On Higher Dimensional Milnor Frames

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Let $\mathfrak{g}$ be a Lie algebra. For each $X \in \mathfrak{g}$, define the linear operator $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$, $U \mapsto [X, U]$. We say that $\mathfrak{g}$ is unimodular if $\mathrm{ad}_X = 0$ for any $X \in \mathfrak{g}$.

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Lemma (4.1 [1], pg. 305)

Let $(\mathfrak{g}, g)$ be a 3-dimensional metric Lie algebra with an orientation. Let $T : \mathfrak{g} \to \mathfrak{g}^1$ be the linear operator defined by $T(U \wedge V) = [U, V]$. Then $\mathfrak{g}$ is unimodular if and only if the linear transformation T is self-adjoint.

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Recall: If a linear transformation T is self-adjoint then there exists an orthonormal frame of eigenvectors.

[1] [1] pg. 299

Corollary ([1], pg. 305)

If $(\mathfrak{g}, g)$ is a unimodular metric Lie algebra, then there exists an orthonormal frame $\{X_1, X_2, X_3\}$ such that $[X_i, X_{\sigma(i)}] = \lambda_{\sigma^2(i)} X_{\sigma^2(i)}$ where $\sigma = (1\,2\,3) \in S_3$.

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If $(\mathfrak{g}, g)$ *is a unimodular metric Lie algebra, then there exists an orthonormal frame* $\{X_1, X_2, X_3\}$ *such that* $[X_i, X_{\sigma(i)}] = \lambda_{\sigma^2(i)} X_{\sigma^2(i)}$ *where* $\sigma = (1\,2\,3) \in S_3$.

Definition

Let $\mathfrak{g}$ be a Lie algebra with a frame $\{X_1, X_2, X_3\}$. If there exists $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that the bracket relation $[X_i, X_{\sigma(i)}] = \lambda_{\sigma^2(i)} X_{\sigma^2(i)}$ holds for any $1 \leq i \leq 3$, then $\{X_1, X_2, X_3\}$ is called a Milnor frame.

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If $\mathfrak{g}$ is a unimodular 3-dimensional metric Lie algebra then $\mathfrak{g}$ admits an orthonormal Milnor frame. [1]

[1] [1] pg. 299

Example

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Denote the 3-dimensional Heisenberg group and its Lie algebra as

$$\mathcal{H}^3 = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}, \mathfrak{h}^3 = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

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$$[X_1, X_2] = 0, \quad [X_2, X_3] = 0, \quad [X_3, X_1] = -X_2$$

Higher Dimensional Milnor Frames

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Definition (H.)

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\sigma \in S_n$, be the permutation $\sigma = (1\,2\,\ldots\,n)$. A linearly independent collection or vectors $\{X_1, \ldots, X_n\}$ is a n-Milnor frame if for i, j,

$$[X_i, X_j] = \begin{cases} \lambda_{\sigma^2(i)} X_{\sigma^2(i)} & j = \sigma(i) \\ -\lambda_{\sigma^2(j)} X_{\sigma^2(j)} & i = \sigma(j) \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda_i \in \mathbb{R}$ for $1 \leq i \leq n$.

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Notice that for $i \in \{1, \ldots, n\}$ such that $\sigma(i) = i$

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Thus we only need to consider the case where σ is cyclic.

Examples

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If $\mathfrak{a}$ is an abelian Lie algebra of dimension $n \geq 1$ then $\mathfrak{h}^3 \oplus \mathfrak{a}$ is a $n \geq 4$ dimensional Lie algebra with a Milnor Frame

Let $\mathfrak{h}^4$ be a fourth dimensional Lie algebra which admits a frame with the following non-trivial bracket relations (up to anti-symmetry)

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_4.$$

$\mathfrak{h}^4$ is sometimes known as the 4-dimensional filiform Lie algebra.

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Theorem (H.)

For any Lie algebra $\mathfrak{g}$ of dimension $n \geq 4$ with a Milnor frame, $\mathfrak{g} \cong (\oplus \mathfrak{h}^3) \oplus (\oplus \mathfrak{h}^4) \oplus \mathfrak{a}$ where $\mathfrak{h}^3$ is the Lie algebra of the Heisenberg Group, $\mathfrak{h}^4$ is a Lie algebra with a Milnor frame and two non-trivial structure constants, and $\mathfrak{a}$ is an abelian Lie Alebra. Moreover, these Lie algebras are at most 3-step nilpotent.

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The Lie algebras $\mathfrak{h}^3$ and $\mathfrak{h}^4$ are at most 3-step nilpotent.

Proposition

Let $\mathfrak{g}$ *be a metric Lie algebra with a Milnor frame* $\{X_1, \ldots, X_n\}$ *and structure constants* $\{\lambda_1, \ldots, \lambda_n\}$ *with* $n \geq 4$*. For each* $1 \leq i \leq n$*,* $\lambda_i \lambda_{i+2} = 0$

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For $X, Y, Z \in \mathfrak{g}$ let

$$J(X, Y, Z) = [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]$$

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Theorem

Let $(\lambda_1, \ldots, \lambda_n) \in \{0, 1\}^n$ such that $\lambda_i \lambda_{i+2} = 0$ for any i. There exists a Lie algebra $\mathfrak{g}$ with a Milnor frame whose structure constants are $\lambda_1, \ldots, \lambda_n$.

Graph Representations for Milnor Frames

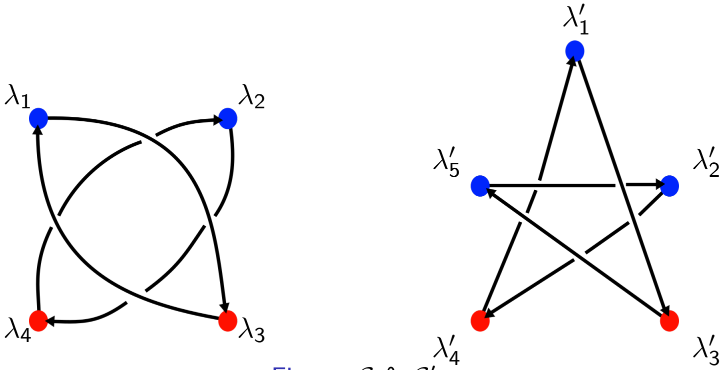

Figure: $\mathcal{G}$ & $\mathcal{G}'$

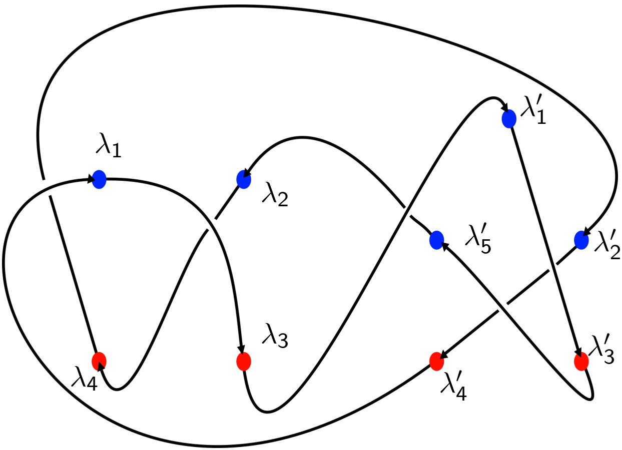

Figure: $\mathcal{G}\#\mathcal{G}'$

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There exists a metric g such that $\mathfrak{h}^4$ does not admit an orthonormal Milnor frame with respect to g. Furthermore there exists a metric g such that $\mathfrak{h}^3 \oplus \mathfrak{h}^3$ does not admit an orthonormal Milnor frame with respect to g.

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Theorem (H.)

Let $\mathfrak{g}$ be a non-abelian Lie algebra with a Milnor frame. If $\mathfrak{g} \not\cong \mathfrak{h}^3 \oplus \mathfrak{a}$ where $\mathfrak{a}$ is an abelian Lie algebra, there exists a metric g on $\mathfrak{g}$ such that $(\mathfrak{g}, g)$ does not admit an orthonormal Milnor frame.

[1] John Milnor. "Curvatures of left invariant metrics on Lie groups". In: *Advances in Math.* 21.3 (1976), pp. 293–329.

Thank you!