Connectivity of Vietoris-Rips complexes of spheres

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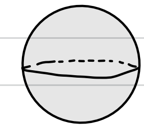
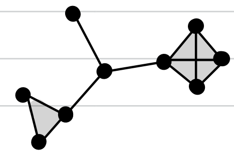
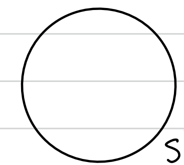
Let S^n be the n-sphere, geodesic metric, diameter π.



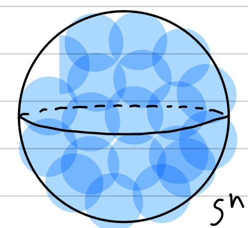
For X a metric space and $r>0$,
the <u>Vietoris-Rips complex</u> $VR(X;r)$ has
- vertex set X
- all simplices $[x_0,\dots,x_k]$ of diameter $< r$.

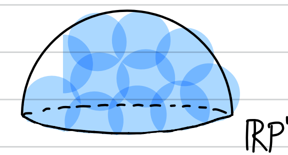




The <u>k-th covering radius</u> of X is
$\mathrm{cov}_X(k) = \inf\{\delta>0 : \exists k \text{ closed } \delta\text{-balls covering } X\}$.

For $\delta>0$, the <u>covering number</u> of X is
$\mathrm{numCov}_X(\delta) = \min\{k\geq 1 : \exists k \text{ closed } \delta\text{-balls covering } X\}$.

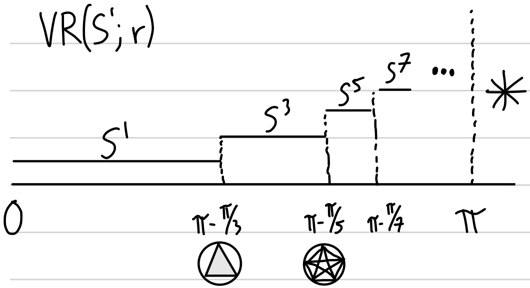




The <u>homotopy connectivity</u> $\mathrm{conn}(Y)$ of a space Y
is the largest k such that $\pi_i(Y)=0 \ \ \forall i\leq k$.

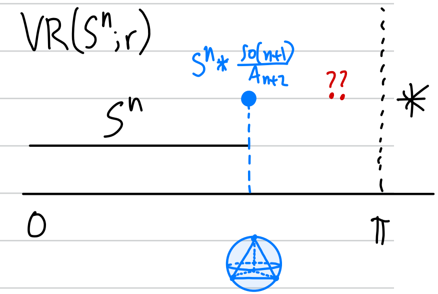
Thm For $\delta>0$, if $\mathrm{conn}(VR(S^n;\pi-\delta))=k-1$, then
$\mathrm{cov}_{S^n}(2k+2)\le\delta\le 2\cdot\mathrm{cov}_{\mathbb{RP}^n}(k)$.

I.e., $\frac{1}{2}\,\mathrm{numCov}_{S^n}(\delta)-2\le\mathrm{conn}(VR(S^n;\pi-\delta))<\mathrm{numCov}_{\mathbb{RP}^n}\left(\frac{\delta}{2}\right)$.



$\mathrm{conn}(VR(S^1;\pi-\delta))=2\ell-2$
for $\delta\in\left[\frac{\pi}{2\ell+1},\frac{\pi}{2\ell-1}\right)\subseteq\left[\frac{\pi}{4\ell},\frac{\pi}{2\ell-1}\right]$.

$\mathrm{conn}(VR(S^1;\pi-\delta))=2\ell-1$
for $\delta\in\emptyset\subseteq\left[\frac{\pi}{4\ell+2},\frac{\pi}{2\ell}\right]$.



Corollary The homotopy type of $VR(S^n;r)$ changes infinitely many times as r increases.

Check
$\pi-r_n\le 2\cdot\mathrm{cov}_{\mathbb{RP}^n}(n)$

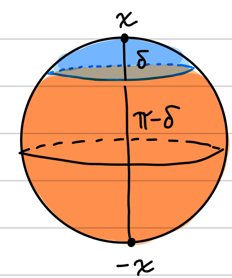
Conjecture Only countably many times.
Conjecture For all n, $\mathrm{conn}(VR(S^n;r))$ is nondecreasing in r.
Remark Čech case

Meshulam 2001: If a finite graph G satisfies $\bigcap_{j=1}^{\ell} N_G(v_j) \neq \emptyset$ for any ℓ vertices $v_1, \ldots, v_\ell$, then $\tilde{H}_i(cl(G)) = 0$ for $0 \leq i \leq \lfloor \frac{\ell}{2} \rfloor - 1$.

Here $N_G(v_j) = \{v : v \sim v_j\}$ is the open neighborhood.
The clique complex $cl(G)$ is the maximal simplicial complex with G as its 1-skeleton.
The proof uses a fancy nerve lemma.

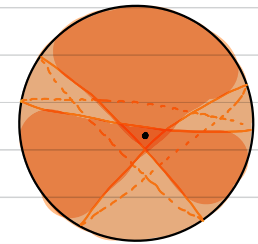
Proof sketch that $\delta < \mathrm{cov}_{S^n}(2k+2) \Rightarrow \mathrm{conn}(VR(S^n; \pi-\delta)) \geq k$.

Note $\delta < \mathrm{cov}_{S^n}(2k+2)$
$\Rightarrow$ No $2k+2$ closed δ-balls cover S^n
$\Rightarrow$ Any $2k+2$ open $(\pi-\delta)$-balls intersect.



Pretending that Meshulam's result held for $G = VR(S^n; \pi-\delta)^{(1)}$ infinite and closed neighborhoods $N_G[v] = N_G(v) \cup \{v\} = B(v; \pi-\delta)$, we'd get
$\tilde{H}_i(cl(G)) = \tilde{H}_i(VR(S^n; \pi-\delta)) = 0$ for $0 \leq i \leq \lfloor \frac{2k+2}{2} \rfloor - 1 = k$.
For $n \geq 2$, use simple-connectedness of $VR(S^n; \pi-\delta)$ and Hurewicz to get
$\pi_i(VR(S^n; \pi-\delta)) = 0$ for $0 \leq i \leq k$.

Pretending is not necessary: Prove $\tilde{H}_i(VR(X;\pi-\delta))=0$ for finite subsets $X \subseteq S^n$ dense enough s.t. any ε-ball in S^n contains $\geq 2k+3$ points in X, where $\varepsilon>0$ satisfies $\delta+\varepsilon < \mathrm{cov}_{S^n}(2k+2)$.
Deduce $\tilde{H}_i(VR(S^n;\pi-\delta))=0$.

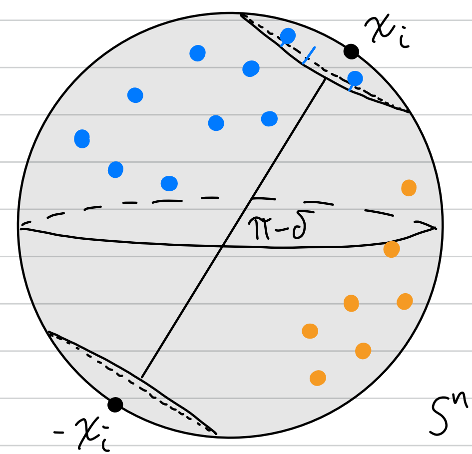


Point in intersection of open $(\pi-\delta-\varepsilon)$-balls

$\underline{\text{Proof sketch}}$ that $\delta > 2\cdot \mathrm{cov}_{\mathbb{RP}^n}(k) \Rightarrow \mathrm{conn}(VR(S^n;\pi-\delta)) \le k-2$.

If the free $\mathbb{Z}/2$ space $VR(S^n;\pi-\delta)$ were $(k-1)$ connected, we'd have $S^k \xrightarrow{\mathbb{Z}/2} VR(S^n;\pi-\delta)$.

By Borsuk-Ulam this contradicts ABF, which says if $\exists$ k open balls $\{B([x_i];\delta/2)\}_{i=1}^k$ covering $\mathbb{RP}^n$, then $\exists$ $VR(S^n;\pi-\delta) \xrightarrow{\mathbb{Z}/2} S^{k-1}$.



Indeed, no $\sum_j \lambda_j y_j \in |VR(S^n;\pi-\delta)|$ has vertex set intersecting both $B(x_i;\delta/2)$ and $B(-x_i;\delta/2)$.
Define $f_i : VR(S^n;\pi-\delta) \to \mathbb{R}$ by

$$f_i\left(\sum_j \lambda_j y_j\right) = \begin{cases} \sum_j \lambda_j d(y_j, S^n \setminus B(x_i;\delta/2)) & \text{if no } y_j \text{ in } B(-x_i;\delta/2) \\ -\sum_j \lambda_j d(y_j, S^n \setminus B(-x_i;\delta/2)) & \quad \ldots \quad B(x_i;\delta/2). \end{cases}$$

Hence $(f_1,\ldots,f_k) : VR(S^n;\pi-\delta) \xrightarrow{\mathbb{Z}/2} \mathbb{R}^k \setminus \{\vec{0}\}$.
Radially project.