf{ is an FFS} & \overset{\not\Longleftarrow}{\Longrightarrow} & S \textbf{ is a BFS} \\
\Updownarrow & & \Updownarrow \\
S[x] \textbf{ is an FFS} & \overset{\not\Longleftarrow}{\Longrightarrow} & S[x] \textbf{ is a BFS} \\
\Updownarrow & & \Updownarrow \\
S[x^{\pm 1}] \textbf{ is an FFS} & \overset{\not\Longleftarrow}{\Longrightarrow} & S[x^{\pm 1}] \textbf{ is a BFS}
\end{array}$$

FIGURE 3. As proved in Theorems 4.3 and 4.4, the vertical implications in the above diagram hold for every semidomain S. The same theorems, along with Example 4.5, ensure that none of the horizontal implications is reversible, which is illustrated by the red marked arrows.

5. Factoriality Properties

It is well known that an integral domain R is a UFD if and only if $R[x]$ is a UFD. However, the same result does not hold for the more general class of semidomains as indicated by the following example.

Example 5.1. While the semidomain $\mathbb{N}_0$ is a UFS, we will verify that the polynomial extension $\mathbb{N}_0[x]$ is not. To do so, consider the polynomial $f(x) := x^5 + x^4 + x^3 + x^2 + x + 1 \in \mathbb{N}_0[x]$. We can factor f in $\mathbb{N}_0[x]$ in the following two ways:

$$f(x) = (x+1)(x^4 + x^2 + 1) \quad \text{and} \quad f(x) = (x^2 + x + 1)(x^3 + 1). \tag{5.1}$$

One can now verify that any decomposition of $x+1$, x^2+x+1, x^3+1, and x^4+x^2+1 as a product of non-constant polynomials in $\mathbb{C}[x]$ must contain a factor that does not belong to $\mathbb{N}_0[x]$. Hence the decompositions in (5.1) are actually distinct factorizations of f in $\mathbb{N}_0[x]$. Thus, $\mathbb{N}_0[x]$ is not a UFS.

Recall that a monoid M is a length-factorial monoid (or an LFM for short) if for all $b \in M$ and $z, z' \in \mathsf{Z}(b)$, the equality $|z| = |z'|$ implies $z = z'$. We say that a semidomain S is a *length-factorial semidomain* (*LFS*) if its multiplicative monoid is an LFM. It is clear that every UFS is an LFS. In addition, it was proved by J. Coykendall and W. W. Smith [18] that an integral domain is an LFS if and only if it is a UFS. As a result, the length-factorial property (somehow vacuously) ascends to (Laurent) polynomial domains. In this section, we prove that the semidomains where the length-factorial property ascends to (Laurent) polynomial semidomains are precisely the integral domains.

For the rest of the section, we identify a semidomain S with a subsemiring of the integral domain $\mathscr{G}(S)$ (see Lemma 2.2). Given a semidomain S, let n be the smallest positive integer such that the sum of n copies of 1 equals 0 in S, and let n be 0 if such a positive integer does not exist. As in the context of commutative rings, we call n the *characteristic* of S. We proceed to show that the irreducible polynomials in Example 5.1 are still irreducible as polynomials in $S[x]$ for any semidomain S that is not an integral domain.

Lemma 5.2. *Let S be a semidomain that is not an integral domain. Then the polynomials*

$$x+1, \quad x^2+x+1, \quad x^3+1, \quad \textit{and} \quad x^4+x^2+1 \tag{5.2}$$

are irreducible in $S[x]$.

Proof. Note that if S had finite characteristic, then every element of S would have an additive inverse, which contradicts the hypothesis that S is not an integral domain. Consequently, S has characteristic 0. Also, observe that any element of S dividing any of the polynomials in (5.2) must be a unit. Let us analyze each polynomial $p(x)$ in (5.2) independently.

Case 1: $p(x) = x+1$. This case immediately follows from the fact that the polynomial $p(x)$ is indecomposable in $S[x]$ along with our previous observation that every constant factor of $p(x)$ in $S[x]$ is a unit.

Case 2: $p(x) = x^2+x+1$. Suppose, towards a contradiction, that the polynomial $p(x)$ is not irreducible in $S[x]$. As in Case 1, we see that $p(x)$ is not divisible in $S[x]$ by any nonunit of S. Then we can write $p(x) = (ax+b)(cx+d)$ for some $a,b,c,d \in S^*$, from which we obtain the identities $ad+bc = bd = ac = 1$. Thus,

$$ab = ab\left((ad)(ac) + (bc)(bd)\right) = abcd(a^2+b^2) = a^2+b^2.$$

Therefore $b^3 = ab^2 - a^2b$ in $\mathscr{G}(S)$, and we can use this identity to obtain

$$b^3c^3 = (ab^2 - a^2b)c^3 = b^2c^2 - bc = bc(bc-1) = bc(-ad) = -1.$$

However $-1 = b^3c^3 \in S$ implies that S is an integral domain, a contradiction.

Case 3: $p(x) = x^3+1$. Suppose, by way of contradiction, that $p(x)$ reduces in $S[x]$. Since $p(x)$ is not divisible in $S[x]$ by any nonunit of S, we can write $p(x) = (ax^2+bx+c)(dx+e)$ for some $a,b,c,d,e \in S$. Expanding the product, we obtain the identities $ce = ad = 1$ and $ae+bd = be+cd = 0$, whence

$$0 = cd(ae+bd) = 1 + bcd^2.$$

This implies that $-1 = bcd^2 \in S$, which contradicts that S is not an integral domain.

Case 4: $p(x) = x^4+x^2+1$. Suppose, by way of contradiction, that $p(x)$ reduces in $S[x]$. As $p(x)$ is not divisible in $S[x]$ by any nonunit of S, it follows that $p(x)$ factors in $S[x]$ either as a polynomial of degree 1 times a polynomial of degree 3, or into two polynomials of degree 2, yielding the following two subcases.

Case 4.1: $p(x) = (ax^3+bx^2+cx+d)(ex+f)$ for some $a,b,c,d,e,f \in S$. After expanding this product, we obtain the identities $ae = df = 1$ and $cf+ed = 0$. Therefore, we see that

$$0 = af(cf+ed) = acf^2 + (ae)(df) = acf^2 + 1.$$

This implies that $-1 = acf^2 \in S$, which contradicts that S is not an integral domain.

Case 4.2: $p(x) = (ax^2+bx+c)(dx^2+ex+f)$ for some $a,b,c,d,e,f \in S$. Observe that if $b = e = 0$, we can produce a contradiction by reducing this case to Case 2. Thus, we can assume, without loss of generality that $e \neq 0$. After unfolding the product $(ax^2+bx+c)(dx^2+ex+f)$, we obtain the identities $ad = cf = af+be+cd = 1$ and $ae+bd = bf+ce = 0$. Since $d \neq 0$ and

$$d(a^2e+b) = (ad)(ae) + bd = ae+bd = 0,$$

the equality $a^2e+b = 0$ holds. Similarly, we can obtain the equality c^2e+b. Therefore the assumption $e \neq 0$ ensures that $a^2 = c^2$. Hence either $a = c$ or $a = -c$ in $\mathscr{G}(S)$. If $a = c$, then

$$1 + be = ad - af - cd + cf = (a-c)(d-f) = 0,$$

contradicting that $-1 \notin S$. On the other hand, if $a = -c$, then

$$3 = ad + cf + (af+be+cd) = a(d+f) + c(d+f) + be = (a+c)(d+f) + be = be,$$

which implies that $-1 = 2 - be = 2 - (1 - af - cd) = af + cd + 1 \in S$, which is a contradiction. $\square$

Corollary 5.3. *Let S be a semidomain. If $S[x]$ is an LFS, then S is an integral domain.*

We say that a semidomain S is a *GCD-semidomain* if S^* is a GCD-monoid. It is well known and not hard to verify that every UFM is a GCD-monoid. Hence every UFS is a GCD-semidomain. We can characterize when a polynomial semidomain $S[x]$ is GCD in terms of S.

Proposition 5.4. *Let S be a semidomain. Then $S[x]$ is a GCD-semidomain if and only if S is a GCD-domain.*

Proof. The reverse implication follows from [27, Theorem 6.4]. For the direct implication, we first observe that the fact that S^* is a divisor-closed submonoid of $S[x]^*$ ensures that S^* is a GCD-monoid. In addition, observe that if S is not an integral domain, then by Lemma 5.2 the polynomial $x+1$ is an irreducible element of $S[x]^*$ that is not prime and, therefore, it follows from [25, Theorem 6.7(2)] that $S[x]^*$ is not a GCD-monoid. Hence the fact that $S[x]$ is a GCD-semidomain also guarantees that S is an integral domain. □

Remark 5.5. Observe that, in the context of semidomains, while being MCD ascends to polynomial extensions, being GCD does not ascend in general.

Now we are in a position to characterize in several ways when $S[x]$ (or $S[x^{\pm 1}]$) is length-factorial.

Theorem 5.6. *Let S be a semidomain. The following statements are equivalent.*

- *(a) S is a UFD.*
- *(b) $S[x]$ is a UFS.*
- *(c) $S[x^{\pm 1}]$ is a UFS.*
- *(d) S is an LFD.*
- *(e) $S[x]$ is an LFS.*
- *(f) $S[x^{\pm 1}]$ is an LFS.*

Proof. (a) $\Leftrightarrow$ (b): The direct implication follows from the well-known fact that the unique factorization property ascends to polynomial rings. For the reverse implication, suppose that $S[x]$ is a UFS. Then $S[x]$ is an atomic GCD-semidomain. Thus, S is atomic by Theorem 3.1. In addition, it follows from Proposition 5.4 that S is a GCD-domain, which concludes our argument given that every atomic GCD-domain is a UFD (see [34, page 114]).

(a) $\Leftrightarrow$ (d): It follows from [18, Corollary 2.11].

(b) $\Leftrightarrow$ (c): Consider the multiplicative monoid $M = \{f \in S[x]^* \mid \operatorname{ord} f = 0\}$, and observe that $S[x]^*$ is isomorphic to the product monoid $\mathbb{N}_0 \times M$ via the map $f \mapsto (\operatorname{ord} f, f/x^{\operatorname{ord} f})$. It is clear that $S[x]$ is a UFS if and only if M is a UFM. Now the equivalence follows from the fact that M and $S[x^{\pm 1}]^*$ have isomorphic reduced monoids.

(b) $\Leftrightarrow$ (e): The direct implication follows from definitions. As for the reverse implication, suppose that $S[x]$ is an LFS. Then one of the polynomials $x+1$, x^3+1, x^2+x+1, and x^4+x^2+1 is not irreducible in $S[x]$; otherwise, the expressions $(x+1)(x^4+x^2+1)$ and $(x^2+x+1)(x^3+1)$ would induce distinct equal-length factorizations of the same element of $S[x]$ (see Example 5.1). In this case, S must be an integral domain by virtue of Lemma 5.2. Hence $S[x]$ is an integral domain, and so a UFD by [18, Corollary 2.11].

(e) $\Leftrightarrow$ (f): This follows similarly to (b) $\Leftrightarrow$ (c), after observing that, under the same notation used to prove the later, $S[x]$ is an LFS if and only if M is an LFM. □

While there are UFSs that are not UFDs (e.g., $\mathbb{N}_0$ and $\mathbb{N}_0[1/n]$), it is not clear whether there exists a *proper* (i.e., not a UFS) LFS. Motivated by this, we pose the following question.

Question 5.7. *Is there a proper LFS?*

Motivated by the terminology in [44], we say that S is a *half-factorial semidomain* (*HFS*) provided that S is atomic and any two factorizations of the same nonzero element have the same length. It is clear that every UFS is an HFS. However, the reverse implication does not hold as the next two examples illustrate.

Example 5.8. Consider the semidomain $S = \mathbb{N}_0 + x\mathbb{Z}[x]$. It is not hard to verify that the equality $\mathscr{A}(S) = \mathscr{A}(\mathbb{Z}[x]) \cap S$ holds. Observe that S is not a UFS. Indeed, we have $x^2 = (-x)^2$, while $-x \not\simeq_S x$. Next we show that S is atomic. Let $f \in S$, and write $f = c_k x^k + \cdots + c_1 x + c_0$, where $k \in \mathbb{N}_0$, $c_k, \ldots, c_1 \in \mathbb{Z}$ and $c_0 \in \mathbb{N}_0$. Let $g_1, \ldots, g_m \in \mathscr{A}(\mathbb{Z}[x])$ such that $f = g_1 \cdots g_m$. We have two possible cases.

Case 1: $c_0 > 0$. In this case, it is easy to see that the set $\{g_i \mid g_i(0) < 0,\ i \in [\![1, m]\!]\}$ has even cardinality. Then we can assume, without loss of generality, that $-g_1, \ldots, -g_{2t}, g_{2t+1}, \ldots, g_m \in \mathscr{A}(\mathbb{Z}[x]) \cap S$. Since $\mathscr{A}(S) = \mathscr{A}(\mathbb{Z}[x]) \cap S$ and $f = (-g_1) \cdots (-g_{2t}) \cdots g_m$, the element f factors into irreducibles in S.

Case 2: $c_0 = 0$. Write $f = (-1)^l x^t f'$, where $l \in \{0, 1\}$, $t \in \mathbb{N}$, and $f' \in S$. By the previous token, f' factors into irreducibles in S, which implies that f also factors into irreducibles.

Hence we can conclude that S is atomic. Let $z = h_1 \cdots h_n$ and $z' = h'_1 \cdots h'_s$ be two factorizations of f in S. Since $\mathscr{A}(S) = \mathscr{A}(\mathbb{Z}[x]) \cap S$, we have that z and z' are two factorizations of f in $\mathbb{Z}[x]$. Consequently, the equality $n = s$ holds as $\mathbb{Z}[x]$ is a UFD. Therefore S is a HFS.

Our next example is a fairly simple "polynomial-like" construction of a non-integrally closed HFD.

Example 5.9. Let F be a field, and let $M = \langle (1, n) \mid n \in \mathbb{N} \rangle \subseteq (\mathbb{N}_0^2, +)$. Denote by $\le$ the lexicograhic ordering, and observe that $(M, \le)$ is totally ordered. Let $D = [\![F^{M,\le}]\!]$ denote the set of all mappings from M to F. We represent an element $f \in D$ as $f = \sum_{m \in M} f(m) x^m$. Since $(M, \le)$ is totally ordered, we can define an addition operation and a convolution product on D just as for formal power series ([19]), and D is an integral domain with respect to these operations ([38, page 84]). By [38, Proposition 2.3], an element $f \in D$ is a unit if and only if $f((0, 0)) \neq 0$. Observe that D is not a UFD as $x^{(1,1)} \cdot x^{(1,3)} = x^{(1,2)} \cdot x^{(1,2)}$. Next we show that D is an HFD. Let $f, g_1, \ldots, g_k$ be nonzero nonunit elements of D satisfying that $f = g_1 \cdots g_k$. Thus,

$$\sum_{m \in M} f(m) x^m = \left(\sum_{m \in M} g_1(m) x^m \right) \cdots \left(\sum_{m \in M} g_k(m) x^m \right).$$

Since $\operatorname{ord} f \neq (0, 0)$ and M is an HFM (see [15, Example 4.23]), there is no loss in assuming that $g_i \in \mathscr{A}(D)$ for all $i \in [\![1, k]\!]$. This implies that $\operatorname{ord} g_i \in \mathscr{A}(M)$ for each $i \in [\![1, k]\!]$; otherwise, $x^{(1,1)} \mid_D g_i$, which contradicts that $g_i \in \mathscr{A}(D)$. Hence $\mathsf{L}(f) = \mathsf{L}(\operatorname{ord} f)$ and, therefore, D is an HFD. It is easy to see that D is not integrally closed.

The half-factorial property does not behave well under polynomial extensions in the context of integral domains (see [45, Theorem 2.4] and [4, Example 5.4]). On the other hand, a semidomain $S[x]$ satisfying that $(S, +)$ is reduced is "far" from being half-factorial in a sense that we now explain.

One of the best-studied arithmetic statistics related to the sets of lengths of an atomic monoid is the elasticity. The elasticity, first studied by J. L. Steffan [41] and R. Valenza [42] in the context of algebraic number theory, measures the deviation of an atomic monoid from being half-factorial. Let M be an atomic monoid. The *elasticity* of a nonunit $x \in M^\bullet$, denoted by $\rho(x)$, is defined as

$$\rho(x) = \frac{\sup \mathsf{L}(x)}{\inf \mathsf{L}(x)}.$$

By convention, we set $\rho(u) = 1$ for every $u \in \mathscr{U}(M)$. Observe that $\rho(x) \in \mathbb{Q}_{\ge 1} \cup \{\infty\}$ for all $x \in M$. In addition, the *elasticity* of the whole monoid M is defined to be

$$\rho(M) := \sup\{\rho(x) \mid x \in M\}.$$

Observe that a monoid M is an HFM if and only if $\rho(M) = 1$. Thus, we can think of the elasticity as an arithmetic statistic to measure how far an atomic monoid is from being an HFM, in which case, an atomic monoid having infinite elasticity is as far from being half-factorial as it can possibly be.

On the other hand, the *set of elasticities* of M is $R(M) := \{\rho(x) \mid x \in M\}$, and M is said to have *full elasticity* provided that $R(M) = (\mathbb{Q} \cup \{\infty\}) \cap [1, \rho(M)]$. As a monoid is half-factorial if and only if its set of elasticities is a singleton, namely $\{1\}$, we observe that for a given monoid having full elasticity provides an indication that it is as far from being an HFM as it can possibly be.

The following proposition is a generalized version of [12, Theorem 2.3], and here we adapt the proof given in [12] to fit the more general setting of polynomials over semidomains.

Proposition 5.10. *An atomic polynomial semidomain $S[x]$ has full and infinite elasticity provided that $(S, +)$ is reduced.*

Proof. As we pointed out before, if S has finite characteristic, then $(S, +)$ is not reduced (it is, in fact, a group). Consequently, S must have characteristic 0. Now let K be a field containing S as a subsemiring. We first claim that for every $n \in \mathbb{N}_{\ge 2}$, the polynomial $(x + n)^n(x^2 - x + 1)$ is an irreducible element in $S[x]$. It follows from [12, Lemma 2.1] that for every $m \in \mathbb{N}_0$ the polynomial $(x+n)^m(x^2-x+1)$ belongs to $\mathbb{N}_0[x]$ if and only if $m \ge n$. This, together with the fact that S is a semidomain whose additive monoid is reduced, guarantees that $(x + n)^m(x^2 - x + 1) \notin S[x]$ when $m < n$. Therefore the fact that $K[x]$ is a UFD guarantees that $(x + n)^n(x^2 - x + 1)$ is an irreducible element in $S[x]$.

By Lemma 5.2, the polynomial $(x^2 - x + 1)(x + 1) = x^3 + 1$ is irreducible in $S[x]$. Now for $n, k \in \mathbb{N}$, consider the polynomial

$$f(x) := (x + n)^n(x^2 - x + 1)(x + 1)^k \in \mathbb{N}_0[x] \subseteq S[x].$$

As every divisor of $f(x)$ in $S[x]$ is a divisor of $f(x)$ in $K[x]$ and $K[x]$ is a UFD, the only two factorizations of $f(x)$ in $S[x]$ are $[(x + n)^n(x^2 - x + 1)] \cdot [x + 1]^k$ and $[x + n]^n \cdot [(x^2 - x + 1)(x + 1)] \cdot [x + 1]^{k-1}$, which have lengths $k + 1$ and $k + n$, respectively. Since $\{(k + n)/(k + 1) \mid k, n \in \mathbb{N}\} = \mathbb{Q}_{\ge 1}$, we conclude that $S[x]$ has full and infinite elasticity. $\square$

Remark 5.11. Transfer Krull monoids (see [21, Section 4] for the definition) play an important role in factorization theory, and they have been investigated in [8,20,23]. In [23, Theorem 3.1], A. Geroldinger and Q. Zhong proved that transfer Krull monoids have full elasticity. However, observe that the additively reduced semidomain $S[x]$ is not transfer Krull by exactly the same argument used in [11, Remark 5.4].

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