On Higher Dimensional Milnor Frames

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AMS Southeastern Sectionals,
Contributed Papers,
March 18, 2023

Introductory Material

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If ∇ is the Levi-Civita connection w.r.t g, for each $i, j, k \in \{1, \ldots, n\}$

$$g(\nabla_{X_i} X_j, X_k) = c_{ij}^\ell g_{\ell k} - c_{jk}^\ell g_{\ell i} + c_{ki}^\ell g_{\ell j}$$

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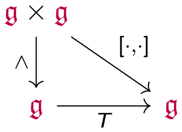
The Ricci tensor is defined to be

$$\mathrm{Ric}_g(X_i, X_j) = g(\nabla_{X_j} \nabla_{X_i} X_i - \nabla_{X_i} \nabla_{X_j} X_i + \nabla_{[X_i, X_j]} X_i, X_j)$$

3-dimensional Unimodular Lie Algebras

Remark 1: Lie Groups are Orientable Manifolds.



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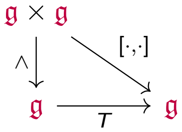
Remark 2: Let $\mathfrak{g}$ be a 3-dimensional metric Lie algebra (a Lie algebra with a left invariant metric) with fixed orientation. For two linearly independent vectors $U, V \in \mathfrak{g}$, define the cross product $U \wedge V$ to be the unique orthogonal positively oriented vector such that

$$g(U \wedge V, U \wedge V) = |U \wedge V|^2 = g(U,U)g(V,V) - g(U,V)^2$$



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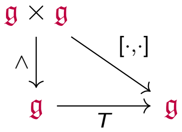
Remark 3: Let $(\mathfrak{g}, g)$ be a 3-dimensional metric Lie algebra with an orientation. By the universal property of alternating bilinear maps, there exists a linear operator $T : \mathfrak{g} \to \mathfrak{g}$ such that the diagram below commutes:



Definition

Let $\mathfrak{g}$ *be a Lie algebra. For each* $X \in \mathfrak{g}$*, define the linear operator* $ad_X : \mathfrak{g} \to \mathfrak{g}$, $U \mapsto [X, U]$. *We say that* $\mathfrak{g}$ *is unimodular if* $ad_X = 0$ *for any* $X \in \mathfrak{g}$.

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Lemma (4.1 Milnor (1976), pg. 305)

Let $(\mathfrak{g}, g)$ be a 3-dimensional metric Lie algebra with an orientation. Let $T : \mathfrak{g} \to \mathfrak{g}^1$ be the linear operator defined by $T(U \wedge V) = [U, V]$. Then $\mathfrak{g}$ is unimodular if and only if the linear transformation T is self-adjoint.

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Recall: If a linear transformation T is self-adjoint then there exists an orthonormal frame of eigenvectors.

Corollary (Milnor (1976), pg. 305)

If $(\mathfrak{g}, g)$ *is a unimodular metric Lie algebra, then there exists an orthonormal frame* $\{X_1, X_2, X_3\}$ *such that* $[X_i, X_{\sigma(i)}] = \lambda_{\sigma^2(i)} X_{\sigma^2(i)}$ *where* $\sigma = (1\,2\,3) \in S_3$.

[1]

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If $\mathfrak{g}$ is a unimodular 3-dimensional metric Lie algebra then we have a Milnor frame $\{X_1, X_2, X_3\}$ with structure constants $\lambda_1, \lambda_2, \lambda_3$. [1]

[1] Milnor, 1976 pg. 299

Denote the 3-dimensional Heisenberg group and its Lie algebra as

$$\mathcal{H}^3 = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}, \mathfrak{h}^3 = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

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If g and g' are two left-invariant metrics on $\mathcal{H}^3$, then there exists $c > 0$ and $\phi \in \mathrm{Aut}(\mathcal{H}^3)$ such that for any $X, Y \in \mathfrak{h}^3$

$$g'(X, Y) = c(\phi.g)(X, Y) = cg(\phi(X), \phi(Y))$$

Higher Dimensional Milnor Frames

Definition (H.)

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\sigma \in S_n$, be the permutation $\sigma = (1\,2\,\ldots\,n)$. A linearly independent collection or vectors $\{X_1, \ldots, X_n\}$ is a n-Milnor frame if for i, j,

$$[X_i, X_j] = \begin{cases} \lambda_{\sigma^2(i)} X_{\sigma^2(i)} & j = \sigma(i) \\ -\lambda_{\sigma^2(j)} X_{\sigma^2(j)} & i = \sigma(j) \\ 0 & \textit{otherwise} \end{cases}$$

where $\lambda_i \in \mathbb{R}$ for $1 \leq i \leq n$.

Proposition

Let $\mathfrak{g}$ be a metric Lie algebra with a Milnor frame $\{X_1, \ldots, X_n\}$ and structure constants $\{\lambda_1, \ldots, \lambda_n\}$ with $n \geq 4$. For each $1 \leq i \leq n$, $\lambda_i \lambda_{\sigma^2(i)} = 0$

For $X, Y, Z \in \mathfrak{g}$ let

$$J(X, Y, Z) = [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]$$

$$0 = J(X_{\sigma(i)}, X_{\sigma^{-2(i)}}, X_{\sigma^{-1}(i)}) = [X_{\sigma(i)}, [X_{\sigma^{-2}(i)}, X_{\sigma^{-1}(i)}]] = \lambda_i [X_{\sigma(i)}, X_i] = -\lambda_i \lambda_{\sigma^2(i)}$$

Theorem

Let $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ such that $\lambda_i \lambda_{\sigma^2(i)} = 0$ for any i. There exists a Lie algebra $\mathfrak{g}$ with a Milnor frame whose structure constants are $\lambda_1, \ldots, \lambda_n$.

Graph Representations for Milnor Frames

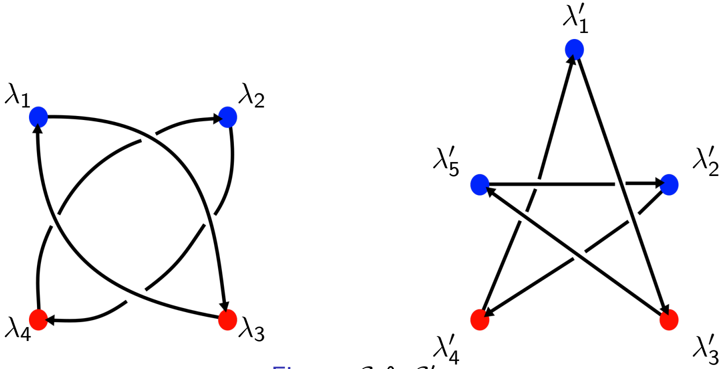

Figure: $\mathcal{G}$ & $\mathcal{G}'$

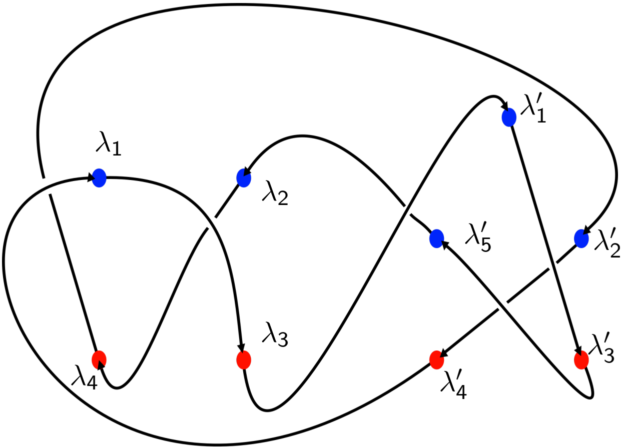

Figure: $\mathcal{G}\#\mathcal{G}'$

Main Theorems

Theorem

For any Lie algebra $\mathfrak{g}$ of dimension $n \geq 4$ with a Milnor frame, $\mathfrak{g} \cong (\oplus\mathfrak{h}^3) \oplus (\oplus\mathfrak{h}^4) \oplus \mathfrak{a}$ where $\mathfrak{h}^3$ is the Lie algebra of the Heisenberg Group, $\mathfrak{h}^4$ is a Lie algebra with a Milnor frame and two non-trivial structure constants, and $\mathfrak{a}$ is an abelian Lie Alebra. Moreover, these Lie algebras are at most 3-step nilpotent.

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The Lie algebras $\mathfrak{h}^3$ and $\mathfrak{h}^4$ are at most 3-step nilpotent.

Theorem

There exists a metric g *such that* $\mathfrak{h}^4$ *does not admit an orthonormal Milnor frame with respect to* g*. Furthermore there exists a metric* g *such that* $\mathfrak{h}^3 \oplus \mathfrak{h}^3$ *does not admit an orthonormal Milnor frame with respect to* g*.*

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Theorem

Let $\mathfrak{g}$ *be a non-abelian Lie algebra with a Milnor frame. If* $\mathfrak{g} \ncong \mathfrak{h}^3 \oplus \mathfrak{a}$ *where* $\mathfrak{a}$ *is an abelian Lie algebra, there exists a metric* g *on* $\mathfrak{g}$ *such that* $(\mathfrak{g}, g)$ *does not admit an orthonormal Milnor frame.*

Lauret, Jorge (2003). “Degenerations of Lie algebras and geometry of Lie groups”. In: *Differential Geom. Appl.* 18.2, pp. 177–194. ISSN: 0926-2245. DOI: 10.1016/S0926-2245(02)00146-8. URL: https://doi.org/10.1016/S0926-2245(02)00146-8.

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