## Outline

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# Computing Persistent Homology*

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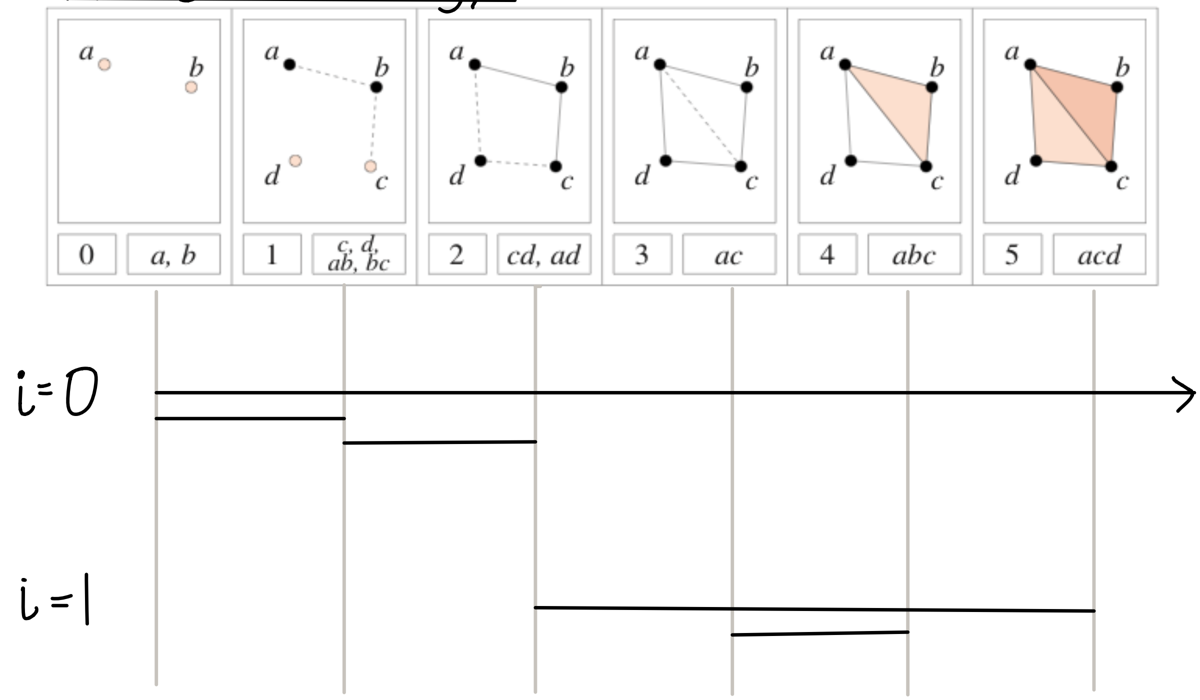
**Abstract.** We show that the persistent homology of a filtered $d$-dimensional simplicial complex is simply the standard homology of a particular graded module over a polynomial ring. Our analysis establishes the existence of a simple description of persistent homology groups over arbitrary fields. It also enables us to derive a natural algorithm for computing persistent homology of spaces in arbitrary dimension over any field. This result generalizes and extends the previously known algorithm that was restricted to subcomplexes of $\mathbb{S}^3$ and $\mathbb{Z}_2$ coefficients. Finally, our study implies the lack of a simple classification over non-fields. Instead, we give an algorithm for computing individual persistent homology groups over an arbitrary principal ideal domain in any dimension.

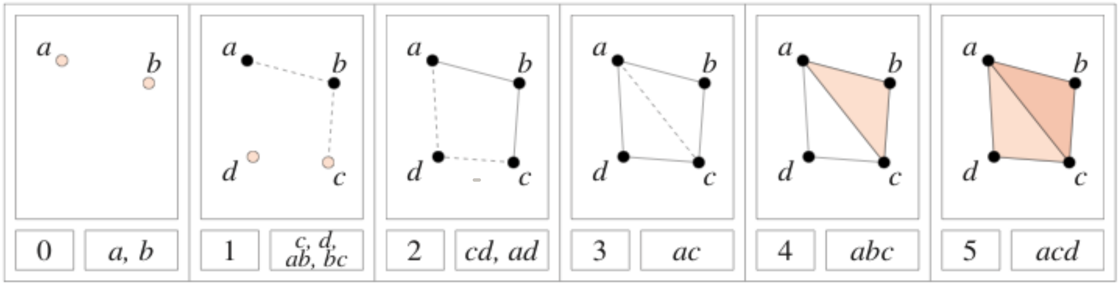
# Persistent homology




Apply $i$-dimensional homology with coefficients in a field $F$ to get

$$H_i(K_0) \to H_i(K_1) \to H_i(K_2) \to H_i(K_3) \to H_i(K_4) \to H_i(K_5)$$

## Equivalence




A persistence module

$$\mathbb{V}: \quad V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \xrightarrow{f_3} V_4 \xrightarrow{f_4} V_5$$

can be thought of as a graded $F[t]$-module with elements $(v_0, v_1, v_2, v_3, v_4, v_5) \in V_0 \oplus \dots \oplus V_5$ and $F[t]$ action given by:

$$3 \cdot (v_0, \dots, v_5) = (3v_0, \dots, 3v_5)$$

$$t \cdot (v_0, \dots, v_5) = (0, f_0(v_0), f_1(v_1), f_2(v_2), f_3(v_3), f_4(v_4))$$

$$t^2 \cdot (v_0, \dots, v_5) = (0, 0, f_1 f_0(v_0), f_2 f_1(v_1), f_3 f_2(v_2), f_4 f_3(v_3))$$

$$(3+t^2) \cdot (v_0, \dots, v_5) = (3v_0, 3v_1, 3v_2 + f_1 f_1(v_0), \dots \qquad )$$

# Decomposition

## Finitely-generated abelian group

$$G \cong \Big(\bigoplus_{i=1}^{n} \mathbb{Z}\Big) \oplus \Big(\bigoplus_{j=1}^{m} \mathbb{Z}/d_j\mathbb{Z}\Big)$$

free torsion

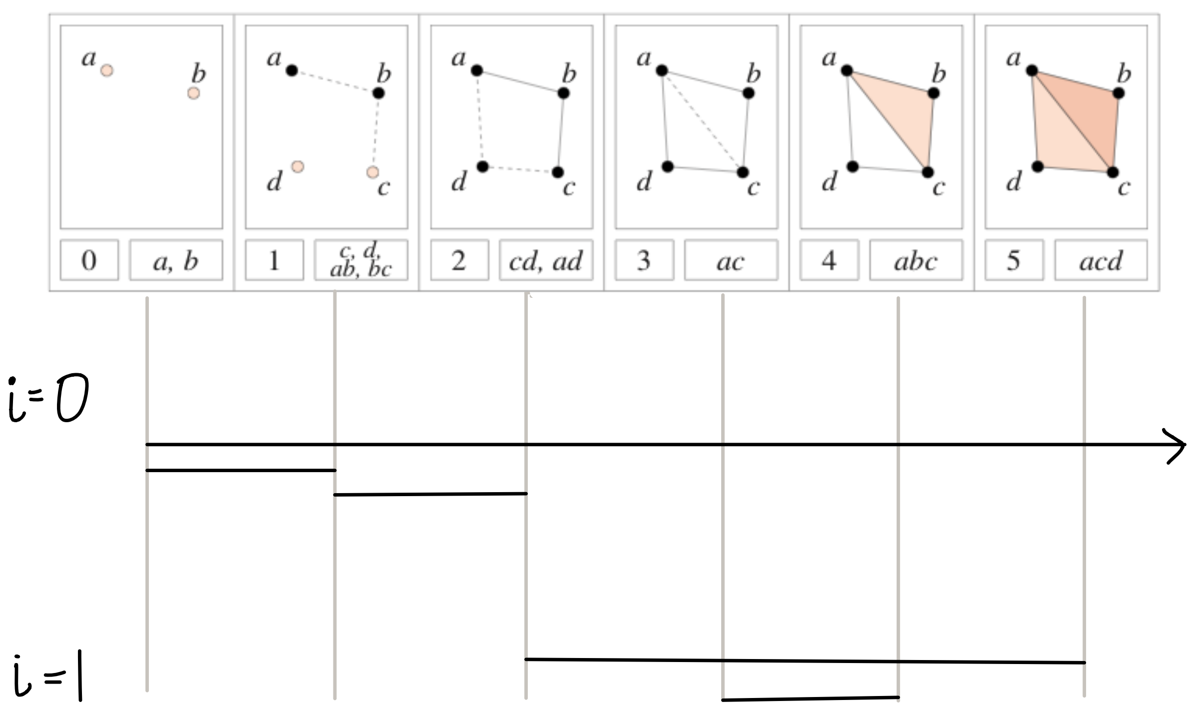
## Finitely-generated $F[t]$-module

$$\mathbb{V} \cong \Big(\bigoplus_{i=1}^{n} F[t]\Big) \oplus \Big(\bigoplus_{j=1}^{m} F[t]/(t^{n_j})\Big)$$

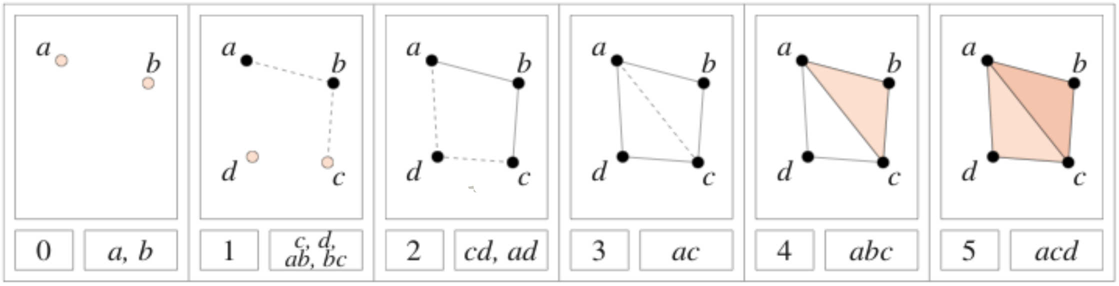
## Finitely-generated graded $F[t]$-module

$$\mathbb{V} \cong \Big(\bigoplus_{i=1}^{n} \Sigma^{\alpha_i} F[t]\Big) \oplus \Big(\bigoplus_{j=1}^{m} \Sigma^{\gamma_j} F[t]/(t^{n_j})\Big)$$

infinite bars finite bars



# Computation



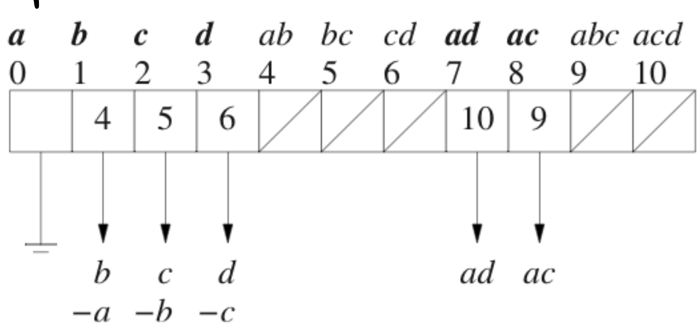

$$M_1 = \begin{array}{c} \\ d \\ c \\ b \\ a \end{array} \begin{array}{c} \begin{array}{ccccc} ab & bc & cd & ad & ac \end{array} \\ \begin{bmatrix} 0 & 0 & t & t & 0 \\ 0 & 1 & t & 0 & t^2 \\ t & t & 0 & 0 & 0 \\ t & 0 & 0 & t^2 & t^3 \end{bmatrix} \end{array}$$

In our example, we have

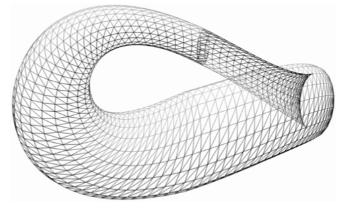
$$\tilde{M}_1 = \begin{array}{c} \\ d \\ c \\ b \\ a \end{array} \begin{array}{c} \begin{array}{ccccc} cd & bc & ab & z_1 & z_2 \end{array} \\ \begin{bmatrix} \boxed{\mathrm{t}} & 0 & 0 & 0 & 0 \\ t & \boxed{1} & 0 & 0 & 0 \\ 0 & t & \boxed{\mathrm{t}} & 0 & 0 \\ 0 & 0 & t & 0 & 0 \end{bmatrix} \end{array}, \qquad (8)$$

where $z_1 = ad - cd - t \cdot bc - t \cdot ab$ and $z_2 = ac - t^2 \cdot bc - t^2 \cdot ab$ form a homogeneous basis for $\mathsf{Z}_1$.

# Sparse computation



# Experiments



Klein bottle
Electrostatic charge
Jet engine

| | $\|K\|$ | Length | Filtration (s) | Persistance (s) |
|---|---|---|---|---|
| K | 12,000 | 1,020 | 0.03 | $<$ 0.01 |
| E | 529,225 | 3,013 | 3.17 | 5.00 |
| J | 3,029,383 | 256 | 24.13 | 50.23 |