# ON THE SUBATOMICITY OF POLYNOMIAL SEMIDOMAINS

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Abstract. A semidomain is an additive submonoid of an integral domain that is closed under multiplication and contains the identity element. Although factorizations and divisibility in atomic domains have been systematically studied for more than thirty years, the same aspects in the more general context of atomic semidomains have been investigated just recently. Here we study subatomicity in the context of semidomains; that is, we study semidomains satisfying divisibility properties weaker than atomicity. We mostly focused on the Furstenberg properties, which is due to P. Clark and motivated by the work of H. Furstenberg on the infinitude of primes, and the almost atomic and quasi-atomic properties, introduced by J. Boynton and J. Coykendall in the context of divisibility in integral domains. We investigate these three properties in the context of semidomain, paying special attention to whether they ascend from a semidomain to its polynomial and Laurent polynomial semidomains.

## 1. Introduction

A semidomain is an additive submonoid of an integral domain that is closed under multiplication and contains the identity element. Let $S$ be a semidomain, and set $S^* := S \setminus \{0\}$; that is, $S^*$ is the multiplicative monoid of $S$. We say that $S$ is atomic provided that every non-invertible element of $S^*$ can be written as a product of atoms (i.e., irreducible elements). Factorizations in atomic domains have been systematically studied for more than three decades, considerably motivated by the landmark paper [3] by D. D. Anderson, D. F. Anderson, and M. Zafrullah. However, factorizations in the more general context of atomic semidomains have been investigated just recently by N. R. Baeth, S. T. Chapman, and the authors [5,16]. In the present paper, we investigate atomic properties that are weaker than being atomic in the setting of semidomains. We put special emphasis on the ascent of such properties from the semidomain $S$ to the polynomial semidomain $S[x]$ and the Laurent polynomial semidomain $S[x^{\pm 1}]$.

Special cases of polynomial semidomains and Laurent polynomial semidomains have been the focus of a great deal of attention lately in the factorization theory community. For instance, methods to factorize polynomials in $\mathbb{N}_0[X]$ were investigated by H. Brunotte in [7] and, more recently, F. Campanini and A. Facchini in [8] carried out a more systematic investigation of factorizations in the semidomain $\mathbb{N}_0[X]$. More generally, semigroup semirings were studied by V. Ponomarenko in [24] from the factorization perspective. The arithmetic of polynomial semidomains with coefficients in $\mathbb{R}_{\geq 0}$ has also been considered; for instance, P. Cesarz et al. in [9] studied the elasticity of $\mathbb{R}_{\geq 0}[X]$, where $\mathbb{R}_{\geq 0}$ is the set of nonnegative real numbers.

Positive semidomains, that is, subsemirings of $\mathbb{R}_{\geq 0}$, have been actively studied in the last few years. Factorizations in positive semidomains consisting of rational numbers were considered in [10] by Chapman et al. and then in [1] by S. Albizu-Campos et al. The same semidomains were studied in [4] by Baeth and the first author in connection with factorizations of matrices. This in turn motivated the paper [5] by Baeth et al., where several examples of positive semidomains were constructed. Positive

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semidomains can be also produced as valuations of polynomial and Laurent polynomial semidomains, and such valuations have also been investigated recently: the arithmetic of factorizations of $\mathbb{N}_0[\alpha]$, where $\alpha$ is a positive algebraic number, was studied recently for rational valuations in [10] by Chapman et al. and for algebraic valuations in [12,22] by J. Correa-Morris et al. On the other hand, the atomic structure of the algebraic valuations of the Laurent polynomial semidomain $\mathbb{N}_0[X^{\pm 1}]$ has been recently studied by S. Zhu in [26].

Following the terminology introduced by P. Clark in [11], we say that the semidomain $S$ is a Furstenberg semidomain if every nonunit element in $S^*$ is divisible by an atom. It is clear that each atomic semidomain is a Furstenberg semidomain. Furstenberg domains have been studied by N. Lebowitz-Lockard in [23] in connection with the properties of almost atomicity and quasi-atomicity, which we define in the next two paragraphs. In addition, Furstenberg domains have been recently considered in [20] by the first author and Zafrullah in connection with idf-domains (i.e., integral domains whose elements have only finitely many irreducible divisors up to associates). Finally, Furstenberg domains have been considered in [19, Section 5] by the first author and B. Li in the context of integer-valued polynomials. In Section 3, we prove that the property of being Furstenberg ascends from the semidomain $S$ to both $S[x]$ and $S[x^{\pm 1}]$. We also construct an example of a Furstenberg semidomain that is neither an integral domain nor an atomic semidomain.

The semidomain $S$ is said to be almost atomic provided that, for every nonunit $b \in S^*$, there exist atoms $a_1, \dots, a_k$ of $S^*$ such that $a_1 \cdots a_k b$ factors into atoms in $S^*$. Observe that each atomic semidomain is almost atomic. The notion of almost atomicity was introduced in [6] by J. Boynton and J. Coykendall, and it was later studied in parallel to various other subatomic properties in [23] by Lebowitz-Lockard. In Section 4, we study almost atomicity in the context of semidomains. Unlike the Furstenberg property, we will show that the property of being almost atomic does not ascend in general from the semidomain $S$ to neither $S[x]$ nor $S[x^{\pm 1}]$. However, it does ascend if we impose that the coefficients of certain polynomials over $S$ have a maximal common divisor, as we will prove.

As the notion of almost atomicity, that of quasi-atomicity was introduced in [6] and further studied in [23] in the context of integral domains. Motivated by this, we say that the semidomain $S$ is quasi-atomic provided that, for every nonunit $b \in S^*$, there exists an element $a$ of $S^*$ such that $ab$ factors into atoms in $S^*$. It follows directly from definitions that each almost atomic semidomain is quasi-atomic. In Section 5, we provide a simple ideal-theoretical characterization of quasi-atomic semidomains. In addition, as for the property of being almost atomic, we prove that the property of being quasi-atomic ascends from the semidomain $S$ to both $S[x]$ and $S[x^{\pm 1}]$ under the same divisibility conditions referred to in the previous paragraph.

## 2. Background

In this section, we introduce the notation and terminology necessary to follow our exposition. Reference material on factorization theory and semiring theory can be found in the monographs [14] by A. Geroldinger and F. Halter-Koch and [17] by J. Golan, respectively. Throughout this paper, we let $\mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ denote the set of integers, rational numbers, and real numbers, respectively. Additionally, we let $\mathbb{N}$ denote the set of positive integers, and we set $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Given $r \in \mathbb{R}$ and $S \subseteq \mathbb{R}$, we set $S_{<r} := \{s \in S \mid s < r\}$, and we define $S_{>r}$ and $S_{\geq r}$ in a similar way. For $m, n \in \mathbb{Z}$, we denote by $[\![m, n]\!]$ the discrete interval from $m$ to $n$, that is, $[\![m, n]\!] := \{k \in \mathbb{Z} \mid m \leq k \leq n\}$.

2.1. **Monoids.** A *monoid*[1] is defined here to be a semigroup with identity that is cancellative and commutative. Since our interest lies in the multiplicative structure of certain semirings, we will use multiplicative notation for monoids unless we specify otherwise. For the rest of this section, let $M$ be

[1] The standard definition of a monoid does not assume the cancellative and the commutative conditions.

a monoid with identity 1. We set $M^\bullet \coloneqq M \setminus \{1\}$, and we let $\mathscr{U}(M)$ denote the group of units (i.e., invertible elements) of $M$. In addition, we let $M_{\text{red}}$ denote the quotient $M/\mathscr{U}(M)$, which is also a monoid. We say that $M$ is *reduced* provided that $\mathscr{U}(M)$ is the trivial group, in which case we identify $M_{\text{red}}$ with $M$. The *Grothendieck group* of $M$, denoted here by $\mathscr{G}(M)$, is the abelian group (unique up to isomorphism) satisfying that any abelian group containing a homomorphic image of $M$ also contains a homomorphic image of $\mathscr{G}(M)$. For a subset $S$ of $M$, we let $\langle S \rangle$ denote the smallest submonoid of $M$ containing $S$, and if $M = \langle S \rangle$, then we say that $S$ is a *generating set* of $M$.

For $b, c \in M$, it is said that $b$ *divides* $c$ *in* $M$ if there exists $b' \in M$ such that $c = bb'$, in which case we write $b \mid_M c$, dropping the subscript precisely when $M = (\mathbb{N}, \times)$. We say that $b, c \in M$ are *associates* if $b \mid_M c$ and $c \mid_M b$. A submonoid $N$ of $M$ is *divisor-closed* if for each $b \in N$ and $d \in M$ the relation $d \mid_M b$ implies that $d \in N$. Let $S$ be a nonempty subset of $M$. An element $d \in M$ is called a *common divisor* of $S$ provided that $d \mid_M s$ for all $s \in S$. A common divisor $d$ of $S$ is called a *greatest common divisor* of $S$ if $d$ is divisible by all common divisors of $S$. Also, a common divisor of $S$ is called a *maximal common divisor* if every greatest common divisor of $S/d$ belongs to $\mathscr{U}(M)$. We let $\gcd_M(S)$ (resp., $\text{mcd}_M(S)$) denote the set consisting of all greatest common divisors (resp., maximal common divisors) of $S$. The monoid $M$ is called a *GCD-monoid* (resp., an *MCD-monoid*) if each finite nonempty subset of $M$ has a greatest common divisor (resp., a maximal common divisor).

An element $a \in M \setminus \mathscr{U}(M)$ is called an *atom* if for all $b, c \in M$ the equality $a = bc$ implies that either $b \in \mathscr{U}(M)$ or $c \in \mathscr{U}(M)$. We let $\mathscr{A}(M)$ denote the set consisting of all atoms of $M$. The monoid $M$ is called *atomic* if each element in $M \setminus \mathscr{U}(M)$ can be written as a (finite) product of atoms. One can readily check that $M$ is atomic if and only if $M_{\text{red}}$ is atomic. Assume for the rest of this paragraph that $M$ is atomic. We let $\mathsf{Z}(M)$ denote the free (commutative) monoid on $\mathscr{A}(M_{\text{red}})$. The elements of $\mathsf{Z}(M)$ are called *factorizations*, and if $z = a_1 \cdots a_\ell \in \mathsf{Z}(M)$ for some $a_1, \dots, a_\ell \in \mathscr{A}(M_{\text{red}})$, then $\ell$ is called the *length* of $z$, which is denoted by $|z|$. Let $\pi \colon \mathsf{Z}(M) \to M_{\text{red}}$ be the unique monoid homomorphism satisfying that $\pi(a) = a$ for all $a \in \mathscr{A}(M_{\text{red}})$. For each $b \in M$, the sets

$$\mathsf{Z}_M(b) \coloneqq \pi^{-1}(b\mathscr{U}(M)) \subseteq \mathsf{Z}(M) \qquad \text{and} \qquad \mathsf{L}_M(b) \coloneqq \{|z| : z \in \mathsf{Z}_M(b)\} \subseteq \mathbb{N}_0 \tag{2.1}$$

are of crucial importance to study the atomicity of $M$. When there seems to be no risk of ambiguity, we drop the subscript $M$ from the notations in (2.1).

Following [11], we say that a monoid is *Furstenberg* provided that every nonunit has a divisor that is an atom. On the other hand, extending the terminology in [6], a monoid $M$ is called *almost atomic* (resp., *quasi-atomic*) provided that, for every nonunit $c \in M$, there exists $a_1, \dots, a_k \in \mathscr{A}(M)$ (resp., $b \in M$) such that $a_1 \cdots a_k c$ (resp., $bc$) can be written as a product of atoms in $M$.

2.2. **Semirings.** A *commutative semiring* $S$ is a nonempty set endowed with two binary operations denoted by '$+$' and '$\cdot$' and called *addition* and *multiplication*, respectively, such that the following conditions hold:

- $(S, +)$ is a monoid with its identity element denoted by $0$;
- $(S, \cdot)$ is a commutative semigroup with an identity element denoted by $1$;
- $b \cdot (c + d) = b \cdot c + b \cdot d$ for all $b, c, d \in S$;
- $0 \cdot b = 0$ for all $b \in S$.

With notation as in the previous definition and for any $b, c \in S$, we write $bc$ instead of $b \cdot c$ when there seems to be no risk of confusion. A more general notion of a 'semiring' $S$ does not assume that the semigroup $(S, \cdot)$ is commutative. However, this more general type of algebraic objects is not of interest in the scope of this paper. Accordingly, from now on we will use the single term *semiring*, tacitly assuming the commutativity of both operations. A subset $S'$ of a semiring $S$ is a *subsemiring* of $S$ if $(S', +)$ is a submonoid of $(S, +)$ that contains 1 and is closed under multiplication. Observe that every subsemiring of $S$ is a semiring.

**Definition 2.1.** We say that a semiring $S$ is a *semidomain* provided that $S$ is a subsemiring of an integral domain.

Let $S$ be a semidomain. We set $S^* := (S \setminus \{0\}, \cdot)$ and call it the *multiplicative monoid* of $S$. In [16, Example 2.3], we exhibited a semiring $S$ that is not a semidomain but still $S^*$ is a monoid. Following standard notation from ring theory, we refer to the units of the multiplicative monoid $S^*$ simply as *units* of $S$, and we denote the set of units of $S$ by $S^\times$. We never consider in this paper the units of the monoid $(S, +)$, so the use of the term 'unit' in the context of the semidomain $S$ should not generate any ambiguity. In addition, we write $\mathscr{A}(S)$ instead of $\mathscr{A}(S^*)$ for the set of atoms of the multiplicative monoid $S^*$, while we let $\mathscr{A}_+(S)$ denote the set of atoms of the additive monoid $(S, +)$. Finally, for any $b, c \in S$ such that $b$ divides $c$ in $S^*$, we write $b \mid_S c$ instead of $b \mid_{S^*} c$. The following lemma was stated and proved in [16], but we include its proof here for the sake of completeness.

**Lemma 2.2.** *[16, Lemma 2.2] For a semiring $S$, the following conditions are equivalent.*

*(a) The multiplication of $S$ extends to $\mathscr{G}(S)$ turning $\mathscr{G}(S)$ into an integral domain.*

*(b) $S$ is a semidomain.*

*Proof.* (a) $\Rightarrow$ (b): This is clear.

(b) $\Rightarrow$ (a): Let $S$ be a semidomain, and suppose that $S$ is embedded into an integral domain $R$. We can identify the Grothendieck group $\mathscr{G}(S)$ of $(S, +)$ with the subgroup $\{r - s \mid r, s \in S\}$ of the underlying additive group of $R$. It is easy to see then that $\mathscr{G}(S)$ is closed under the multiplication it inherits from $R$, and it contains the multiplicative identity because $0, 1 \in S$. Hence $\mathscr{G}(S)$ is an integral domain having $S$ as a subsemiring. $\square$

We say that a semidomain $S$ is *atomic* (resp., *Furstenberg*, *almost atomic*, *quasi-atomic*) if its multiplicative monoid $S^*$ is atomic (resp., Furstenberg, almost atomic, quasi-atomic). We denote by $\langle \mathscr{A}(S) \rangle$ the submonoid of $S^*$ generated by the atoms and units of $S$. A subset $I$ of $S$ is an *ideal*[2] of $S$ provided that $(I, +)$ is a submonoid of $(S, +)$ and $IS \subseteq I$. We say that an ideal $I$ is *prime* if $I \neq S$ and, for $b, c \in S$, the containment $bc \in I$ implies that either $b \in I$ or $c \in I$. Although a semidomain $S$ can be embedded into an integral domain $R$, the semidomain $S$ may not inherit any (sub)atomic property from $R$ as, after all, the integral domain $\mathbb{Q}[x]$ is a UFD but it contains as a subring the integral domain $\mathbb{Z} + x\mathbb{Q}[x]$, which is not even quasi-atomic (see [23, Lemma 17]).

The set consisting of all polynomial expressions with coefficients in the semiring $S$ is also a semiring, which we denote by $S[x]$ and call the *semiring of polynomials over* $S$. Additionally, if $S$ is a semidomain embedded into an integral domain $R$, then it is clear that $S[x]$ is also a semidomain, and the elements of $S[x]$ are, in particular, polynomials in $R[x]$. Consequently, when $S$ is a semidomain all the standard terminology for polynomials can be applied to elements of $S[x]$, including *constant polynomial*, *degree*, *order*, and *leading coefficient*. Observe that $S^*$ is a divisor-closed submonoid of $S[x]^*$ and, therefore, $S[x]^\times = S^\times$ and $\mathscr{A}(S[x]) \cap S = \mathscr{A}(S)$. Following [25], we say that a nonzero polynomial in $S[x]$ is *indecomposable* if it is not a product of two nonconstant polynomials in $S[x]$.

Following the terminology in [5], we call a subsemiring of $\mathbb{R}$ consisting of nonnegative numbers a *positive semiring*. The fact that underlying additive monoids of positive semirings are reduced makes them more tractable. The reader can check the recent paper [5] for several examples of positive semirings. The class of semidomains clearly contains those of integral domains and positive semirings.

[2] Golan [17] defines an ideal in a more restrictive way: if $I$ is an ideal of a semiring $S$, then by definition $I \neq S$. Consequently, any result we cite from [17] is interpreted here as a statement about the proper ideals of a semiring.

## 3. Furstenberg Semidomains

In this section, we analyze under which conditions the Furstenberg property ascends from a semidomain to its semidomain of (Laurent) polynomials.

The Furstenberg property is, evidently, a relaxation of being atomic, and the reader can find interesting examples of non-atomic Furstenberg domains in [19, Section 5] and [23, Section 4]. We now provide an example of a non-atomic positive semiring that is Furstenberg.

**Example 3.1.** Let $p$ be a prime number such that there exists $k \in \mathbb{N}$ satisfying the inequalities $1 < k < p-1$, and consider the monoid

$$M = \left\langle p\left(\frac{k}{p}\right)^n, \ (p-k)\left(\frac{k}{p}\right)^n \ \middle| \ n \in \mathbb{N}_0 \right\rangle.$$

By virtue of [18, Theorem 6.2] and the identity

$$p\left(\frac{k}{p}\right)^n = (p-k)\left(\frac{k}{p}\right)^n + p\left(\frac{k}{p}\right)^{n+1},$$

we have $\mathscr{A}(M) = \{(p-k)(k/p)^n \mid n \in \mathbb{N}_0\}$, which implies that $M$ is a non-atomic Furstenberg monoid. Now consider the additive monoid $E(M) := \langle e^m \mid m \in M\rangle$, which is free on the set $\{e^m \mid m \in M\}$ by the Lindemann-Weierstrass theorem stating that, for distinct algebraic numbers $\alpha_1, \dots, \alpha_n$, the set $\{e^{\alpha_1}, \dots, e^{\alpha_n}\}$ is linearly independent over the algebraic numbers. Observe that $E(M)$ is closed under multiplication and, consequently, it is a positive semiring[3]. We argue that $E(M)$ is a non-atomic semidomain that is Furstenberg. Clearly, the multiplicative submonoid $e(M) := \{e^m \mid m \in M\}$ is isomorphic to $(M, +)$, which implies that $e(M)$ is not atomic. Since $e(M)$ is a divisor-closed submonoid of $E(M)$, the semiring $E(M)$ is not atomic either. Now let $x \in E(M)$, and write $x = c_1e^{m_1} + \dots + c_ke^{m_k}$, where $c_1, \dots, c_k \in \mathbb{N}$ and $m_1, \dots, m_k \in M$. Thus,

$$x = e^m(c_1e^{m_1-m} + \dots + c_ke^{m_k-m}) = e^m y_1 \cdots y_t,$$

where $m$ is a common divisor of $m_1, \dots, m_k$ in $M$ and $y_i \in E(M) \setminus \{1\}$ for each $i \in [\![1, t]\!]$. If $m > 0$, then there exists an atom $a \in \mathscr{A}(M)$ such that $a \mid_M m$ which, in turn, implies that $e^a \mid_{E(M)} x$. Note that $e^a \in \mathscr{A}(E(M))$. On the other hand, if $m_1, \dots, m_k$ have no nonzero common divisor in $M$, then no element of the form $e^m$ (with $m \in M^\bullet$) divides $y_i$ in $E(M)$ for each $i \in [\![1, t]\!]$, which implies that $t \le \log_2(c_1 + \dots + c_k)$; in this case, $x \in \langle \mathscr{A}(E(M))\rangle$. Therefore $E(M)$ is Furstenberg, which concludes our argument.

**Remark 3.2.** For an example of a Furstenberg semidomain that is not quasi-atomic, see Proposition **??** and Remark **??**.

Next we show that the Furstenberg property ascends from a semidomain to its semidomain of (Laurent) polynomials.

**Theorem 3.3.** *For a semidomain $S$, the following statements are equivalent.*

*(a) $S$ is Furstenberg.*
*(b) $S[x]$ is Furstenberg.*
*(c) $S[x^{\pm 1}]$ is Furstenberg.*

*Proof.* (a) $\Rightarrow$ (b): Suppose that $S$ is Furstenberg. Take a nonzero nonunit $f \in S[x]$. If $f \in S$, then the fact that $S^*$ is a Furstenberg monoid guarantees the existence of $a \in \mathscr{A}(S)$ with $a \mid_S f$. As $S^*$ is a divisor-closed submonoid of $S[x]$, the element $a$ is also an atom of $S[x]$, and so $f$ is divisible by an atom in $S[x]$. Now suppose that $\deg f \ge 1$. Take the largest $m \in \mathbb{N}$ such that $f = rg_1 \cdots g_m$ for some

[3]This construction was introduced in [5].

$r \in S^*$ and $g_1, \dots, g_m \in S[x]$ with $\deg g_i \ge 1$ for every $i \in [\![1, m]\!]$. If $g_1 \in \mathscr{A}(S[x])$ we are done. If $g_1$ is reducible, then we can write $g_1 = s(g_1/s)$, where $s \in S^*$ is a nonunit element dividing $g_1$ in $S[x]$. As $s$ is a nonunit of $S^*$, it must be divisible by an atom $b$ in $S$. Since $b$ is an atom of $S[x]$, $f$ is divisible by an atom in $S[x]$. Hence $S[x]$ is Furstenberg.

(b) $\Rightarrow$ (c): First, observe that every irreducible $f$ in $S[x]$ with $\operatorname{ord} f = 0$ is an irreducible in $S[x^{\pm 1}]$. Now take a nonzero nonunit $g \in S[x^{\pm 1}]$, and write $g = x^d h$ for some $d \in \mathbb{Z}$ and $h \in S[x]$ with $\operatorname{ord} h = 0$. As $g$ is not a unit in $S[x^{\pm 1}]$, we see that $h$ is not a unit in $S[x]$, and so there is an $a \in \mathscr{A}(S[x])$ such that $a \mid_{S[x]} h$. Note that $\operatorname{ord} a = 0$ because the same holds for $h$. Thus, $a$ is an irreducible in $S[x^{\pm 1}]$ dividing $g$. Therefore $S[x^{\pm 1}]$ is Furstenberg.

(c) $\Rightarrow$ (a): This follows from the fact that $\{sx^n \mid s \in S^* \text{ and } n \in \mathbb{Z}\}$ is a divisor-closed submonoid of $S[x^{\pm 1}]$ whose reduced monoid is isomorphic to that of $S^*$. □

Observe that Theorem 3.3 can help us identifying non-atomic semidomains that are Furstenberg. For instance, Roitman [25] provided the first example of an atomic domain $A$ such that $A[x]$ is not atomic. By virtue of Theorem 3.3, we can now assert that $A[x]$ is a non-atomic Furstenberg domain.

## 4. Almost Atomic Semidomains

Clearly, atomic semidomains are almost atomic. However, the reverse implication does not hold as the next example illustrates.

**Example 4.1.** Let $S = \{(1/2)^{n+2} \mid n \in \mathbb{N}\}$, and let $p_1, p_2, \dots$ be the (ascending) sequence of prime numbers greater than 4. Set

$$S' := \left\{ \frac{1}{p_n}, \ \frac{1}{2^{n+2}} + \frac{1}{2} - \frac{1}{p_n} \ \middle| \ n \in \mathbb{N} \right\},$$

and consider the monoid $M = \langle S \cup S' \rangle$. It is not hard to see that $\mathscr{A}(M) = S'$, which implies that $M$ is not atomic. Note that every element $x \in M$ can be written as $x = c(1/2)^N + x'$, where $c \in \mathbb{N}_0$, $N \in \mathbb{N}_{\ge 3}$, and $x' \in \langle \mathscr{A}(M) \rangle$; thus, we have $x + c = x' + 2c\big((1/2)^{N+1} + 1/2\big)$. Therefore $M$ is almost atomic. Now pick an arbitrary element $x \in M$. Evidently, we can write

$$x = \frac{c}{2^n} + \frac{c_1}{p_{n_1}} + \dots + \frac{c_k}{p_{n_k}} + d_1 \left( \frac{1}{2^{m_1+2}} + \frac{1}{2} - \frac{1}{p_{m_1}} \right) + \dots + d_l \left( \frac{1}{2^{m_l+2}} + \frac{1}{2} - \frac{1}{p_{m_l}} \right), \tag{4.1}$$

where either $c = 0$ or $\gcd(c, 2^n) = 1$, $0 < c_i < p_{n_i}$, $0 < d_j < p_{m_j}$, and $p_{n_i} \neq p_{m_j}$ for $i \in [\![1, k]\!]$ and $j \in [\![1, l]\!]$; we also assume that if $p_{n_i} = p_{n_{i'}}$ (resp., $p_{m_j} = p_{m_{j'}}$) for $i, i' \in [\![1, k]\!]$ (resp., for $j, j' \in [\![1, l]\!]$), then we have $i = i'$ (resp., $j = j'$). We claim that $x \in M$ has finitely many representations of the form (4.1). Consider another representation of $x$ having this form

$$x = \frac{\alpha}{2^\ell} + \frac{\alpha_1}{p'_{n_1}} + \dots + \frac{\alpha_t}{p'_{n_t}} + \beta_1 \left( \frac{1}{2^{m'_1+2}} + \frac{1}{2} - \frac{1}{p_{m'_1}} \right) + \dots + \beta_r \left( \frac{1}{2^{m'_r+2}} + \frac{1}{2} - \frac{1}{p_{m'_r}} \right). \tag{4.2}$$

After cancelling similar terms in expressions (4.1) and (4.2), we may assume that either $\alpha = 0$ or $c = 0$ and $p_{n_i} \neq p'_{n_j}$ for any $i \in [\![1, k]\!]$ and any $j \in [\![1, t]\!]$. Then $p_{m'_j} = p_{n_1}$ for some $j \in [\![1, r]\!]$; otherwise, we would obtain a contradiction after clearing denominators. This, in turn, implies that $\beta_j = p_{n_1} - c_1$. Using an inductive argument, it is not hard to see that, if we fix the representation (4.1), then the representation (4.2) is completely determined by the similar terms we can cancel in both expressions, which proves our claim. Consequently, $M$ is an MCD-monoid. Indeed, if $(\alpha_n)_{n \in \mathbb{N}}$ is a nonconstant sequence of common divisors of $x_1, \dots, x_m \in M$ for some $m \in \mathbb{N}_{>1}$ such that $\alpha_n \mid_M \alpha_{n+1}$ for all $n \in \mathbb{N}$, then $x_j$ has infinitely many representations of the form (4.1) for some $j \in [\![1, m]\!]$. As in Example 3.1, consider the positive semiring $E(M)$. As we mentioned before, the multiplicative monoid $e(M)$ is isomorphic to $(M, +)$, which implies that $e(M)$ is not atomic. Since $e(M)$ is a divisor-closed submonoid of $E(M)$, the monoid $E(M)$ is not atomic either. Let $x$ be a nonzero nonunit element of $E(M)$, and write $x = c_1 e^{m_1} + \cdots + c_k e^{m_k}$, where $c_1, \ldots, c_k \in \mathbb{N}$ and $m_1, \ldots, m_k \in M$. There is no loss in assuming that $c_1, \ldots, c_k$ are relatively prime positive integers. Let $m \in \operatorname{mcd}_M(m_1, \ldots, m_k)$, and notice that

$$x = e^m(c_1 e^{m_1 - m} + \cdots + c_k e^{m_k - m}) = e^m y_1 \cdots y_t,$$

where $y_i \in E(M) \setminus \{0, 1\}$ for each $i \in [\![1, t]\!]$. Since no element of the form $e^{m'}$ (with $m' \in M^\bullet$) divides $y_i$ in $E(M)$ for any $i \in [\![1, t]\!]$, we have that the inequality $t \le \log_2(c_1 + \cdots + c_k)$ holds which, in turn, implies that there is no loss in assuming that $y_1, \ldots, y_t \in \langle \mathscr{A}(E(M)) \rangle$. Since $M$ is almost atomic, there exists $m^* \in \langle \mathscr{A}(M) \rangle$ such that $m^* + m \in \langle \mathscr{A}(M) \rangle$. We can conclude that $xe^{m^*} \in \langle \mathscr{A}(E(M)) \rangle$. Hence $E(M)$ is almost atomic.

For almost atomic semidomains, we have a result similar to Theorem 3.3.

**Theorem 4.2.** *For a semidomain $S$, each of the following statements implies the next.*

*(a) $S$ is almost atomic and $\operatorname{mcd}(s_1, \ldots, s_n) \neq \emptyset$ for any coefficients $s_1, \ldots, s_n$ of an indecomposable polynomial in $S[x]$.*

*(b) $S[x]$ is almost atomic.*

*(c) $S[x^{\pm 1}]$ is almost atomic.*

*Moreover, conditions (b) and (c) are equivalent.*

*Proof.* (a) $\Rightarrow$ (b): Let $f$ be a nonzero nonunit element of $S[x]$ such that $\deg f = n$ for some $n \in \mathbb{N}_0$. If $n = 0$, then our result follows from the fact that $S$ is almost atomic and $\langle \mathscr{A}(S) \rangle \subseteq \langle \mathscr{A}(S[x]) \rangle$. Consequently, we may assume that $n > 0$. Write $f = f_1 \cdots f_m$, where $f_i \in S[x]$ and $\deg f_i > 0$ for each $i \in [\![1, m]\!]$. Without loss of generality, assume that $m$ is maximal. Fix an arbitrary $j \in [\![1, m]\!]$. Since $m$ is maximal, the polynomial $f_j$ is indecomposable in $S[x]$. Now write $f_j = s_1 x^{n_1} + \cdots + s_k x^{n_k}$ with coefficients $s_1, \ldots, s_k \in S^*$ and exponents $n_1, \ldots, n_k \in \mathbb{N}_0$. Take $s \in \operatorname{mcd}(s_1, \ldots, s_k)$, and note that $s^{-1} f_j \in \mathscr{A}(S[x])$. Since $S$ is almost atomic, there exists $\beta_j \in \langle \mathscr{A}(S) \rangle$ such that $s\beta_j \in \langle \mathscr{A}(S) \rangle$. Consequently, there exists $\beta := \prod_{i=1}^m \beta_i \in \langle \mathscr{A}(S[x]) \rangle$ such that $\beta f \in \langle \mathscr{A}(S[x]) \rangle$. Hence $S[x]$ is almost atomic.

(b) $\Rightarrow$ (c): First, observe that $\mathscr{A}(S[x]) \setminus \{x\} \subseteq \mathscr{A}(S[x^{\pm 1}])$. In fact, assume towards a contradiction that $f = g \cdot h$, where $f \in \mathscr{A}(S[x]) \setminus \{x\}$ and $g, h \in S[x^{\pm 1}] \setminus S[x^{\pm 1}]^\times$. Then $f = x^n g'(x) \cdot x^m h'(x)$ for some $m, n \in \mathbb{Z}$ and $g', h' \in S[x]$ such that $\operatorname{ord} g' = \operatorname{ord} h' = 0$. It is not hard to see that $n + m = 0$, which implies that either $g'$ or $h'$ is a unit of $S[x]$. This, in turn, implies that either $g$ or $h$ is a unit of $S[x^{\pm 1}]$, a contradiction. Now let $f$ be a nonzero nonunit element of $S[x^{\pm 1}]$, and write $f = x^k g$, where $k \in \mathbb{Z}$, $g \in S[x]$, and $\operatorname{ord} g = 0$. Since $S[x]$ is almost atomic, there exists $h \in \langle \mathscr{A}(S[x]) \rangle$ such that $hg \in \langle \mathscr{A}(S[x]) \rangle$. Observe that $\langle \mathscr{A}(S[x]) \rangle \subseteq \langle \mathscr{A}(S[x^{\pm 1}]) \rangle$ because $\mathscr{A}(S[x]) \setminus \{x\} \subseteq \mathscr{A}(S[x^{\pm 1}])$ and $\{x\} \cup S[x]^\times \subseteq S[x^{\pm 1}]^\times$. Since $x^k \in S[x^{\pm 1}]^\times$ for every $k \in \mathbb{Z}$, our result follows.

(c) $\Rightarrow$ (b): Let $f$ be a nonzero nonunit element of $S[x]$. Given that $x \in \mathscr{A}(S[x])$, there is no loss in assuming that $\operatorname{ord} f = 0$. There exists $g \in \langle \mathscr{A}(S[x^{\pm 1}]) \rangle$ such that $gf \in \langle \mathscr{A}(S[x^{\pm 1}]) \rangle$. As we mentioned above, $x^k \in S[x^{\pm 1}]^\times$ for every $k \in \mathbb{Z}$, so we may assume $\operatorname{ord} g = 0$ (consequently, we have $\operatorname{ord} gf = 0$ since $S$ contains no zero divisors). If $g \in S[x^{\pm 1}]^\times$, then $g \in S^\times$, which implies that $g \in \langle \mathscr{A}(S[x]) \rangle$. Otherwise, we can write $g = g_1 \cdots g_n$, where $g_i \in \mathscr{A}(S[x^{\pm 1}])$ for each $i \in [\![1, n]\!]$. Again, without loss of generality, we can assume that $\operatorname{ord} g_i = 0$ for every $i \in [\![1, n]\!]$ which, in turn, implies that $g_i \in \mathscr{A}(S[x])$ for all $i \in [\![1, n]\!]$. Hence $g \in \langle \mathscr{A}(S[x]) \rangle$. By the same argument, $gf \in \langle \mathscr{A}(S[x]) \rangle$. Therefore $S[x]$ is almost atomic. $\square$

In general, we do not know whether the polynomial extension of an almost atomic semidomain is almost atomic, so we raise the following conjecture.

**Conjecture 4.3.** *There exists an almost atomic semidomain $S$ such that $S[x]$ is not almost atomic.*

Following [13], we say that a semidomain $M$ is *antimatter* provided that the set of atoms $\mathscr{A}(S)$ is empty. We conclude this section by providing an example of an antimatter semidomain $S$ whose polynomial extension $S[x]$ is almost atomic.

**Example 4.4.** Consider the positive semiring $S = \{0\} \cup \mathbb{Q}_{\geq 1}$, which is antimatter ([5, Example 3.10]). We shall prove that $S[x]$ is almost atomic. Take an arbitrary nonzero nonunit element $f \in S[x]$, and observe that we can write $f = cg$, where $c \in \mathbb{Q}_{\geq 1}$ and $g = c_n x^n + \cdots + c_1 x + c_0$ with $c_j = 1$ for some $j \in [\![0, n]\!]$. Then our problem reduces to show that every element of $\mathbb{Q}_{\geq 1}$ and every polynomial $g = c_n x^n + \cdots + c_1 x + c_0$ with $c_j = 1$ for some $j \in [\![0, n]\!]$ can be expressed as a quotient of a (finite) product of atoms of $S[x]$. Let us start with the latter case: write $g = f_1 \cdots f_m$ as a product of indecomposable polynomials $f_1, \dots, f_m \in S[x]$. Note that every $c \in \mathbb{Q}_{\geq 1}$ dividing all coefficients $c_0, c_1, \dots, c_n$ is, necessarily, a unit of $S$, which means that $f_i$ is an atom of $S[x]$ for each $i \in [\![1, m]\!]$. To tackle the first case, observe that every $c \in \mathbb{Q}_{\geq 1}$ can be written as

$$c = \frac{(cx+1)(x+c)}{x^2 + \left(c + \frac{1}{c}\right) x + 1}, \tag{4.3}$$

where each of the polynomials in Equation (4.3) factors into atoms of $S[x]$ by the previous argument. Thus $S[x]$ is almost atomic.

## 5. Quasi-atomic Semidomains

As mentioned in the introduction, in this section, we provide an ideal-theoretical characterization of quasi-atomic semidomains and study when quasi-atomicity ascends from a semidomain to its semidomain of (Laurent) polynomials.

While the fact that almost atomic semidomains are quasi-atomic follows immediately from definitions, no simple counterexamples to the reverse implication is known. Next we generalize a construction that can be found in [23, Example 7] for integral domains.

**Example 5.1.** Before proceeding with our example, let us introduce a couple of definitions. A *semifield* is a semiring in which every nonzero element has a multiplicative inverse, while a *bounded factorization semidomain* is an atomic semidomain satisfying that $\mathsf{L}(b)$ is finite for all $b \in M$. Now let $S$ be a bounded factorization semidomain that is not a semifield (e.g, $\mathbb{N}_0$). Let $K$ be a field properly containing the field of fractions of $\mathscr{G}(S)$, and consider the semidomain $R = S[x] + x^2 K[x]$. Take an arbitrary $f = c_n x^n + \cdots + c_1 x + c_0 \in R^*$ with $n \in \mathbb{N}_0$, and suppose that $\operatorname{ord} f = m$ for some $m \in \mathbb{N}_0$.

We shall prove that $f \in \langle \mathscr{A}(R) \rangle$ if and only if $c_m \in S$. Assume that $c_m \notin S$, and write $f = g_1 \cdots g_t$ with $g_1, \dots, g_t \in R^*$. For some $j \in [\![1, t]\!]$, the coefficient corresponding to the term $x^{\operatorname{ord} g_j}$ in $g_j$ is not an element of $S$. Consequently, every element of $S^*$ divides $g_j$ in $R$. Observe that $R^\times = S^\times$. Since $S$ is not a semifield, some nonunit of $S$ divides $g_j$ in $R$; in other words, $f \notin \langle \mathscr{A}(R) \rangle$. As for the reverse implication, write $f = g_1 \cdots g_t$, where $g_i \notin R^\times$ for any $i \in [\![1, t]\!]$. If $m = 0$ (resp., $m = 1$), then the inequality $t \leq n + \mathsf{L}(c_0)$ (resp., $t \leq n + \mathsf{L}(c_1)$) holds. Indeed, for every $i \in [\![1, t]\!]$, we have that either $\deg g_i > 0$ or $g_i$ is a divisor of $c_0$ (resp., $c_1$) in $S$ that is not a unit. Consequently, if either $m = 0$ or $m = 1$, then $f \in \langle \mathscr{A}(R) \rangle$. On the other hand, if $m > 1$, then $f = x^{m-1} g$ with $g = c_n x^{n-m+1} + c_{n-1} x^{n-m} + \cdots + c_m x$, and the reverse implication follows from the fact that $x \in \mathscr{A}(R)$.

Observe now that if $c_m \notin S$, then $f \cdot (1/c_m) x^2 \in \langle \mathscr{A}(R) \rangle$, which implies that $R$ is quasi-atomic. On the other hand, if $c_m$ is an element of $K$ that is not in the field of fractions of $\mathscr{G}(S)$, then $fg \notin \langle \mathscr{A}(R) \rangle$ for any $g \in \langle \mathscr{A}(R) \rangle$. Consequently, the semidomain $R$ is not almost atomic.

We are now in a position to characterize quasi-atomic semidomains. To do so, we mimick the proof of [23, Theorem 8].

**Theorem 5.2.** *A semidomain $S$ is quasi-atomic if and only if every nonzero prime ideal of $S$ contains an irreducible element.*

*Proof.* Suppose that $S$ is quasi-atomic, and let $P$ be a nonzero prime ideal of $S$. Take a nonzero $x \in P$ (clearly, $x \notin S^\times$). Since $S$ is quasi-atomic, there exist $\beta \in S^*$ and $a_1, \dots, a_n \in \mathscr{A}(S)$ such that $x\beta = a_1 \cdots a_n$. As a consequence, we have that $a_i \in P$ for some $i \in [\![1, n]\!]$. As for the remaining implication, assume towards a contradiction that there exists $x \in S^*$ such that $xS \cap \langle \mathscr{A}(S) \rangle = \emptyset$. Let $P$ be an ideal of $S$ that is maximal among those disjoint from $\langle \mathscr{A}(S) \rangle$. By virtue of [17, Proposition 7.12], the ideal $P$ is prime and, clearly, it contains no irreducible elements. This contradiction concludes our proof. □

For quasi-atomic semidomains, we have a result similar to Theorem 4.2.

**Theorem 5.3.** *For a semidomain $S$, each of the following statements implies the next.*

- *(a) $S$ is quasi-atomic and $\mathrm{mcd}(s_1, \dots, s_n) \neq \emptyset$ for any coefficients $s_1, \dots, s_n$ of an indecomposable polynomial in $S[x]$.*
- *(b) $S[x]$ is quasi-atomic.*
- *(c) $S[x^{\pm 1}]$ is quasi-atomic.*

*Moreover, conditions (b) and (c) are equivalent.*

*Proof.* (a) $\Rightarrow$ (b): Let $f$ be a nonzero nonunit element of $S[x]$ such that $\deg f = n$ for some $n \in \mathbb{N}_0$. If $n = 0$, then our result follows from the fact that $S$ is quasi-atomic and $\langle \mathscr{A}(S) \rangle \subseteq \langle \mathscr{A}(S[x]) \rangle$. Consequently, we may assume that $n > 0$. Write $f = f_1 \cdots f_m$, where $f_i \in S[x]$ and $\deg f_i > 0$ for each $i \in [\![1, m]\!]$. Without loss of generality, assume that $m$ is maximal. Fix an arbitrary $j \in [\![1, m]\!]$. Since $m$ is maximal, the polynomial $f_j$ is indecomposable. Now write $f_j = s_1 x^{n_1} + \dots + s_k x^{n_k}$ with coefficients $s_1, \dots, s_k \in S^*$ and exponents $n_1, \dots, n_k \in \mathbb{N}_0$. Take $s \in \mathrm{mcd}(s_1, \dots, s_k)$, and note that $s^{-1} f_j \in \mathscr{A}(S[x])$. Since $S$ is quasi-atomic, there exists $\beta_j \in S^*$ such that $s\beta_j \in \langle \mathscr{A}(S) \rangle$. Consequently, there exists $\beta := \prod_{i=1}^m \beta_i \in S[x]^*$ such that $\beta f \in \langle \mathscr{A}(S[x]) \rangle$. Therefore $S[x]$ is quasi-atomic.

(b) $\Rightarrow$ (c): We already established that $\mathscr{A}(S[x]) \setminus \{x\} \subseteq \mathscr{A}(S[x^{\pm 1}])$. Now let $f$ be a nonzero nonunit element of $S[x^{\pm 1}]$, which can be written as $f = x^k g$, where $k \in \mathbb{Z}$, $g \in S[x]$, and $\mathrm{ord}\, g = 0$. Since $S[x]$ is quasi-atomic, there exists $h \in S[x]^*$ such that $hg \in \langle \mathscr{A}(S[x]) \rangle$. Note that $\langle \mathscr{A}(S[x]) \rangle \subseteq \langle \mathscr{A}(S[x^{\pm 1}]) \rangle$ because $\mathscr{A}(S[x]) \setminus \{x\} \subseteq \mathscr{A}(S[x^{\pm 1}])$ and $\{x\} \cup S[x]^\times \subseteq S[x^{\pm 1}]^\times$. Since $x^k \in S[x^{\pm 1}]^\times$ for every $k \in \mathbb{Z}$, our result follows.

(c) $\Rightarrow$ (b): Let $f$ be a nonzero nonunit element of $S[x]$. Given that $x \in \mathscr{A}(S[x])$, there is no loss in assuming that $\mathrm{ord}\, f = 0$. There exists $g \in S[x^{\pm 1}]^*$ such that $gf \in \langle \mathscr{A}(S[x^{\pm 1}]) \rangle$. As we mentioned above, $x^k \in S[x^{\pm 1}]^\times$ for every $k \in \mathbb{Z}$, so we may assume $\mathrm{ord}\, g = 0$ (which, in turn, implies that $\mathrm{ord}\, gf = 0$). If $gf \in S[x^{\pm 1}]^\times$, then $gf \in S^\times$, which implies that $gf \in \langle \mathscr{A}(S[x]) \rangle$. Otherwise, we can write $gf = g_1 \cdots g_n$, where $g_i \in \mathscr{A}(S[x^{\pm 1}])$ for each $i \in [\![1, n]\!]$. Again, without loss of generality, we can assume that $\mathrm{ord}\, g_i = 0$ for every $i \in [\![1, n]\!]$ which, in turn, implies that $g_i \in \mathscr{A}(S[x])$ for all $i \in [\![1, n]\!]$. Hence $gf \in \langle \mathscr{A}(S[x]) \rangle$. Therefore $S[x]$ is quasi-atomic. □

As a corollary of Theorem 5.3, we obtain that, in a GCD-semidomain $S$, quasi-atomicity ascends from $S$ to its semidomain of (Laurent) polynomials. The following result sheds light upon this observation.

**Proposition 5.4.** *Let $M$ be a (cancellative and commutative) monoid. Then $M$ is a UFM if and only if it is quasi-atomic and GCD.*

*Proof.* The direct implication clearly holds. As for the reverse implication, it is known that an atomic GCD-monoid is a UFM (see, for example, [21, Section 10.7]). Then our problem reduces to show that $M$ is atomic. Let $x$ be a nonzero nonunit element of $M$. Since $M$ is quasi-atomic, there exists $x' \in M^\bullet$ such that $xx' \in \langle \mathscr{A}(M) \rangle$. Consequently, we have $xx' = up_1 \cdots p_n$, where $u \in \mathscr{U}(M)$ and $p_1, \dots, p_n \in \mathscr{A}(M)$.

For every $i \in [\![1, n]\!]$, we have that each $p_i$ is prime by [15, Theorem 6.7(2)], which implies that either $p_i \,|_M\, x$ or $p_i \,|_M\, x'$. Since $M$ is cancellative and $x$ is not a unit, we have that $x \in \langle \mathscr{A}(M) \rangle$. Hence $M$ is atomic, which concludes our argument. $\square$

**Corollary 5.5.** *Let $S$ be a semidomain. Then $S$ is a UFS if and only if it is quasi-atomic and GCD.*

To ensure that a GCD-monoid is a UFM, some sort of subatomic property needs to be assumed as the following example illustrates.

**Example 5.6.** Let $\alpha = \frac{\sqrt{5}-1}{2}$, and consider the additive monoid $\mathbb{N}_0[\alpha]$. Observe that $\alpha$ is an algebraic number with minimal polynomial $m_\alpha(X) = X^2 + X - 1$. Since $1 \notin \mathscr{A}_+(\mathbb{N}_0[\alpha])$, the monoid $\mathbb{N}_0[\alpha]$ is antimatter by [12, Theorem 4.1]. Next we show that $\mathbb{N}_0[\alpha]$ is a GCD-monoid. We start by proving that $\gcd(m\alpha^n, k\alpha^{n+1}) = \min(m\alpha^n, k\alpha^{n+1})$ for all $k, m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Let

$$S = \left\{(m,k) \in \mathbb{N} \times \mathbb{N} \mid \gcd(m\alpha^n, k\alpha^{n+1}) \neq \min(m\alpha^n, k\alpha^{n+1}) \text{ for some } n \in \mathbb{N}_0\right\}.$$

By way of contradiction, assume that $S$ is nonempty. Let $(m', k') \in S$ such that $m' + k'$ is minimal. Clearly, there exists $n' \in \mathbb{N}_0$ such that $\gcd(m'\alpha^{n'}, k'\alpha^{n'+1}) \neq \min(m'\alpha^{n'}, k'\alpha^{n'+1})$. Observe that $k' > m' > 0$ since the equality $\alpha^{n'} = \alpha^{n'+1} + \alpha^{n'+2}$ holds. Thus,

$$\gcd((k'-m')\alpha^{n'+1}, m'\alpha^{n'+2}) = \min((k'-m')\alpha^{n'+1}, m'\alpha^{n'+2})$$

which, in turn, implies that $m'\alpha^{n'+1} + \min((k'-m')\alpha^{n'+1}, m'\alpha^{n'+2})$ is a common divisor of $m'\alpha^{n'}$ and $k'\alpha^{n'+1}$ in $\mathbb{N}_0[\alpha]$. Now if the inequality $(k'-m')\alpha^{n'+1} < m'\alpha^{n'+2}$ holds, then $\gcd(m'\alpha^{n'}, k'\alpha^{n'+1}) = k'\alpha^{n'+1}$, which is a contradiction. We obtain a similar contradiction if $(k'-m')\alpha^{n'+1} \geq m'\alpha^{n'+2}$. Consequently, $S$ is an empty set. Let $x, y$ be nonzero elements of $\mathbb{N}_0[\alpha]$. Since $\alpha^n = \alpha^{n+1} + \alpha^{n+2}$ for all $n \in \mathbb{N}_0$, it is not hard to see that there exist $m \in \mathbb{N}$ and $c_1, c_2, c_3, c_4 \in \mathbb{N}_0$ such that $x = c_1\alpha^m + c_2\alpha^{m+1}$ and $y = c_3\alpha^m + c_4\alpha^{m+1}$. We may assume that $c_1 \geq c_3$. If $c_2 \geq c_4$ then it follows readily that $\gcd(x,y) = \min(x,y)$. On the other hand, if $c_2 < c_4$ then $c_3\alpha^m + c_2\alpha^{m+1} + \gcd((c_1-c_3)\alpha^m, (c_4-c_2)\alpha^{m+1})$ is a common divisor of $x$ and $y$ in $\mathbb{N}_0[\alpha]$. Since $\gcd((c_1-c_3)\alpha^m, (c_4-c_2)\alpha^{m+1}) = \min((c_1-c_3)\alpha^m, (c_4-c_2)\alpha^{m+1})$, a simple computation shows that $\gcd(x,y) = \min(x,y)$. By [15, Corollary 6.3], $\mathbb{N}_0[\alpha]$ is a GCD-monoid.

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