On Higher Dimensional Milnor Frames

Hayden Hunter

UNIVERSITY of FLORIDA

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Outline of the Talk

- Part 1: General Knowledge of Riemannian Manifolds
- Part 2: Left Invariant Metrics
- Part 3: Higher Dimensional Milnor Frames

Part 1: General Knowledge of Riemannian Manifolds

We will talk about the following:

– *n*-dimensional Manifolds and Vector Fields
– Affine Connections [Covariant Derivatives] and the Levi-Civita Connection
– Sectional and Ricci Curvature
– Lie Groups with Left Invariant Metrics

References:

Carmo (1992) & Petersen (2016)

Differentiable Manifolds

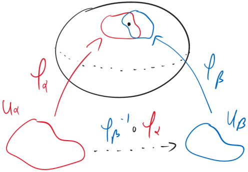
Definition

An n-dimensional differentiable manifold M is a topological manifold equipped with a collection of charts $\{(\phi_\alpha, U_\alpha)\}_\alpha$, $\phi_\alpha : U_\alpha \subset \mathbb{R}^n \to M$, ϕ_α a homeomorphism, such that

(i). $\cup_\alpha \phi_\alpha(U_\alpha) = M$

(ii). For α, β *where* $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) \neq \emptyset$ *the map* $\phi_\beta^{-1} \circ \phi_\alpha$ *is differentiable.*

(iii). The family $\{(\phi_\alpha, U_\alpha)\}_\alpha$ *is maximal.*



Tangent Bundles and Vector Fields

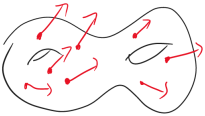
For each point $p \in M$ we define an n dimensional vector space

$$T_pM = \{\alpha'(0) \mid \alpha : (-\epsilon, \epsilon) \to M, \alpha(0) = p, \alpha \text{ differentiable}\}$$

The tangent bundle is defined as $TM := \sqcup_{p \in M} T_pM$.

A vector field, X, is a section of the tangent bundle

$$\begin{array}{c} T_pM \\ \downarrow \\ TM \\ X \curvearrowright \Big\downarrow (p,v) \to p \\ M \end{array}$$



Affine Connections

Definition

An affine connection is a smooth operator $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$, $\nabla(X, Y) = \nabla_X Y$, *which is bilinear over* $\mathbb{R}$ *and for smooth functions,* $f, g \in C^\infty(M)$ *and smooth vector fields* $X, Y, Z \in \Gamma(TM)$

(i). $\nabla_{fX} Y = f \nabla_X Y$

(ii). $\nabla_X(fY) = X(f)Y + f\nabla_X Y$

Let M^n be a smooth n-manifold. A Riemannian metric g on M is a a positive definite inner product on each T_pM varying smoothly with $p \in M$.

A connection ∇ is compatible with g if for $X, Y, Z \in \Gamma(TM)$,

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

∇ is symmetric if $[X, Y] = XY - YX = \nabla_X Y - \nabla_Y X$.

∇ is the Levi-Civita connection if ∇ is both compatible with the metric g, and is symmetric. Such a ∇ exists and it is unique.

Sectional Curvature

$$R(X, Y, Z) = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z = \nabla_{[X,Y]} Z - [\nabla_X, \nabla_Y] Z$$

is a multilinear map.

The sectional curvature $\kappa : T_p M \times T_p M \to \mathbb{R}$ is defined to be.

$$\kappa(X, Y) = \frac{g(R(X, Y, X), Y)}{|X \wedge Y|}, \quad |X \wedge Y| = g(X, X) g(Y, Y) - g(X, Y)^2$$

κ induces a map on the Grassmannian $\tilde{\kappa} : G_{2,n}(T_p M) \to \mathbb{R}$

Lie Groups with Left Invariant Metrics

Definition

A Lie Group is a differentiable manifold with a smooth group structure.

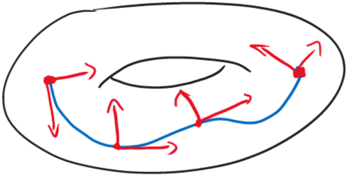
$\mathbb{R}^n$, S^1, $\mathcal{H}^3$, $SL(n, \mathbb{R})$, and any closed subgroup of $GL(n, \mathbb{R})$

For $h \in G$ define $L_h : G \to G$, $L_h(p) = hp$.

Let $\{E_i\}_{i=1}^n$ be a basis for T_eG and define vector fields $X_i(h) = dL_h(E_i)$.

A Vector Field $X \in \Gamma(TM)$ is said to be left invariant if $dL_h(X) = X$.

We say that g is left invariant if for any $X, Y \in \Gamma(TG)$ and $h \in G$,
$g(X, Y) = g(dL_h(X), dL_h(Y))$.



$\{E_i\}_{i=1}^n$ is a basis for T_eG, then we can define $g_e(E_i, E_j) = g_{ij}$.

If g is left invariant, $\{dL_h(E_i)\}$ forms a basis at $T_h(G)$.

We form a frame on G by setting $X_i(h) = dL_h(E_i)$. Each X_i is a left invariant vector field and the function $h \to g_h(X_i, X_j)$ is constant.

Consequentially, $Y(g(X_i, X_j)) = 0$ for every vector field $Y \in \Gamma(TG)$.

The Levi-Civita Connection and Left Invariant Metrics

Theorem (Levi-Civita, see Carmo (1992), Theorem 3.6 pg. 55)

For any Riemannian manifold (M, g)*, there exists a unique affine connection* ∇ *with following properties: For* $X, Y, Z \in \Gamma(TM)(M)$*,*

(i). $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

(ii). $\nabla_X Y - \nabla_Y X = [X, Y] = XY - YX$

Letting $\{X_1, \ldots, X_n\}$ be a local frame:

$$\nabla_X Y = \sum_k g(\nabla_X Y, X_k) X_k$$

$$= \sum_k \frac{1}{2} \Big[X(g(Y, X_k)) + Y(g(X_k, X)) - X_k(g(X, Y))$$

$$+ g([X, Y], X_k) - g([Y, X_k], X) + g([X_k, X], Y) \Big] X_k$$

If g is left invariant and $\{X_1, \ldots, X_n\}$ is a left invariant frame on G then the previous equation reduces to

$$g(\nabla_{x_i} X_j, X_k) = \frac{1}{2}\Big(g([X_i, X_j], X_k) - g([X_j, X_k], X_i) + g([X_k, X_i], X_j)\Big)$$

Letting $[X_i, X_j] = c_{ij}^{\ell} X_\ell$ gives us the expression

$$g([X_i, X_j], X_k) = c_{ij}^{\ell} g_{\ell k} \implies g(\nabla_{X_i} X_j, X_k) = \frac{1}{2}\left(c_{ij}^{\ell} g_{\ell k} - c_{jk}^{\ell} g_{\ell i} + c_{ki}^{\ell} g_{\ell j}\right)$$

If $\{X_1, \ldots, X_n\}$ are orthonormal then

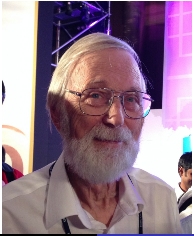
$$g(\nabla_{X_i} X_j, X_k) = \frac{1}{2}\left(c_{ij}^{k} - c_{jk}^{i} + c_{ki}^{j}\right)$$

Part 2: Left Invariant Metrics

As shown in Curvatures of Left Invariant Metrics on Lie Groups (Milnor, 1976), one can compute the curvatures of Lie Groups equipped with left invariant metrics in terms of the structure constants of their associated Lie algebras. Topics will Include:

- Showing the relationship between structure constants, sectional curvature, and Ricci curvature.
- Geometric consequences for non-commutative nilpotent Lie algebras
- 3-dimensional unimodular Lie algebras.

Reference: Milnor (1976)



For the remainder of the talk G will be a Lie group with a left-invariant metric g and Lie algebra $\mathfrak{g}$.

Given that g is determined by elements of $\mathfrak{g}$, we may consider metric Lie algebras $(\mathfrak{g}, g)$.

For the remainder of the presentation all frames will be a frame of left-invariant vector fields.

Lemma (1.1 Milnor (1976) pg. 295)

Given an orthonormal frame $\{X_1, \ldots, X_n\}$ of the metric g,

$$\kappa(X_1, X_2) = \sum_k \frac{1}{2} c_{12}^k (c_{k1}^2 + c_{2k}^1 - c_{12}^k) - \frac{1}{4}(c_{12}^k - c_{2k}^1 - c_{1k}^2)(c_{1k}^2 + c_{12}^k + c_{2k}^1) - c_{k1}^1 c_{k2}^2$$

Example: The Heisenberg Group

$$\mathcal{H}^3 = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}, \mathfrak{h}^3 = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

We have the following basis at $T_I \mathcal{H}^3$:

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

If $\{X_1, X_2, X_3\}$ are the canonical left invariant vector fields with $X_i(I) = E_i$ then

$$[X_1, X_2] = 0, \ [X_2, X_3] = 0, \ [X_3, X_1] = -X_2$$

$$\kappa(X_1, X_2) = 1/4, \quad \kappa(X_1, X_3) = -3/4, \quad \kappa(X_2, X_3) = 1/4$$

The Ricci Tensor and Sectional Curvature

Recall: $R(X, Y, Z) = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$

Definition

Let $\{X_1, \dots, X_n\}$ be an orthonormal frame for G. The Ricci tensor Ric_g is defined to be

$$Ric_g(X, Y) = \sum_i g(R(X, X_i)Y, X_i)$$

$$\mathrm{Ric}_g(X, X) = \sum_i g(R(X, X_i)X, X_i) = \sum_i |X \wedge X_i| \kappa(X, X_i)$$

$$X_j \in \{X_1, \dots, X_n\} \implies \mathrm{Ric}_g(X_j, X_j) = \sum_{i \neq j} \kappa(X_j, X_i)$$

The Ricci Tensor and Nilpotent Lie algebras

Define $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] = \{[X, Y] \mid X, Y \in \mathfrak{g}\}$.

Inductively define $\mathfrak{g}^n = [\mathfrak{g}, \mathfrak{g}^{n-1}]$. We say that $\mathfrak{g}$ is nilpotent if there exists n such that $\mathfrak{g}^n = 0$.

Theorem (Theorem 2.5 Milnor (1976) pg. 302)

Suppose that $Lie(G) = \mathfrak{g}$ is nilpotent but not commutative. If g is a left invariant metric then there exists $X, Y \in \mathfrak{g}$ with $Ric_g(X, X) > 0$, $Ric_g(Y, Y) < 0$.

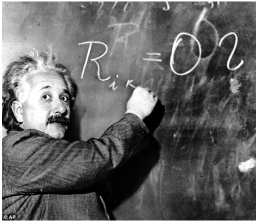
Consider the Heisenberg Lie algebra $\mathfrak{h}^3$ with the same frame as before $\{X_1, X_2, X_3\}$. If we choose a left invariant metric g which makes $\{X_1, X_2, X_3\}$ an orthonormal frame then

$$\mathrm{Ric}_g(X_1, X_1) = -1/2, \quad \mathrm{Ric}_g(X_2, X_2) = 1/2, \quad \mathrm{Ric}_g(X_3, X_3) = -1/2$$

Einstein Metrics

Definition

An Einstein Metric is a metric, g, such that $Ric_g = \lambda g$, where $\lambda \in \mathbb{R}$.



Corollary (Milnor, 1976)

Every nilpotent non-commutative Lie algebra does not admit a left invariant Einstein Metric.

Ricci Soliton Equations: An Extension of Einstein Metrics

Definition

A complete Riemannian manifold (M, g) is a Ricci Soliton if there exists a smooth vector field X and $\lambda \in \mathbb{R}$ such that

$$Ric_g = \lambda g + \mathcal{L}_X g$$

For Lie groups with left invariant metrics this is equivalent to the expression

$$\mathrm{Ric}_g = \lambda I + D$$

where for any $X, Y \in \mathfrak{g}$, $D[X, Y] = [DX, Y] + [X, DY]$.

An Additional Condition: Unimodularity

Definition (Milnor, 1976,pg. 318)

A Lie algebra $\mathfrak{g}$ is unimodular if for every $U \in \mathfrak{g}$, the linear operator $ad_U : \mathfrak{g} \to \mathfrak{g}^1$, $V \stackrel{ad_U}{\to} [U, V]$ has trace 0.

The Lie algebra for the Heisenberg group is

$$\mathfrak{h}^3 = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

and so $\mathfrak{h}^3$ is unimodular.

Remark 1: Lie Groups are Orientable Manifolds.

Remark 2: Let $\mathfrak{g}$ be a 3-dimensional metric Lie algebra (a Lie algebra with a left invariant metric) with fixed orientation. For two linearly independent vectors $U, V \in \mathfrak{g}$, define the cross product $U \wedge V$ to be the unique orthogonal positively oriented vector such that

$$g(U \wedge V, U \wedge V) = |U \wedge V|^2 = g(U, U)g(V, V) - g(U, V)^2$$

Lemma (4.1 Milnor (1976), pg. 305)

Let (G, g) be a connected 3-dimensional Lie Group with left invariant metric g and a canonical orientation. Let $T : \mathfrak{g} \to \mathfrak{g}^1$ by the linear operator defined by $T(U \wedge V) = [U, V]$. Then $\mathfrak{g}$ is unimodular if and only if the linear transformation T is self-adjoint.

Corollary (Milnor (1976), pg. 305)

If $\mathfrak{g}$ is unimodular, then there exists an orthonormal frame $\{X_1, X_2, X_3\}$ relative to the left invariant metric g such that for $\sigma = (1\,2\,3) \in S_3$, $[X_i, X_{\sigma(i)}] = \lambda_{\sigma^2(i)} X_{\sigma^2(i)}$.

We call $\{X_1, X_2, X_3\}$ a Milnor frame with structure constants $\lambda_1, \lambda_2, \lambda_3$.

Diagonalization of the Ricci Quadratic Form, J. Milnor

If $\mathfrak{g}$ is a unimodular 3-dimensional metric Lie algebra then we have a Milnor frame $\{X_1, X_2, X_3\}$ with structure constants $\lambda_1, \lambda_2, \lambda_3$.

Let $\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i$.

$$\mathrm{Ric}_g(X_i, X_j) = 2\mu_{\sigma(i)}\mu_{\sigma^2(i)}\delta_{ij}$$

[1]

[1] Milnor, 1976 pg. 299

The Heisenberg Lie Algebra Revisited

Let E_{ij} denote the matrix with 1 in the i, j^{th} entry and 0 everywhere else.

Let $\mathfrak{h}^3$ be equipped with the metric g which orthonormalizes $\{X_1 = E_{12}, X_2 = E_{13}, X_3 = E_{23}\}$.

The structure constants are $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = 0$.

$$\mu_1 = -1/2, \quad \mu_2 = 1/2, \quad \mu_3 = -1/2$$

$$\mathrm{Ric}_g(X_1, X_1) = 2 \cdot (1/2) \cdot (-1/2) = -1/2$$
$$\mathrm{Ric}_g(X_2, X_2) = 2 \cdot (-1/2) \cdot (-1/2) = 1/2$$
$$\mathrm{Ric}_g(X_3, X_3) = 2 \cdot (-1/2) \cdot (1/2) = -1/2$$

which coincides with our computations from before.

Part 3: New Results and Future Research

For this portion of the talk we will talk about the following:

- Generalization of Milnor frames to higher (finite) dimensions.
- The relationship between Milnor frames and the diagonalization of the Ricci tensor.
- Algebraic restrictions on the existence of higher dimensional Milnor frames.
- Milnor Graphs
- Metrics on Higher Dimensional Milnor Frames.

References: Lauret and Will (2013) & Malcev (1951)

Definition (H.)

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\sigma \in S_n$, be the permutation $\sigma = (1\,2\,\ldots\,n)$. A linearly independent collection or vectors $\{X_1, \ldots, X_n\}$ is a n-Milnor frame if for i, j,

$$[X_i, X_j] = \begin{cases} \lambda_{\sigma^2(i)} X_{\sigma^2(i)} & j = \sigma(i) \\ -\lambda_{\sigma^2(j)} X_{\sigma^2(j)} & i = \sigma(j) \\ 0 & \textit{otherwise} \end{cases}$$

where $\lambda_i \in \mathbb{R}$ for $1 \leq i \leq n$.

Other equivalent definitions include:

$$[X_i, X_j] = \delta_{\sigma(i)j} \lambda_{\sigma^2(i)} X_{\sigma^2(i)} - \delta_{\sigma(j)i} \lambda_{\sigma^2(j)} X_{\sigma^2(j)}$$

$$\mathrm{ad}_{X_i} = \lambda_{\sigma^2(i)} E_{\sigma^2(i)\sigma(i)} - \lambda_{\sigma(i)} E_{\sigma(i)\sigma^{-1}(i)}$$

For any abelian Lie algebra $\mathfrak{a}$, $\mathfrak{h}^3 \oplus \mathfrak{a}$ admits a Milnor frame.

Let $\mathfrak{h}^4$ be the fourth dimensional Lie algebra which admits a frame $\{X_1, X_2, X_3, X_4\}$ with the following non-trivial bracket relations:

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_4$$

The above frame is a Milnor frame

Because $\mathfrak{h}^3$ is 2-step nilpotent and $\mathfrak{h}^4$ is 3-step nilpotent, $\mathfrak{h}^4 \ncong \mathfrak{h}^3 \oplus \mathfrak{a}$.

Does the higher-dimensional analogue have a diagonalizable Ricci tensor?

Definition (Lauret and Will (2013), pg. 3652)

Let $\mathfrak{g}$ be a nilpotent Lie algebra. A frame $\{X_1, \ldots, X_n\}$ is nice if $[X_i, X_j] = c_{ij}^k X_k$ have the property that $\forall i, j$, $c_{ij}^k \neq 0$ for at most one k and $\forall i, k$ there exists at most one j with $c_{ij}^k \neq 0$

Nilpotent Milnor frames are nice.

Theorem (Lauret and Will (2013), pg. 3652, 3655-3656)

An orthogonal frame, $\{X_1, \ldots, X_n\}$, for a nilpotent Lie algebra $\mathfrak{g}$ diagonalizes the Ricci Tensor if and only if it is nice.

Proposition

Let $\mathfrak{g}$ be a metric Lie algebra with a Milnor frame $\{X_1, \ldots, X_n\}$ and structure constants $\{\lambda_1, \ldots, \lambda_n\}$ with $n \geq 4$. For each $1 \leq i \leq n$, $\lambda_i \lambda_{\sigma^2(i)} = 0$

For $X, Y, Z \in \mathfrak{g}$ let

$$J(X, Y, Z) = [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]$$

$$0 = J(X_{\sigma(i)}, X_{\sigma^{-2}(i)}, X_{\sigma^{-1}(i)}) = [X_{\sigma(i)}, [X_{\sigma^{-2}(i)}, X_{\sigma^{-1}(i)}]] = \lambda_i [X_{\sigma(i)}, X_i] = -\lambda_i \lambda_{\sigma^2(i)}$$

Theorem

Let $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ such that $\lambda_i \lambda_{\sigma^2(i)} = 0$ for any i. There exists a Lie algebra $\mathfrak{g}$ with a Milnor frame whose structure constants are $\lambda_1, \ldots, \lambda_n$.

Graph Representations of Milnor Frames

Suppose we have a tuple of real numbers $(\lambda_1, \dots, \lambda_n)$ with the property that $\lambda_i \lambda_{\sigma^2(i)} = 0$.

We want to determine the maximal number of non-zero real values in the tuple so that the above property is attained.

Construct the graph $\mathcal{G} = (V, E)$ where V, $|V| = n$, is the collection of vertices which represent the real numbers in our tuple and $E = \{\{v_i, v_{\sigma^2(i)}\} \mid 1 \leq i \leq n\}$.

Partition V into sets A and B where A will be the collection of vertices v_i such that $\lambda_i = 0$ and $B = V \setminus A$.

For each edge $e \in E$, we require that $e \cap A \neq \emptyset$.

We may use these graphs as a tool to determine the maximal number of non-zero structure constants a Lie algebra with a Milnor frame can have.

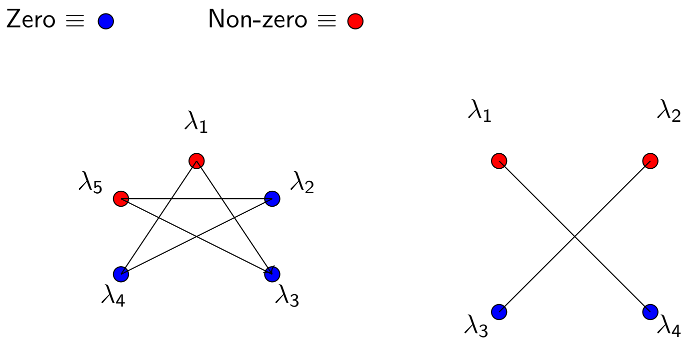


Figure: Examples

We refer to these graphs as Milnor Graphs.

Proposition

Let $\mathfrak{g}$ be an n-dimensional Lie algebra with a Milnor frame $\{X_1, \ldots, X_n\}$. The maximal number of non-trivial structure constants of $\{X_1, \ldots, X_n\}$ is $\lfloor n/2 \rfloor$.

Proof.

Let $\mathcal{G} = (V = A \cup B, E)$ be a Milnor graph where for each edge $e \in E$, $e \cap A \neq \emptyset$. For each $v_i \in B$, define $f : B \to A$ as $f(v_i) = v_{\sigma^2(i)}$. Because f is injective

$$2|B| \leq |B| + |A| = n \implies |B| \leq \frac{n}{2}$$

where $|B| \in \mathbb{Z}$ implies that $|B| \leq \lfloor \frac{n}{2} \rfloor$. □

Whenever $4|n$, the upper bound can be attained.

What information do we obtain when we direct the edges?

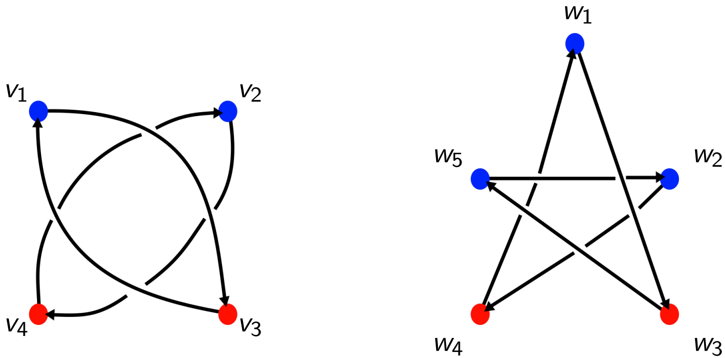


Figure: $\mathcal{G}$ & $\mathcal{G}'$

Theorem

Let $\mathcal{G} = (V = A \cup B, E)$ be a directed graph such that $V = \{v_1, \ldots, v_n\}$, $E = \{(v_i, v_{\sigma^2(i)}) \mid 1 \leq i \leq n\}$ and for any $(v_i, v_{\sigma^2(i)}) \in E$ either v_i or $v_{\sigma^2(i)} \in A$. Define $\lambda_i = 0$ if $v_i \in A$ and $\lambda_i = 1$ if $v_i \in B$. There exists a Lie algebra $\mathfrak{g}$ with a Milnor frame $\{X_1, \ldots, X_n\}$ whose structure constants are $\lambda_1, \ldots, \lambda_n \in \{0, 1\}$ such that $\mathcal{G}$ is the directed Milnor graph of $\mathfrak{g}$.

Theorem

For any Lie algebra $\mathfrak{g}$ with a Milnor frame $\{X_1, \ldots, X_n\}$, there exists a Milnor frame $\{c_1X_1 = Y_1, \ldots, c_nX_n = Y_n\}$ such that the structure constants of $\{Y_1, \ldots, Y_n\}$ are either 0 or 1.

Corollary

Lie algebras with Milnor frames are in bijective correspondence with directed Milnor graphs.

Lemma

Let $\mathfrak{g}$, dimension $n \geq 6$, be a Lie algebra with a Milnor frame whose structure constants are $\lambda_1, \ldots, \lambda_n$. Then there exist $i, j \in \{1, \ldots, n\}$ such that $\{i, \sigma(i)\} \cap \{j, \sigma(j)\} = \emptyset$ and $\lambda_i, \lambda_{\sigma(i)}, \lambda_j, \lambda_{\sigma(j)} = 0$.

Let $\mathcal{G} = (V = A \cup B, E)$ and $\mathcal{G}' = (V' = A' \cup B', E')$ be two directed Milnor Graphs with $|V| = n \geq 3$ and $|V'| = m \geq 3$.

Suppose further that $\exists i$ and j such that $v_i, v_{\sigma(i)} \in A$ and $w_j, w_{\sigma(j)} \in A'$.

Let $U = \{v_1, \ldots, v_n, w_1, \ldots, w_m\}$.

Let $D = \{(v_{\sigma^{-2}(i)}, v_i), (v_{\sigma^{-1}(i)}, v_{\sigma(i)}), (w_{\sigma^{-2}(j)}, w_j), (w_{\sigma^{-1}(j)}, w_{\sigma(j)})\}$

Let $D' = \{(v_{\sigma^{-2}(i)}, w_j), (v_{\sigma^{-1}(i)}, w_{\sigma(j)}), (w_{\sigma^{-2}(j)}, v_i), (w_{\sigma^{-1}(j)}, v_{\sigma(i)})\}$

Define $F = [(E \cup E') \setminus D] \cup D'$

The graph $\mathcal{G} \# \mathcal{G}' = (U = [A \cup A'] \cup [B \cup B'], F)$ which we call the sum of $\mathcal{G}$ and $\mathcal{G}'$ is a Milnor Graph.

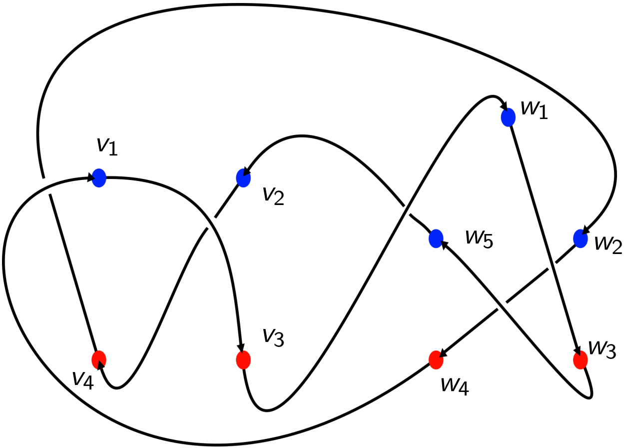


Figure: $\mathcal{G}\#\mathcal{G}'$

Proposition

If $\mathfrak{g}$ and $\mathfrak{g}'$ are two which correspond to directed Milnor graphs $\mathcal{G}$ and $\mathcal{G}'$ respectively, then $\mathcal{G}\#\mathcal{G}'$ corresponds to $\mathfrak{g} \oplus \mathfrak{g}'$.

Theorem

Any directed Milnor graph $\mathcal{G}$ with $n \geq 6$ vertices is the (finite) sum of Milnor graphs with 3, 4, or 5 vertices.

A 5-dimensional Lie algebra with a Milnor frame is isomorphic to $\mathfrak{h}^4 \oplus \mathfrak{a}$.

Corollary

For any Lie algebra $\mathfrak{g}$ of dimension $n \geq 4$ with a Milnor frame, $\mathfrak{g} \cong (\oplus \mathfrak{h}^3) \oplus (\oplus \mathfrak{h}^4) \oplus \mathfrak{a}$ where $\mathfrak{h}^3$ is the Lie algebra of the Heisenberg Group, $\mathfrak{h}^4$ is a Lie algebra with a Milnor frame and 2 non-trivial structure constants, and $\mathfrak{a}$ is an abelian Lie Alebra.

Lemma

There exists a metric g on the Lie algebra $\mathfrak{h}^4$ such that $(\mathfrak{h}^4, g)$ does not admit an orthonormal Milnor frame.

Lemma

There exists a metric g on the Lie algebra $(\mathfrak{h}^3 \oplus \mathfrak{h}^3)$ such that $(\mathfrak{h}^3 \oplus \mathfrak{h}^3, g)$ does not admit an orthonormal Milnor frame.

Theorem

For any Lie algebra with a Milnor frame which is not isomorphic to $\mathfrak{h}^3 \oplus \mathfrak{a}$, $\mathfrak{a}$ abelian, there exists a metric g on $\mathfrak{g}$ such that $(\mathfrak{g}, g)$ does not have an orthonormal Milnor frame.

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Thank you for your time!