ON THE ATOMIC STRUCTURE OF EXPONENTIAL PUISEUX MONOIDS AND SEMIRINGS

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Abstract. We say that a Puiseux monoid is exponential provided that it is generated by some of the powers of a rational number. Here we study the atomic properties of exponential Puiseux monoids and semirings. First, we characterize atomic exponential Puiseux monoids, and we prove that the finite factorization property, the bounded factorization property, and the ACCP coincide in this context. Then we proceed to offer necessary and sufficient conditions for an exponential Puiseux monoid to satisfy the ACCP. We conclude by describing the exponential Puiseux monoids that are semirings.

1. Introduction

An integral domain R is atomic provided that every nonzero nonunit element is a product of finitely many atoms (i.e., irreducibles) of R, and it satisfies the ACCP if for each sequence $(x_n)_{n\geq 1}$ in R satisfying that x_{n+1} divides x_n in R for each $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that x_n and x_k are associates for each $n \geq k$. Clearly, an integral domain R satisfying the ACCP is also atomic. Indeed, take a nonzero nonunit element $x_0 \in R$. Then either x_0 is an atom or it can be written as $x_0 = x_1x_2$, where neither x_1 nor x_2 are units of R. If both elements x_1 and x_2 are atoms then we stop; otherwise, either x_1 or x_2 is not an atom of R, and we can repeat the process by factoring that element. The fact that R satisfies the ACCP forces this procedure to stop, which means that every nonzero nonunit element is a product of atoms of R.

In 1974, A. Grams disproved P. Cohn's assumption that every atomic integral domain satisfies the ACCP [17]. Since then, several papers have studied the interplay between the atomic property and the ACCP in the context of integral domains (see, for instance, [1, 22]). We can investigate these properties not only for integral domains but in the more general context of commutative and cancellative monoids, where it is not hard to find examples of atomic monoids that do not satisfy the ACCP (see, for example, [6, Corollary 5.5]). Additionally, atomic monoids not satisfying the ACCP play an important role in understanding when being atomic is transferred from a commutative monoid M to its monoid ring $R[M]$, a question posed by R. Gilmer in [11,

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page 189] and partially answered by M. Roitman in [20] and J. Coykendall and F. Gotti in [7].

In this article, we study the atomic structure of exponential Puiseux monoids, that is, additive submonoids of the nonnegative cone of rational numbers $\mathbb{Q}$ generated by some of the powers of a positive rational number. These monoids are a generalization of rational cyclic monoids (which are also semirings), which were introduced in [16] and deeper studied in [5]. The class of exponential Puiseux monoids provides a fertile ground to study the interplay between the atomic property and the ACCP: this class consists mostly of atomic monoids and contains a large subclass of atomic monoids that do not satisfy the ACCP.

We begin the next section by introducing not only the necessary background but also the notation we shall be using throughout this paper. In Section 3, we characterize the exponential Puiseux monoids that are atomic and show that the finite factorization property, the bounded factorization property, and the ACCP agree in the context of exponential Puiseux monoids. In Section 4, we provide a necessary condition and a sufficient condition for an exponential Puiseux monoid to satisfy the ACCP. We conclude by describing, in Section 5, the exponential Puiseux monoids that are semirings.

2. Background

In this section we introduce the concepts and notation related to our exposition. Reference material on non-unique factorization theory can be found in the monograph [10] by A. Geroldinger and F. Halter-Koch.

2.1. **Notation.** We let $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive and nonnegative integers, respectively, and let $\mathbb{P}$ denote the set of prime numbers. In addition, if X is a subset of the rational numbers then we set $X_{<q} \coloneqq \{x \in X \mid 0 \le x < q\}$. In the same way we define $X_{\le q}$, $X_{>q}$ and $X_{\ge q}$. Additionally, if $n \in \mathbb{N}$ and $X \subseteq \mathbb{N}_0$ then we set $X - n \coloneqq \{x - n \mid x \in X_{\ge n}\}$. For a rational number $r = n/d$ with n and d relatively prime positive integers, we call n the *numerator* and d the *denominator* of r, and we set $\mathsf{n}(r) \coloneqq n$ and $\mathsf{d}(r) \coloneqq d$. For k, m nonnegative integers such that $k \le m$, we denote by $[\![k, m]\!]$ the set of integers between k and m, i.e., $[\![k, m]\!] \coloneqq \{s \in \mathbb{N}_0 \mid k \le s \le m\}$.

2.2. **Puiseux monoids.** A monoid is defined to be a semigroup with identity, and we tacitly assume that every monoid we refer to here is cancellative, commutative, and reduced (i.e., its only invertible element is the identity). Unless we specify otherwise, we shall use additive notation for monoids. Now let M be a monoid. We denote by $\mathcal{A}(M)$ the set consisting of those elements $a \in M^\bullet \coloneqq M \setminus \{0\}$ such that if $a = x + y$ for some $x, y \in M$ then either $x = 0$ or $y = 0$. The elements of $\mathcal{A}(M)$ are called *atoms*. For a subset $S \subseteq M$, we denote by $\langle S \rangle$ the minimal submonoid of M containing S, and if $M = \langle S \rangle$ then we say that S is a *generating set* of M. We call the monoid M *atomic* provided that $M = \langle \mathcal{A}(M) \rangle$. On the other hand, it is said that M is *antimatter* provided that $\mathcal{A}(M) = \emptyset$. For $x, y \in M$, it is said that x *divides* y *in* M if there exists

$x' \in M$ such that $y = x + x'$ in which case we write $x \mid_M y$ and drop the subscript M whenever $M = (\mathbb{N}, \cdot)$. A subset I of M is an *ideal* of M on condition that $I + M \subseteq I$. An ideal I is *principal* if $I = x + M$ for some $x \in M$. Furthermore, it is said that M satisfies the *ascending chain condition on principal ideals* (or *ACCP*) if every increasing sequence of principal ideals of M eventually stabilizes. If M satisfies the ACCP then it is atomic [10, Proposition 1.1.4].

A *numerical monoid* N is an additive submonoid of $\mathbb{N}_0$ whose complement in $\mathbb{N}_0$ is finite. If $N \neq \mathbb{N}_0$ then the greatest integer that is not an element of N is called the *Frobenius number* of N and is denoted by $F(N)$. It is well known that numerical monoids are always finitely generated and, therefore, atomic. An introduction to numerical monoids can be found in [8]. A *Puiseux monoid* is an additive submonoid of $\mathbb{Q}_{\geq 0}$. Clearly, Puiseux monoids are a natural generalization of numerical monoids. The atomic structure of Puiseux monoids has received considerable attention during the past few years (see, for instance, [13, 15, 16]). In particular, some authors have studied rational cyclic semirings, i.e., Puiseux monoids generated by the elements of a finite geometric progression (see [5, 16, 18]). Unlike Puiseux monoids in general, rational cyclic semirings have a tractable atomic structure, and this allows to nicely compute some of their factorization invariants (for some of these computations, see [5]).

Definition 2.1. The *rational cyclic semiring* over $r \in \mathbb{Q}_{>0}$ is the Puiseux monoid generated by the nonnegative powers of r, i.e., $S_r = \langle r^n \mid n \in \mathbb{N}_0 \rangle$.

Most rational cyclic semirings are atomic as the next theorem indicates.

Theorem 2.2. [16, Theorem 6.2] *Let $r \in \mathbb{Q}_{>0}$ and $S_r = \langle r^n \mid n \in \mathbb{N}_0 \rangle$. The following statements hold.*

1. *If $\mathsf{d}(r) = 1$ then S_r is atomic with $\mathcal{A}(S_r) = \{1\}$.*
2. *If $\mathsf{d}(r) > 1$ and $\mathsf{n}(r) = 1$ then S_r is antimatter.*
3. *If $\mathsf{d}(r) > 1$ and $\mathsf{n}(r) > 1$ then S_r is atomic with $\mathcal{A}(S_r) = \{r^n \mid n \in \mathbb{N}_0\}$.*

2.3. **Factorizations.** Let M be a commutative, cancellative, reduced, and atomic monoid. The *factorization monoid* of M, denoted by $\mathsf{Z}(M)$, is the free commutative monoid on $\mathcal{A}(M)$. The elements of $\mathsf{Z}(M)$ are called *factorizations*, and if $z = a_1 + \cdots + a_n \in \mathsf{Z}(M)$ for $a_1, \ldots, a_n \in \mathcal{A}(M)$ then we say that the length of z, denoted by $|z|$, is n. The unique monoid homomorphism $\pi\colon \mathsf{Z}(M) \to M$ satisfying that $\pi(a) = a$ for all $a \in \mathcal{A}(M)$ is called the *factorization homomorphism* of M. For each $x \in M$, there are two important sets associated to x:

$$\mathsf{Z}_M(x) \coloneqq \pi^{-1}(x) \subseteq \mathsf{Z}(M) \qquad \text{and} \qquad \mathsf{L}_M(x) \coloneqq \{|z| : z \in \mathsf{Z}_M(x)\},$$

which are called the *set of factorizations* of x and the *set of lengths* of x, respectively; if the monoid M is clear from the context then we drop the subscript. In addition, the collection $\mathcal{L}(M) \coloneqq \{\mathsf{L}(x) \mid x \in M\}$ is called the *system of sets of lengths* of M. See [9] for a survey on sets of lengths. We say that M satisfies the *finite factorization property* (or *FFP*) provided that $\mathsf{Z}(x)$ is nonempty and finite for all $x \in M$. In this case we also say that M is an *FFM*. Similarly, M satisfies the *bounded factorization property* (or *BFP*) if $\mathsf{L}(x)$ is nonempty and finite for all $x \in M$, and in this case we say that M is a *BFM*. It is clear that each FFM is a BFM, and it follows from [10, Corollary 1.3.3] that each BFM satisfies the ACCP.

3. Atomic Structure of Exponential Puiseux Monoids

In this section we investigate the atomic structure of *exponential Puiseux monoids*. Specifically, we show that most exponential Puiseux monoids are atomic, and we prove that the FFP, the BFP, and the ACCP are equivalent in this context. But first let us define the object of study of this paper.

Definition 3.1. Take $r \in \mathbb{Q}_{>0}$ and let $S = \{s_0 = 0 < s_1 < s_2 < \cdots\}$ be a subset of $\mathbb{N}_0$. We let $M_{r,S}$ denote the Puiseux monoid

$$M_{r,S} \coloneqq \langle r^{s_n} \mid n \in \mathbb{N}_0 \rangle,$$

which we called *exponential*.

Remark 3.2. With notation as in Definition 3.1 and for each $n \in \mathbb{N}_0$, we denote by s_n the $(n+1)$th smallest element of S and set $\delta_n \coloneqq s_{n+1} - s_n$. Then it is easy to see that the equality $\sum_{i=0}^{m-1} \delta_i = s_m$ holds.

Remark 3.3. Finitely generated Puiseux monoids are isomorphic to numerical monoids by [14, Proposition 3.2]. As it is clear that numerical monoids satisfy the finite factorization property, we assume that exponential Puiseux monoids are not finitely generated unless we specify otherwise.

In the literature we can find many instances in which particular families of exponential Puiseux monoids have been studied. Consider the following examples.

Example 3.4. Let $N = (\mathbb{N}_0, +)$. Clearly, N is an exponential Puiseux monoid. Furthermore, $\mathbb{N}_0 \subseteq M$ for all exponential Puiseux monoids M. As mentioned before, the atomic structure of N is not hard to describe: it is an FFM.

Example 3.5. Let r be a positive rational number and consider the Puiseux monoid generated by the set $\{r^n \mid n \in \mathbb{N}_0\}$. These monoids, introduced in [16], are called rational cyclic semirings because they are also closed under multiplication. The atomic structure of rational cyclic semirings is not very rich since they are almost always atomic (Theorem 2.2) and satisfy the ACCP if and only if $r \geq 1$ ([6, Corollary 4.4], Theorem 2.2, and [13, Theorem 5.6]).

Example 3.6. Let $\mathcal{B}$ be a finite subset of $\mathbb{Q}_{>0}$ and set $M_{\mathcal{B}} \coloneqq \langle b^n \mid b \in \mathcal{B},\, n \in \mathbb{N}_0 \rangle$. We say that $M_{\mathcal{B}}$ is the rational multicyclic monoid over $\mathcal{B}$ provided that $\mathcal{B}$ is minimal, that is, if $\mathcal{B}' \subsetneq \mathcal{B}$ then $M_{\mathcal{B}'} \subsetneq M_{\mathcal{B}}$. These monoids are a direct generalization of rational cyclic semirings, and they were studied by the third author in [18]. Many rational multicyclic monoids are exponential. Indeed, it is not hard to see that $M_{\mathcal{B}}$ is an exponential Puiseux monoid if and only if the elements of $\mathcal{B}$ are powers of the same positive rational number r. The atomic structure of rational multicyclic monoids is somewhat similar to that one of rational cyclic semirings (see [18, Theorem 3.7]).

As is the case for rational cyclic semirings, it is straightforward to describe the exponential Puiseux monoids that are atomic.

Proposition 3.7. *Let $M_{r,S}$ be an exponential Puiseux monoid. Then the following statements hold.*

(1) *If $\mathsf{d}(r) = 1$ then $M_{r,S} = \mathbb{N}_0$ and so $\mathcal{A}(M_{r,S}) = \{1\}$.*
(2) *If $\mathsf{n}(r) = 1$ and $\mathsf{d}(r) > 1$ then $M_{r,S}$ is antimatter and so $\mathcal{A}(M_{r,S}) = \emptyset$.*
(3) *If $\mathsf{n}(r) > 1$ and $\mathsf{d}(r) > 1$ then $M_{r,S}$ is atomic and $\mathcal{A}(M_{r,S}) = \{r^s \mid s \in S\}$.*

Proof. It is easy to see that if $\mathsf{d}(r) = 1$ then $M_{r,S} = \langle 1 \rangle = \mathbb{N}_0$. On the other hand, the equation $\mathsf{d}(r)^{-s_n} = \mathsf{d}(r)^{\delta_n} \mathsf{d}(r)^{-s_{n+1}}$ holds for all $n \in \mathbb{N}_0$ from which (2) follows since $\mathsf{n}(r) = 1$. As for (3), if $r^s \notin \mathcal{A}(M_{r,S})$ for some $s \in S$ then $r^s \notin \mathcal{A}(S_r)$, where $S_r = \langle r^n \mid n \in \mathbb{N}_0 \rangle$, contradicting Theorem 2.2. $\square$

As we indicated above, a rational cyclic semiring S_r with $r < 1$ does not satisfy the ACCP by [6, Corollary 4.4] and Theorem 2.2. On the other hand, if $r \geq 1$ then S_r is an FFM by virtue of [13, Theorem 5.6]. Consequently, the FFP, the BFP, and the ACCP coincide in the context of rational cyclic semirings. As we show next, this result can be extended to exponential Puiseux monoids. First we collect two technical lemmas.

Lemma 3.8. *Let x be a nonzero element of an atomic exponential Puiseux monoid $M_{r,S}$ with $r \in \mathbb{Q}_{<1}$ and consider a factorization $z = \sum_{i=0}^{n} c_i r^{s_i} \in \mathsf{Z}(x)$ with coefficients $c_0, \dots, c_n \in \mathbb{N}_0$. The following conditions hold.*

(1) $\min \mathsf{L}(x) = |z|$ *if and only if $c_i < \mathsf{d}(r)^{\delta_{i-1}}$ for each $i \in [\![1, n]\!]$.*
(2) *There exists exactly one factorization z_0 in $\mathsf{Z}(x)$ of minimum length.*
(3) $\max \mathsf{L}(x) = |z|$ *if and only if $c_i < \mathsf{n}(r)^{\delta_i}$ for all $i \in [\![0, n]\!]$.*
(4) *There exists, at most, one factorization of maximum length of x.*
(5) *If $c_i < \mathsf{n}(r)$ for each $i \in [\![0, n]\!]$ then $|\mathsf{Z}(x)| = 1$.*

Proof. The proofs of (1) and (2) are left to the reader as they follow the proof of [5, Lemma 3.1]. To prove the direct implication of (3) note that if $c_i \geq \mathsf{n}(r)^{\delta_i}$ for some $i \in [\![0, n]\!]$ then by using the transformation $\mathsf{n}(r)^{\delta_i} r^{s_i} = \mathsf{d}(r)^{\delta_i} r^{s_{i+1}}$ we can generate a factorization $z^* \in \mathsf{Z}(x)$ such that $|z^*| > |z|$ since $\mathsf{d}(r) > \mathsf{n}(r)$. Conversely, consider a factorization $z' = \sum_{i=0}^{t} d_i r^{s_i} \in \mathsf{Z}(x)$ with coefficients $d_0, \dots, d_t \in \mathbb{N}_0$ and suppose, by way of contradiction, that $|z'| > |z|$. There is no loss in assuming that $n \leq t$. By applying the identity $\mathsf{n}(r)^{\delta_m} r^{s_m} = \mathsf{d}(r)^{\delta_m} r^{s_{m+1}}$ with $m \in \mathbb{N}_0$ finitely many times, we can generate factorizations $z' = z_1, \dots, z_k = \sum_{i=0}^{q} e_i r^{s_i} \in \mathsf{Z}(x)$ such that $|z_j| < |z_{j+1}|$ for $j \in [\![1, k-1]\!]$ and $e_i < \mathsf{n}(r)^{\delta_i}$ for $i \in [\![0, t]\!]$. Note that $t \leq q$. Let $l \in [\![0, n]\!]$ be the smallest index such that $c_l \neq e_l$. Note that such an index l exists given that the inequalities $|z| < |z'| \leq |z_k|$ hold. This implies that z and z_k are two different factorizations of x. Thus,

$$(c_l - e_l)r^{s_l} = \sum_{i=l+1}^{n} (e_i - c_i)r^{s_i} + \sum_{i=n+1}^{q} e_i r^{s_i};$$

which implies that $\mathsf{n}(r)^{\delta_l} \mid c_l - e_l$. This contradiction proves that our hypothesis is untenable. Consequently, z is a factorization of maximum length of x. Note that (4) follows readily from (3) and (5) is a direct consequence of (1) and (3). $\square$

Lemma 3.9. *Let $M_{r,S}$ be as in Lemma 3.8 and let $x = \pi(k_0\,\mathsf{d}(r)^{\delta_i} r^{s_{i+1}})$ be an element of $M^{\bullet}_{r,S}$ for some $k_0, i \in \mathbb{N}_0$. If x does not have a factorization of maximum length then $x = y + x'$ with $y \in M^{\bullet}_{r,S}$ and $x' = \pi(k\,\mathsf{d}(r)^{\delta_j} r^{s_{j+1}})$ for some $k, j \in \mathbb{N}$. Furthermore, x' does not have a factorization of maximum length.*

Proof. We start by describing a process that will generate a sequence of factorizations of x each having the form $z_m = k_m \mathsf{d}(r)^h r^{s_{m+i+1}}$ for some positive integers k_m and h. Let $z_0 = k_0 \mathsf{d}(r)^{\delta_i} r^{s_{i+1}}$. Assume that $z_j = k_j \mathsf{d}(r)^{h_j} r^{s_{i+j+1}}$ was already defined for some $j \in \mathbb{N}_0$. Now if $\mathsf{n}(r)^{\delta_{i+j+1}} \nmid k_j$ then the process stops, and we obtain a sequence of factorizations $z_0, \ldots, z_j \in \mathsf{Z}(x)$. On the other hand, if $\mathsf{n}(r)^{\delta_{i+j+1}} \mid k_j$ then by applying the transformation $\mathsf{n}(r)^{\delta_{i+j+1}} r^{s_{i+j+1}} = \mathsf{d}(r)^{\delta_{i+j+1}} r^{s_{i+j+2}}$ we generate a new factorization $z_{j+1} = k_{j+1}\mathsf{d}(r)^{h_{j+1}} r^{s_{i+j+2}} \in \mathsf{Z}(x)$ for some positive integers k_{j+1} and h_{j+1} such that $k_j > k_{j+1}$. Since there is no infinite strictly decreasing sequence of positive integers, this process eventually stops, and it yields a sequence of factorizations $z_0, \ldots, z_n = k_n \mathsf{d}(r)^h r^{s_{n+i+1}} \in \mathsf{Z}(x)$, where $\mathsf{n}(r)^{\delta_{n+i+1}} \nmid k_n$.

Next note that since x has no factorization of maximum length, the inequality $\mathsf{n}(r)^{\delta_{n+i+1}} \leq k_n \mathsf{d}(r)^h$ holds by Lemma 3.8 (part (3)). We have $k_n \mathsf{d}(r)^h = k' \mathsf{n}(r)^{\delta_{n+i+1}} + l$ with $k' \in \mathbb{N}$ and $l \in [\![0, \mathsf{n}(r)^{\delta_{n+i+1}} - 1]\!]$. Since $\mathsf{n}(r)$ and $\mathsf{d}(r)$ are relatively prime numbers for each $r \in \mathbb{Q}_{>0}$, we have $l \neq 0$. Thus,

$$k'\mathsf{d}(r)^{\delta_{n+i+1}} \cdot r^{s_{n+i+2}} + l \cdot r^{s_{n+i+1}} \in \mathsf{Z}(x).$$

Note that $z = l\, r^{s_{n+i+1}}$ is a factorization of maximum length of $y = \pi(z)$ by Lemma 3.8. We conclude by proving that $x' = \pi(k'\mathsf{d}(r)^{\delta_{n+i+1}} \cdot r^{s_{n+i+2}})$ has no factorization of maximum length. Suppose, by way of contradiction, that x' has a factorization of maximum length $z^* \in \mathsf{Z}(x')$. There exists a sequence of factorizations $z_0, \ldots, z_t = z^*$, where $z_0 = k'\mathsf{d}(r)^{\delta_{n+i+1}} r^{s_{n+i+2}}$. In fact, if z_j is already defined for some $j \in \mathbb{N}_0$ then either z_j has a summand of the form $\mathsf{n}(r)^{\delta_k} r^{s_k}$ (with k a nonnegative integer) in which case we generate a factorization z_{j+1} using the transformation $\mathsf{n}(r)^{\delta_k} r^{s_k} = \mathsf{d}(r)^{\delta_k} r^{s_{k+1}}$ or z_j does not have such a summand in which case $z_j = z^*$ by Lemma 3.8 (parts (3) and (4)). The aforementioned replacements not only yield each time a factorization of bigger length, which means that we cannot carry out these transformations infinitely many times, but also increase the exponents of the summands involved. Consequently, $z^* + z$ is a factorization of maximum length of x by Lemma 3.8. This contradiction proves that x' has no factorization of maximum length. $\square$

Now we are in a position to prove the main result of this section.

Theorem 3.10. *Let $M_{r,S}$ be an atomic exponential Puiseux monoid. Then the following statements are equivalent.*

1. *$M_{r,S}$ satisfies the FFP.*
2. *$M_{r,S}$ satisfies the BFP.*
3. *$M_{r,S}$ satisfies the ACCP.*

Proof. Recall that, for commutative and cancellative monoids, (1) implies (2) by definition and (2) implies (3) by [10, Corollary 1.3.3]. Then our proof reduces to showing that (3) implies (1). If $r \geq 1$ then our result follows from [13, Theorem 5.6]; consequently, we may assume $r < 1$. Assume that $M_{r,S}$ is not an FFM, and let $x \in M_{r,S}^\bullet$ with $|\mathsf{Z}(x)| = \infty$.

By way of contradiction, suppose that x has a factorization of maximum length $z = \sum_{i=0}^m c_i r^{s_i} \in \mathsf{Z}(x)$ with coefficients $c_0, \ldots, c_m \in \mathbb{N}_0$. Note that the set $S' = \{s \in S : r^s \,|_{M_{r,S}}\, x\}$ has infinite cardinality since $|\mathsf{Z}(x)| = \infty$. Clearly, there exists $s_j \in S'$ such that $s_j > s_m$. This implies that there exists $z_1 = \sum_{i=0}^l d_i r^{s_i} \in \mathsf{Z}(x)$, where $l, d_i \in \mathbb{N}_0$, $j < l$, $d_j > 0$, and $s_i \in S$ for each $i \in [\![0, l]\!]$. Since $d_j > 0$, we have $z_1 \neq z$. By virtue of Lemma 3.8, the inequality $d_k \geq \mathsf{n}(r)^{\delta_k}$ holds for some $k \in [\![0, l]\!]$. Consequently, one can obtain a factorization z_2 from z_1 by applying the transformation $\mathsf{n}(r)^{\delta_k} r^{s_k} = \mathsf{d}(r)^{\delta_k} r^{s_{k+1}}$. Note that $|z_2| > |z_1|$. Moreover, $z_2 \neq z$ as either s_j or s_{j+1} shows up in the factorization z_2. Repeating this process for z_2 we can obtain a factorization $z_3 \in \mathsf{Z}(x)$ such that $|z_3| > |z_2| > |z_1|$ and $z_3 \neq z$. It follows by induction that there exists a sequence $z_1, z_2, \ldots$ of elements of $\mathsf{Z}(x)$ such that $|z_n| < |z_{n+1}| < |z|$ for each $n \in \mathbb{N}$, a contradiction. Hence x has no factorization of maximum length.

Now let $z = \sum_{i=0}^m c_i r^{s_i} \in \mathsf{Z}(x)$ with coefficients $c_0, \ldots, c_m \in \mathbb{N}_0$. By applying the transformation $\mathsf{n}(r)^{\delta_i} r^{s_i} = \mathsf{d}(r)^{\delta_i} r^{s_{i+1}}$ finitely many times over all summands $c_i r^{s_i}$ with $i \in [\![0, m-1]\!]$ we can generate a factorization $z' = \sum_{i=0}^m d_i r^{s_i} \in \mathsf{Z}(x)$ with $d_i \in \mathbb{N}_0$ for each $i \in [\![0, m]\!]$ such that $d_i < \mathsf{n}(r)^{\delta_i}$ for each $i \in [\![0, m-1]\!]$. Note that $d_m = h\,\mathsf{n}(r)^{\delta_m} + l$, where $h \in \mathbb{N}$ and $l \in [\![0, \mathsf{n}(r)^{\delta_m} - 1]\!]$; otherwise, z' would be the factorization of maximum length of x by Lemma 3.8, which is impossible. Then $x' = \pi(h\mathsf{n}(r)^{\delta_m} r^{s_m}) = \pi(h\mathsf{d}(r)^{\delta_m} r^{s_{m+1}})$ has no factorization of maximum length. It is not hard to see that using Lemma 3.9 one can generate a sequence $(y_n)_{n \geq 1}$ of elements of $M_{r,S}$ such that $y_1 > y_2 > \cdots$ and $y_{n+1} \,|_{M_{r,S}}\, y_n$ for all $n \in \mathbb{N}$. Therefore, $M_{r,S}$ does not satisfy the ACCP. $\square$

If no exponential Puiseux monoid $M_{r,S}$ with $r < 1$ satisfies the ACCP then Theorem 3.10 holds trivially by [13, Theorem 5.6]. However, this is far from being the case as we will show in the next section.

4. Exponential Puiseux Monoids and the ACCP

Now we proceed to study the ACCP in the context of exponential Puiseux monoids. We show that there are infinitely many exponential Puiseux monoids that satisfy the ACCP and infinitely many that do not. In this section, we provide a necessary condition and a sufficient condition for an exponential Puiseux monoid to satisfy the ACCP.

Proposition 4.1. *Let $M_{r,S}$ be an atomic exponential Puiseux monoid with $r < 1$. If $M_{r,S}$ satisfies the ACCP then*

$$\mathsf{d}(r) \le \mathsf{n}(r) \cdot \limsup_{n\to\infty} \sqrt[s_n]{\mathsf{n}(r)^{\delta_n}}. \tag{4.1}$$

Proof. We prove that if $M_{r,S}$ satisfies the ACCP then

$$\sum_{k=0}^{\infty} \left(\mathsf{n}(r)^{\delta_k} - 1\right) r^{s_k} \tag{4.2}$$

does not converge. For the sake of a contradiction, suppose that the series (4.2) (of positive terms) converges to a real number $R > 0$, and let $K \in \mathbb{N}$ such that

$$K > R = r^{\delta_{-1}} \sum_{k=0}^{\infty} \left(\mathsf{n}(r)^{\delta_k} - 1\right) r^{s_k};$$

where $\delta_{-1} = 0$. In addition, let $z_0 = Kr^{s_0} \in \mathsf{Z}(K)$. We now prove that if $z_m = \sum_{i=0}^{m} c_i r^{s_i} \in \mathsf{Z}(K)$ is a factorization of K such that

$$c_m > \prod_{j=-1}^{m-1} r^{-\delta_j} \sum_{k=m}^{\infty} \left(\mathsf{n}(r)^{\delta_k} - 1\right) r^{s_k}$$

then there exists a factorization $z_{m+1} = \sum_{i=0}^{m+1} d_i r^{s_i} \in \mathsf{Z}(K)$, where

$$d_{m+1} > \prod_{j=-1}^{m} r^{-\delta_j} \sum_{k=m+1}^{\infty} \left(\mathsf{n}(r)^{\delta_k} - 1\right) r^{s_k}.$$

Note that $c_m > r^{s_m}\left(\mathsf{n}(r)^{\delta_m} - 1\right) \prod_{j=-1}^{m-1} r^{-\delta_j} = \mathsf{n}(r)^{\delta_m} - 1$, where the equality holds by Remark 3.2. This implies that $c_m = h + H\mathsf{n}(r)^{\delta_m}$ with $H \ge 1$ and $h \in [\![0, \mathsf{n}(r)^{\delta_m} - 1]\!]$. Using the identity $\mathsf{n}(r)^{\delta_n} r^{s_n} = \mathsf{d}(r)^{\delta_n} r^{s_{n+1}}$ we obtain a factorization z_{m+1} from z_m in the following manner:

$$\sum_{i=0}^{m-1} c_i r^{s_i} + c_m r^{s_m} = \sum_{i=0}^{m-1} c_i r^{s_i} + h r^{s_m} + H\mathsf{d}(r)^{\delta_m} r^{s_{m+1}};$$

where

$$
\begin{aligned}
H\mathsf{d}(r)^{\delta_m} &= r^{-\delta_m}(c_m - h) \ge r^{-\delta_m}\left(c_m + 1 - \mathsf{n}(r)^{\delta_m}\right) \\
&> r^{-\delta_m}\left(1 - \mathsf{n}(r)^{\delta_m} + \prod_{j=-1}^{m-1} r^{-\delta_j} \sum_{k=m}^{\infty}\left(\mathsf{n}(r)^{\delta_k} - 1\right) r^{s_k}\right) \\
&= \prod_{j=-1}^{m} r^{-\delta_j} \sum_{k=m+1}^{\infty}\left(\mathsf{n}(r)^{\delta_k} - 1\right) r^{s_k}.
\end{aligned} \tag{4.3}
$$

The last equality in Equation (4.3) follows from Remark 3.2. By induction, we have $|\mathsf{Z}(K)| = \infty$, which implies that $M_{r,S}$ does not satisfy the ACCP by Theorem 3.10. This contradiction proves that our hypothesis is untenable; hence the series 4.2 does not converge. This, in turn, implies that the series $\sum_{k=0}^{\infty} \mathsf{n}(r)^{\delta_k} r^{s_k}$ does not converge either, and our result follows by [21, Theorem 3.39]. $\square$

Corollary 4.2. *Let $M_{r,S}$ be an atomic exponential Puiseux monoid with $r < 1$. If there exists $k \in \mathbb{N}$ such that $\delta_n < k$ for all $n \in \mathbb{N}$ then $M_{r,S}$ does not satisfy the ACCP.*

Note that Corollary 4.2 is a generalization of [6, Corollary 4.4]. On the other hand, the converse of Proposition 4.1 does not hold as the following example illustrates.

Example 4.3. Let $a, b \in \mathbb{N}$ such that $1 < a < b$. We start by constructing, iteratively, a sequence of rational numbers $(q_n)_{n\ge 1}$ converging to $\log_a b$ such that $\mathsf{n}(q_n) = \mathsf{d}(q_{n+1})$ and $1 < q_n < \log_a b$ for all $n \in \mathbb{N}$. Let $n_1 \in \mathbb{N}$ such that $1/n_1 < \log_a b - 1$. It is not hard to see that there exists a rational number q_1 such that $\log_a b - 1/n_1 \le q_1 < \log_a b$ and $\mathsf{d}(q_1) = n_1$. Note that $\mathsf{n}(q_1) > n_1$ since $q_1 > 1$. Now assume that $q_1, \dots, q_k \in \mathbb{Q}_{>1}$ have been defined such that $\log_a b - 1/\mathsf{d}(q_i) \le q_i < \log_a b$ for all $i \in [\![1, k]\!]$ and $\mathsf{n}(q_i) = \mathsf{d}(q_{i+1})$ for all $i \in [\![1, k-1]\!]$. Then there exists a rational number q_{k+1} such that $\log_a b - 1/\mathsf{n}(q_k) \le q_{k+1} < \log_a b$ and $\mathsf{d}(q_{k+1}) = \mathsf{n}(q_k) < \mathsf{n}(q_{k+1})$. It is easy to see that the sequence $(q_n)_{n\ge 1}$ satisfies the aforementioned requirements.

Consider the exponential Puiseux monoid $M_{r,S}$ where $r = a/b$ and $\delta_n = \mathsf{d}(q_{n+1})$ for all $n \in \mathbb{N}_0$. Clearly, $M_{r,S}$ is atomic by Proposition 3.7. On the other hand, note that $\delta_{n+1}/\delta_n = q_{n+1}$ for all $n \in \mathbb{N}_0$. Since $\delta_{n+1}/\delta_n < \log_{\mathsf{n}(r)} \mathsf{d}(r)$ for all $n \in \mathbb{N}_0$, it is not hard to see that $\mathsf{d}(r)^{\delta_n} > \mathsf{n}(r)^{\delta_{n+1}}$ for all $n \in \mathbb{N}_0$, which implies that $M_{r,S}$ does not satisfy the ACCP by virtue of the identity

$$\mathsf{n}(r)^{\delta_n} r^{s_n} = \left(\mathsf{d}(r)^{\delta_n} - \mathsf{n}(r)^{\delta_{n+1}}\right) r^{s_{n+1}} + \mathsf{n}(r)^{\delta_{n+1}} r^{s_{n+1}}.$$

Let $R = \log_{\mathsf{n}(r)} \mathsf{d}(r)$. Next, we proceed to show (through some cumbersome computations) that $\limsup_{n\to\infty} \delta_n/s_n = R - 1$ holds. We have

$$
\begin{aligned}
\limsup_{n\to\infty} \frac{\delta_n}{s_n} &= \limsup_{n\to\infty} \frac{\delta_n}{\delta_0 + \cdots + \delta_{n-1}} \quad \text{(by Remark 3.2)}\\
&= \limsup_{n\to\infty} \frac{\delta_n}{\delta_{n-1}} \left(\frac{1}{\frac{\delta_0}{\delta_{n-1}} + \cdots + \frac{\delta_{n-2}}{\delta_{n-1}} + 1} \right)\\
&= \limsup_{n\to\infty} \frac{\delta_n}{\delta_{n-1}} \left(\frac{1}{\frac{\delta_0}{\delta_1}\frac{\delta_1}{\delta_2}\cdots\frac{\delta_{n-2}}{\delta_{n-1}} + \cdots + \frac{\delta_{n-2}}{\delta_{n-1}} + 1} \right)\\
&\leq \limsup_{n\to\infty} R \left(\frac{1}{\frac{1}{R^{n-1}} + \cdots + \frac{1}{R} + 1} \right) \quad (\text{since } \delta_{n+1}/\delta_n < R \text{ for all } n \in \mathbb{N}_0)\\
&= \limsup_{n\to\infty} R \left(\frac{R^{n-1}}{1 + R + \cdots + R^{n-1}} \right)\\
&= \limsup_{n\to\infty} \frac{R^{n+1} - R^n}{R^n - 1} = R - 1.
\end{aligned}
$$

Now let $\epsilon > 0$. Since the sequence $(\delta_n/\delta_{n+1})_{n\geq 0}$ converges to R^{-1} and $\delta_n/\delta_{n+1} > R^{-1}$ for all $n \in \mathbb{N}_0$, there exists $t \in \mathbb{N}$ such that $\delta_k/\delta_{k+1} - R^{-1} < \epsilon/R$ for all $k \in \mathbb{N}_{\geq t}$. Let $T \in \mathbb{N}_{\geq t}$ such that $\delta_T/\delta_{T+1} \geq \delta_k/\delta_{k+1}$ for all $k \in \mathbb{N}_{\geq t}$. Thus,

$$
\begin{aligned}
\limsup_{n\to\infty} \frac{\delta_n}{s_n} &= \limsup_{n\to\infty} \frac{\delta_n}{\delta_0 + \cdots + \delta_{n-1}}\\
&= \limsup_{n\to\infty} \frac{\delta_n}{\delta_{n-1}} \left(\frac{1}{\frac{\delta_0}{\delta_{n-1}} + \cdots + \frac{\delta_{n-2}}{\delta_{n-1}} + 1} \right)\\
&= \limsup_{n\to\infty} \frac{\delta_n}{\delta_{n-1}} \left(\frac{1}{\frac{\delta_t}{\delta_{n-1}} + \cdots + \frac{\delta_{n-2}}{\delta_{n-1}} + 1} \right)\\
&= \limsup_{n\to\infty} \frac{\delta_n}{\delta_{n-1}} \left(\frac{1}{\frac{\delta_t}{\delta_{t+1}}\frac{\delta_{t+1}}{\delta_{t+2}}\cdots\frac{\delta_{n-2}}{\delta_{n-1}} + \cdots + \frac{\delta_{n-2}}{\delta_{n-1}} + 1} \right)\\
&\geq R \cdot \limsup_{n\to\infty} \left(\frac{1}{\frac{\delta_T}{\delta_{T+1}}^{n-t-1} + \cdots + \frac{\delta_T}{\delta_{T+1}} + 1} \right)\\
&= R \cdot \limsup_{n\to\infty} \frac{1 - \frac{\delta_T}{\delta_{T+1}}}{1 - \left(\frac{\delta_T}{\delta_{T+1}}\right)^{n-t}}
\end{aligned}
$$

$$= R \cdot \left(1 - \frac{\delta_T}{\delta_{T+1}}\right) = R \cdot \left(1 - \frac{1}{R} + \frac{1}{R} - \frac{\delta_T}{\delta_{T+1}}\right)$$
$$= R - 1 - R \cdot \left(\frac{\delta_T}{\delta_{T+1}} - \frac{1}{R}\right) \geq R - 1 - \epsilon.$$

Since ϵ is an arbitrary positive real number, $\limsup_{n\to\infty} \delta_n/s_n \geq R - 1$. Note that $\mathsf{n}(r)\limsup_{n\to\infty} \sqrt[s_n]{\mathsf{n}(r)^{\delta_n}} = \mathsf{n}(r)^R = \mathsf{d}(r)$, which implies that $M_{r,S}$ satisfies Equation (4.1). However, the monoid $M_{r,S}$ does not satisfy the ACCP as was already proved.

We now provide a sufficient condition for an exponential Puiseux monoid to satisfy the ACCP, but first let us prove a lemma.

Lemma 4.4. *Let $M_{r,S}$ be an atomic exponential Puiseux monoid, and let s_i be an element of S. Then $M_{r,S}$ satisfies the ACCP if and only if $M_{r,S-s_i}$ satisfies the ACCP.*

Proof. If $r \geq 1$ then our result follows trivially by [13, Theorem 5.6]. Consequently, one may assume without loss that $r < 1$. Suppose that $M_{r,S-s_i}$ does not satisfy the ACCP, and let $(x'_n)_{n\geq 1}$ be a sequence of elements of $M^\bullet_{r,S-s_i}$ such that $x'_n = x'_{n+1} + y'_n$ for some $y'_n \in M^\bullet_{r,S-s_i}$ and for all $n \in \mathbb{N}$. Now set $x_n = r^{s_i}x'_n$ and $y_n = r^{s_i}y'_n$ for all $n \in \mathbb{N}$. Note that $(x_n)_{n\geq 1}$ is a sequence of elements of $M^\bullet_{r,S}$ such that $x_n = x_{n+1} + y_n$ for some $y_n \in M^\bullet_{r,S}$. Then $M_{r,S}$ does not satisfy the ACCP, and the direct implication follows.

We use a similar argument for the reverse implication: Let $(x_n)_{n\geq 1}$ be a sequence of elements of $M^\bullet_{r,S}$ such that $x_n = x_{n+1} + y_n$ for some $y_n \in M^\bullet_{r,S}$ and for all $n \in \mathbb{N}$. Fix $K \in \mathbb{N}_0$, and let $C_K = \{m \in \mathbb{N} : mr^K \mid_{M_{r,S}} x_n \text{ for some } n \in \mathbb{N}\}$. Since $(x_n)_{n\geq 1}$ is a bounded sequence, C_K is a finite set. Consequently, there are infinitely many elements of the sequence $(x_n)_{n\geq 1}$ whose factorization of minimum length share the same coefficient for the atom r^K. This, in turn, implies that there is no loss in assuming that r^K does not show up in the factorization of minimum length of x_n for any $n \in \mathbb{N}$. By the same token, we may assume that, for each $n \in \mathbb{N}$, r^{s_m} does not show up in the factorization of minimum length of x_n for any $m \in [\![0, i]\!]$. Now set $x'_n = r^{-s_i}x_n$ and $y'_n = r^{-s_i}y_n$ for all $n \in \mathbb{N}$. It is not hard to see that $(x'_n)_{n\geq 1}$ is a sequence of elements of $M^\bullet_{r,S-s_i}$ such that $x'_n = x'_{n+1} + y'_n$ for some $y'_n \in M^\bullet_{r,S-s_i}$ and our proof concludes. □

Proposition 4.5. *Let $M_{r,S}$ be an atomic exponential Puiseux monoid with $r < 1$. If there exists $m \in \mathbb{N}_0$ such that $\mathsf{d}(r)^{\delta_n} < \mathsf{n}(r)^{\delta_{n+1}}$ for $n \geq m$ then $M_{r,S}$ satisfies the ACCP.*

Proof. By virtue of Lemma 4.4, one may assume that $\mathsf{d}(r)^{\delta_n} < \mathsf{n}(r)^{\delta_{n+1}}$ for all $n \in \mathbb{N}_0$. By way of contradiction, suppose that $M_{r,S}$ does not satisfy the ACCP. Then there exists a sequence $(x_n)_{n\geq 1}$ of elements of $M^\bullet_{r,S}$ such that $x_n = x_{n+1} + y_n$ for some $y_n \in$

$M^\bullet_{r,S}$ and for all $n \in \mathbb{N}$. Let $z_n = \sum_{i=0}^{m_n} c_{n,i}\, r^{s_i}$ with $m_n, c_{n,i} \in \mathbb{N}_0$ for each $i \in [\![0, m_n]\!]$ be the factorization of minimum length of x_n for each $n \in \mathbb{N}$. Take an arbitrary $n \in \mathbb{N}$. Clearly, x_n has no factorization of maximum length which implies that there exists $j \in [\![0, m_n]\!]$ such that $c_{n,j} \geq \mathsf{n}(r)^{\delta_j}$ by Lemma 3.8 (part (3)). Then $j = 0$; otherwise, z_n is not the factorization of minimum length of x_n given that $\mathsf{n}(r)^{\delta_j} > \mathsf{d}(r)^{\delta_{j-1}}$. Without loss of generality, we may assume that there exists $i_n \in [\![0, m_{n+1}]\!]$ such that $c_{n,i_n} = 0$ and $c_{n+1,i_n} \neq 0$.

We shall prove that $c_{n,0} > c_{n+1,0}$. Take $z \in \mathsf{Z}(y_n)$ and let $z'_1 = z_{n+1} + z \in \mathsf{Z}(x_n)$. By applying the identity $\mathsf{d}(r)^{\delta_k} r^{s_{k+1}} = \mathsf{n}(r)^{\delta_k} r^{s_k}$ with $k \in \mathbb{N}_0$ finitely many times, one can obtain a sequence $z'_1, z'_2, \ldots, z'_l$ of distinct factorizations of x_n such that $|z'_i| > |z'_{i+1}|$ for all $i \in [\![1, l-1]\!]$ and $z'_l = z_n$. Since $c_{n,i_n} = 0$ and $c_{n+1,i_n} \neq 0$ for some $i_n \in [\![0, l_{n+1}]\!]$, we have $z'_1 \neq z'_l$. Note that, due to the nature of the transformation $\mathsf{d}(r)^{\delta_k} r^{s_{k+1}} = \mathsf{n}(r)^{\delta_k} r^{s_k}$, where $k \in \mathbb{N}_0$, the coefficient of r^0 in z'_i is less than or equal to the corresponding coefficient in z'_{i+1} for all $i \in [\![1, l-1]\!]$. Since $c_{n,i} < \mathsf{d}(r)^{\delta_{i-1}} < \mathsf{n}(r)^{\delta_i}$ for each $i \in [\![1, m_n]\!]$ by Lemma 3.8, we have that $c_{n,0}$ is strictly bigger than the coefficient of r^0 in z'_{l-1} which, in turn, implies that $c_{n,0} > c_{n+1,0}$. This is a contradiction as there is no infinite strictly decreasing sequence of positive integers. Hence $M_{r,S}$ satisfies the ACCP. $\square$

Remark 4.6. Note that if there exists $m \in \mathbb{N}$ such that $\mathsf{d}(r)^{\delta_n} > \mathsf{n}(r)^{\delta_{n+1}}$ for $n \geq m$ then $M_{r,S}$ does not satisfy the ACCP since

$$\mathsf{n}(r)^{\delta_n} r^{s_n} = \left(\mathsf{d}(r)^{\delta_n} - \mathsf{n}(r)^{\delta_{n+1}}\right) r^{s_{n+1}} + \mathsf{n}(r)^{\delta_{n+1}} r^{s_{n+1}}$$

as we pointed out in Example 4.3.

Using Proposition 4.5 we can easily construct many different examples of exponential Puiseux monoids satisfying the ACCP. On the other hand, Remark 4.6 provides us with a large number of exponential Puiseux monoids not satisfying the ACCP. Consider the following example.

Example 4.7. Let $p(x)$ be a non-constant polynomial with positive integer coefficients, and consider the exponential Puiseux monoid $M_{r,S}$ with $r \in \mathbb{Q}_{<1}$ and $\delta_n = p(n)$ for all $n \in \mathbb{N}_0$. Note that $\mathsf{d}(r)^{\delta_n} > \mathsf{n}(r)^{\delta_{n+1}}$ if and only if $\frac{\ln \mathsf{d}(r)}{\ln \mathsf{n}(r)} > \frac{\delta_{n+1}}{\delta_n}$. Since

$$\lim_{n\to\infty} \frac{\delta_{n+1}}{\delta_n} = \lim_{n\to\infty} \frac{p(n+1)}{p(n)} = 1,$$

there exists $N \in \mathbb{N}$ such that $\frac{\ln \mathsf{d}(r)}{\ln \mathsf{n}(r)} > \frac{\delta_{n+1}}{\delta_n}$ for all $n \geq N$. Then $M_{r,S}$ does not satisfy the ACCP.

5. Exponential Puiseux Semirings

Exponential Puiseux monoids are generalizations of rational cyclic Puiseux monoids which are closed under multiplication and thus semirings. Motivated by this, we study the exponential Puiseux monoids that are also closed under the standard multiplication of $\mathbb{Q}$. Throughout this section, we allow exponential Puiseux monoids to be finitely generated.

A *commutative semiring* S is a nonempty set endowed with two binary operations (called addition and multiplication and denoted by $+$ and $\cdot$ respectively) satisfying the following conditions:

(1) $(S,+)$ is a monoid with identity element 0;
(2) $(S,\cdot)$ is a commutative semigroup with identity element 1;
(3) for $a,b,c \in S$ we have $(a+b)\cdot c = a\cdot c + b\cdot c$;
(4) $0 \cdot a = 0$ for all $a \in S$.

The more general definition of a 'semiring' does not assume that the semigroup $(S,\cdot)$ is commutative; however, this more general type of algebraic object is not of interest for us here. Accordingly, from now on we use the single term *semiring*, implicitly assuming commutativity. For extensive background information on semirings, we refer readers to the monograph [12] of J. S. Golan.

During the last decade, semirings have received some attention in the context of factorization theory. For instance, in [4] the authors investigated, for quadratic algebraic integers τ, the elasticity of the multiplicative structure of the semiring $\mathbb{N}_0[\tau]$ and in [19], factorization aspects of semigroup semirings were studied. More recently, a systematic investigation of the factorizations of the multiplicative structure of the semiring $\mathbb{N}_0[x]$ was provided in [3]. Finally and most relevant to this work, the atomicity of the additive and multiplicative structures of rational cyclic semirings were considered in [5] and [2, Section 3], respectively.

We now show that the subsets N of $\mathbb{N}_0$ for which an exponential Puiseux monoid $M_{r,N}$ is a semiring are precisely the numerical monoids.

Proposition 5.1. *Take $r \in \mathbb{Q}_{>0} \setminus \mathbb{N}$ such that $\mathsf{n}(r) \neq 1$ and let N be a nonempty subset of $\mathbb{N}_0$. Consider the Puiseux monoid $(S_{r,N},+) = \langle r^n \mid n \in N\rangle$. Then N is isomorphic to a numerical monoid if and only if $S_{r,N}$ is a semiring.*

Proof. First, we verify that $(S_{r,N},+)$ is atomic with $\mathcal{A}(S_{r,N},+) = \{r^n \mid n \in N\}$. It follows from Proposition 3.7 that $\mathcal{A}(S_r,+) = \{r^n \mid n \in \mathbb{N}_0\}$, where $(S_r,+)$ is the rational cyclic semiring parameterized by r. Because the inclusion $\mathcal{A}(S_r,+) \cap S_{r,N} \subseteq \mathcal{A}(S_{r,N},+)$ holds, $r^n \in \mathcal{A}(S_{r,N},+)$ for every $n \in N$. This, in turn, implies that $(S_{r,N},+)$ is atomic.

To argue the direct implication suppose that N is isomorphic to a numerical monoid. Since N is closed under addition, the set $\{r^n \mid n \in N\}$ is closed under multiplication. This immediately implies that $(S_{r,N},+)$ is multiplicatively closed. As $0 \in N$, the multiplicative semigroup $(S^\bullet_{r,N},\cdot)$ has an identity. Hence $S_{r,N}$ is a semiring.

To prove the reverse implication suppose that $S_{r,N}$ is a semiring. Take $n,m \in N$. So we have $r^{n+m} = r^n r^m \in S_{r,N}$. Since $r^{n+m} \in \mathcal{A}(S_r,+) \cap S_{r,N}$, we also have that

$r^{n+m} \in \mathcal{A}(S_{r,N}, +)$. This guarantees that $n + m \in N$. Hence N is closed under addition. Because $S_{r,N}$ is a semiring, $1 \in S_{r,N}$. So we can write

$$1 = \sum_{i=1}^{k} c_i r^{n_i} \tag{5.1}$$

for some index $k \in \mathbb{N}$, coefficients $c_1, \dots, c_k \in \mathbb{N}$, and exponents $n_1, \dots, n_k \in N$. One can assume without loss that $n_1 < n_2 < \cdots < n_k$. After cleaning denominators, we obtain $\mathsf{d}(r)^{n_k} = \sum_{i=1}^{k} c_i \mathsf{n}(r)^{n_i} \mathsf{d}(r)^{n_k - n_i}$, whose left-hand side is not divisible by $\mathsf{n}(r)$. As a result $n_1 = 0$, which implies that $0 \in N$. Hence N is isomorphic to a numerical monoid. $\square$

Before continuing, let us make a definition to avoid long descriptions.

Definition 5.2. Let $r \in \mathbb{Q}_{>0}$ and $N \subseteq \mathbb{N}_0$. We say that $(S_{r,N}, +) = \langle r^n \mid n \in N \rangle$ is an *exponential Puiseux semiring* provided that N is a numerical monoid.

Next we prove that whether or not the ACCP, the BFP, and the FFP hold for the multiplicative monoid of an exponential Puiseux semiring can be completely determined by whether or not the same property holds for the corresponding rational cyclic semiring. First let us collect some lemmas.

Lemma 5.3. *Let $S_{r,N}$ be an exponential Puiseux semiring with $\mathsf{n}(r) \neq 1$. Then $(S^\bullet_{r,N}, \cdot)$ is a monoid.*

Proof. It is clear that the commutative semigroup $(S^\bullet_{r,N}, \cdot)$ is cancellative. So it suffices to verify that it is reduced. By way of contradiction, suppose that $(S^\bullet_{r,N}, \cdot)$ is not reduced. So there exists $q \in S^\bullet_{r,N} \setminus \{1\}$ such that q^{-1} is also an element of $S^\bullet_{r,N}$. It follows from Proposition 3.7 that either q or q^{-1} is not an additive atom of the exponential Puiseux monoid $(S_{r,N}, +)$. This implies that 1 is not an atom of $(S_{r,N}, +)$. Indeed if, for example, $q^{-1} = x + y$ then $1 = qq^{-1} = qx + qy$. Then the exponential Puiseux monoid $(S_{r,N}, +)$ is antimatter. However, this contradicts Proposition 3.7. $\square$

Lemma 5.4. *Let $S_{r,N}$ be an exponential Puiseux semiring with $r < 1 < \mathsf{n}(r)$. Then for each $x \in S^\bullet_{r,N}$ there exists $m \in \mathbb{N}$ such that $r^n \nmid_{(S^\bullet_{r,N}, \cdot)} x$ for any $n \in \mathbb{N}_{\geq m}$.*

Proof. Since $r < 1 < \mathsf{n}(r)$, the monoid $(S_{r,N}, +)$ is atomic by Proposition 3.7. Suppose, by way of contradiction, there exists $x \in S^\bullet_{r,N}$ for which the set $N' = \{n \in \mathbb{N} : r^n \mid_{(S^\bullet_{r,N}, \cdot)} x\}$ has infinite cardinality. Then, for each $n \in N'$, there exists an additive factorization $\sum_{i=n}^{m_n} d_i r^i \in \mathsf{Z}_{(S_{r,N}, +)}(x)$ for some index $m_n \in \mathbb{N}_{\geq n}$, coefficients $d_n, \dots, d_{m_n} \in \mathbb{N}_0$, and exponents $n, \dots, m_n \in \mathbb{N}$. This implies that $\mathsf{n}(r)^n \mid \mathsf{n}(x)$ for all $n \in N'$, which is a contradiction given that N' has infinite cardinality. This contradiction proves that our hypothesis is untenable, and our result follows. $\square$

Now we are in a position to prove the main result of this section.

Theorem 5.5. *Let $S_{r,N}$ be an exponential Puiseux semiring, and let S_r be the corresponding rational cyclic semiring. The following statements hold.*

(1) *$(S^\bullet_{r,N}, \cdot)$ satisfies the ACCP if and only if $(S^\bullet_r, \cdot)$ satisfies the ACCP.*
(2) *$(S^\bullet_{r,N}, \cdot)$ satisfies the BFP if and only if $(S^\bullet_r, \cdot)$ satisfies the BFP.*
(3) *$(S^\bullet_{r,N}, \cdot)$ satisfies the FFP if and only if $(S^\bullet_r, \cdot)$ satisfies the FFP.*

Proof. If either $\mathsf{n}(r) = 1$ or $\mathsf{d}(r) = 1$ then $S^\bullet_{r,N} = S^\bullet_r$, so there is no loss in assuming that $\mathsf{n}(r) \neq 1$ and $r \notin \mathbb{N}$. The reverse implication of (1) follows from the fact that submonoids of monoids satisfying the ACCP also satisfy the ACCP, while the reverse implications of (2) and (3) follow for [10, Corollary 1.3.3] and [10, Corollary 1.5.7], respectively. On the other hand, if $r > 1$ then $(S^\bullet_r, \cdot)$ is atomic by [2, Proposition 3.11], and since the atoms of $(S^\bullet_r, \cdot)$ are strictly bigger than 1, $(S^\bullet_r, \cdot)$ is an FFM. Consequently, we may assume without loss of generality that $r < 1 < \mathsf{n}(r)$.

To prove the direct implication of (1), suppose that $(S^\bullet_r, \cdot)$ does not satisfy the ACCP. Then there exist sequences $(x_m)_{m\geq 1}$ and $(y_m)_{m\geq 1}$ of elements of $S^\bullet_r \setminus \{1\}$ such that $x_m = x_{m+1} \cdot y_m$ for all $m \in \mathbb{N}$. First note that by Lemma 5.4 there is $k \in \mathbb{N}$ so that $y_m > 1$ for each $m > k$. Thus we may assume that $y_m > 1$ for each $m \in \mathbb{N}$. Also note that the sequence $(y_m)_{m\geq 1}$ converges to 1. Indeed, if there exists a rational number $q > 1$ such that $y_m > q$ for infinitely many indices m then it is not hard to see that 0 is a limit point of the divisors of x_1 in $S^\bullet_r$, contradicting Lemma 5.4. Finally, since the sequence $(y_m)_{m\geq 1}$ converges to 1, there exists $h \in \mathbb{N}$ such that $y_m \in S^\bullet_{r,N}$ for $m > h$. Now consider the sequence $(r^{F(N)+1} x_m)_{m\geq 1}$, where $F(N)$ is the Frobenius number of the numerical monoid N. Clearly, $r^{F(N)+1} x_m \in S^\bullet_{r,N}$ for all $m \in \mathbb{N}$. Moreover, $r^{F(N)+1} x_m = (r^{F(N)+1} x_{m+1}) \cdot y_m$ for each $m \in \mathbb{N}$, and since $y_m \in S^\bullet_{r,N}$ for $m > h$, it follows that $(S^\bullet_{r,N}, \cdot)$ does not satisfy the ACCP, from which (1) follows.

To prove the direct implications of (2) and (3), note first that $(S^\bullet_r, \cdot)$ is atomic. Indeed, since $(S^\bullet_{r,N}, \cdot)$ is a BFM, it satisfies the ACCP. Then, by (1), the monoid $(S^\bullet_r, \cdot)$ also satisfies the ACCP, so it is atomic. Moreover, $r \in \mathcal{A}(S^\bullet_r, \cdot)$. Now let $x \in S^\bullet_r \setminus \{1\}$ and consider a factorization $z = a_1 \cdots a_n \in \mathsf{Z}_{(S^\bullet_r, \cdot)}(x)$. By Lemma 5.4, there exists a positive integer m, which does not depend on z, such that $|\{a_i : a_i = r\}| < m$; on the other hand,

$$\left|\left\{a_i \in S^\bullet_r \setminus S^\bullet_{r,N} \mid a_i \neq r\right\}\right| < \left\lceil \log_d \left(x r^{-m}\right)\right\rceil,$$

where $d = 1 + r^{F(N)}$. In other words, a factorization $z \in \mathsf{Z}_{(S^\bullet_r, \cdot)}(x)$ contains as factors at most $h = m + \lceil \log_d(x r^{-m})\rceil$ atoms of $(S^\bullet_r, \cdot)$ that are not elements of $S^\bullet_{r,N}$. Consequently, $y = r^{h(F(N)+1)} x$ is an element of $S^\bullet_{r,N}$. It is not hard to see that if $\mathsf{L}_{(S^\bullet_r, \cdot)}(x)$ is unbounded then $\mathsf{L}_{(S^\bullet_{r,N}, \cdot)}(y)$ is also unbounded, which concludes the proof of (2). As for the direct implication of (3), if $|\mathsf{Z}_{(S^\bullet_r, \cdot)}(x)| = \infty$ then there are infinitely many atoms of $(S^\bullet_r, \cdot)$ dividing x by [10, Proposition 1.5.5]. Since r is the only element of $\mathcal{A}(S^\bullet_r, \cdot)_{<1}$, the set $A(x) = \{a \in \mathcal{A}(S^\bullet_r, \cdot)_{>1} : a \mid_{(S^\bullet_r, \cdot)} x\}$ has infinite cardinality. This, in turn, implies that

y has infinitely many divisors in $(S^{\bullet}_{r,N}, \cdot)$ as $r^{(F(N)+1)} \cdot a$ is a divisor of y in $(S^{\bullet}_{r,N}, \cdot)$ for all $a \in A(x)$. Then (3) follows from [10, Proposition 1.5.5].

$\square$

Corollary 5.6. *Let $(S_{r,N}, +)$ be an exponential Puiseux monoid with $\mathsf{n}(r) > 1$. Then $(S^{\bullet}_{r,N}, \cdot)$ satisfies the ACCP provided that $\mathsf{d}(r) = p^k$ for some $p \in \mathbb{P}$ and $k \in \mathbb{N}$.*

Proof. If $r > 1$ then our result follows from [2, Proposition 3.11] and Theorem 5.5. Consequently, we may assume that $r < 1$, which implies that $(S_r, +)$ is atomic by Proposition 3.7. By way of contradiction, suppose that $(S^{\bullet}_r, \cdot)$ does not satisfy the ACCP. As we have already established in the proof of Theorem 5.5, there exist sequences $(x_m)_{m \ge 1}$ and $(y_m)_{m \ge 1}$ of elements of $S^{\bullet}_r \setminus \{1\}$ such that $y_m > 1$, $x_m = x_{m+1} \cdot y_m$ for all $m \in \mathbb{N}$, and $(y_m)_{m \ge 1}$ converges to 1. Then $x_1 y_1^{-1} \cdots y_n^{-1} \in S^{\bullet}_r$ for all $n \in \mathbb{N}$. For each $j \in \mathbb{N}$, let $z_j = \sum_{i=0}^{n_j} c_{j,i} r^i$ be the factorization of minimum length of y_j in $(S_r, +)$ with $n_j, c_{j,i} \in \mathbb{N}_0$. By virtue of Lemma 5.4, one can assume that $c_{j,0} \neq 0$ for all $j \in \mathbb{N}$. Thus,

$$y_j = \frac{c_{j,0}\mathsf{d}(r)^{n_j} + \sum_{i=1}^{n_j} c_{j,i}\mathsf{n}(r)^i \mathsf{d}(r)^{n_j - i}}{\mathsf{d}(r)^{n_j}} = \frac{h \cdot \mathsf{n}(y_j)}{p^{k n_j}}$$

for some $h \in \mathbb{N}$. It is easy to see that if $p \mid h\, \mathsf{n}(y_j)$ then $p \mid c_{j,n_j}$. Since $c_{j,n_j} < \mathsf{d}(r)$ by Lemma 3.8, there exists a prime number q such that $q \mid h\, \mathsf{n}(y_j)$ and $q \nmid \mathsf{d}(r)$. Consequently, $\mathsf{n}(x_1)$ is divisible by infinitely many (counting repetitions) prime numbers, a contradiction. Therefore, $(S^{\bullet}_{r,N}, \cdot)$ satisfies the ACCP by Theorem 5.5. $\square$

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