Persistent Homology of Products and Gromov-Hausdorff Distances Between Hypercubes and Spheres

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First Part of the Dissertation: Persistent Homology of Products

- Spectral Sequences and Persistent Homology
- Analysis of the Categorical Product Filtration $X \times Y$.

Basu, Saugata and Parida, Laxmi; "Spectral sequences, exact couples, and persistent homology of filtrations"; *Expo Math* 35 (2017) 119-132.

Theorem (Basu and Parida)

Let

$$X : \quad \emptyset = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{N-1} \subseteq X_N$$

be an increasing filtration of simplicial complexes, where $X_i = \emptyset$ *if* $i < 0$ *and* $X_i = X_N$ *if* $i > N$*. Then for every integer* r, p, q *with* $r \geq 1$*,*

$$rank\ (E^r_{p,q}(X)) = b_n^{p,p+r-1}(X) - b_n^{p-1,p+r-1}(X) + b_{n-1}^{p-r,p-1}(X) - b_{n-1}^{p-r,p}(X),$$

where $p + q = n$*,* $b_n^{s,t} = rank(H_n^{s,t}(X))$*, and each* $H_n^{s,t}(X)$ *is a finitely generated vector space.*

The sequence below was used to determine the rank in the previous theorem.

$$\cdots \to H_n^{p,p+r-1}(X) \xrightarrow{j^{(r)}_{p+r-\mathbf{1},q-r+\mathbf{1}}} E^{(r)}_{p,q}(X) \xrightarrow{\partial^{(r)}_{p,q}} H_{n-1}^{p-r,p-1}(X) \xrightarrow{i^{(r)}_{p-\mathbf{1},q}} H_{n-1}^{p-r+1,p}(X) \to \cdots,$$

where $\mathrm{Im}(i^{(r)}_{p+r-1,q-r+1}) = H_n^{p,p+r}(X)$.

Analysis of the Categorical Product Filtration $X \times Y$.

Theorem (Vargas-Rosario)

For each $r \geq 1$ and $n = p + q$, the groups $E_{*,*}^{(*)}(X \times Y)$ and the persistent homology groups $H_{*,*}^{(*)}(X)$ and $H_{*,*}^{(*)}(Y)$ are related by a long exact sequence of the following form.

$$\ldots \to \bigoplus_{l+j=n} H_l^{p,p+r-1}(X) \otimes H_j^{p,p+r-1}(Y) \to E_{p,q}^{(r)}(X \times Y)$$

$$\to \bigoplus_{l+j=n-1} H_l^{p-r,p-1}(X) \otimes H_j^{p-r,p-1}(Y)$$

$$\to \bigoplus_{l+j=n-1} H_l^{p-r+1,p}(X) \otimes H_j^{p-r+1,p}(Y) \to \ldots .$$

Analysis of the Categorical Product Filtration $X \times Y$.

$$
\begin{array}{ccccc}
\cdots \bigoplus_{l+j=n} H_l^{p,p+r-\mathbf{1}}(X) \otimes H_j^{p,p+r-\mathbf{1}}(Y) & \longrightarrow & E^{(r)}_{p,q}(X \times Y) & \longrightarrow & \bigoplus_{l+j=n-\mathbf{1}} H_l^{p-r,p-\mathbf{1}}(X) \otimes H_j^{p-r,p-\mathbf{1}}(Y) \cdots \\
\downarrow & & \downarrow{\scriptstyle id} & & \downarrow \\
\cdots H_n^{p,p+r-\mathbf{1}}(X \times Y) & \xrightarrow{j^{(r)}_{p+r-\mathbf{1},q-r+\mathbf{1}}} & E^{(r)}_{p,q}(X \times Y) & \xrightarrow{\partial^{(r)}_{p,q}} & H^{p-r,p-\mathbf{1}}_{n-\mathbf{1}}(X \times Y) \overset{i^{(r)}_{p-\mathbf{1},q}}{\rightarrow} \cdots .
\end{array}
$$

Example

Consider X to be the filtration $S^0 \subset S^1$ (so $X_0 = S^0$ and $X_1 = S^1$), and Y is the same filtration $S^0 \subset S^1$.

$$
H_n^{s,t}(X \times Y) \cong \bigoplus_{l+j=n} H_l^{s,t}(X) \otimes H_j^{s,t}(Y).
$$

Analysis of the Categorical Product Filtration $X \times Y$.

- Inclusions $X_p \hookrightarrow X_{p+r-1}$ and $Y_p \hookrightarrow Y_{p+r-1}$

  Then the Künneth formula is natural:

$$
\begin{array}{ccc}
\bigoplus_{l+j=n} H_l(X_p) \otimes H_j(Y_p) & \longrightarrow & \bigoplus_{l+j=n} H_l(X_{p+r-1}) \otimes H_l(Y_{p+r-1}) \\
\downarrow & & \downarrow \\
H_n(X_p \times Y_p) & \longrightarrow & H_n(X_{p+r-1} \times Y_{p+r-1}).
\end{array}
$$

  and the vertical arrows are isomorphisms.

Analysis of the Categorical Product Filtration $X \times Y$.

- We restrict

$$\alpha : \bigoplus_{l+j=n} H_l^{p,p+r-1}(X) \otimes H_j^{p,p+r-1}(Y) \to H_n^{p,p+r-1}(X \times Y),$$

where $\alpha\left(\sum_{l+j=n} ([z_l] \otimes [w_j]) \right) = \sum_{l+j=n} [z_l \otimes w_j]$ is an isomorphism, and H is a Persistent Homology.

$$\begin{array}{ccccc}
\cdots \bigoplus_{l+j=n} H_l^{p,p+r-\mathbf{1}}(X) \otimes H_j^{p,p+r-\mathbf{1}}(Y) & \longrightarrow & E^{(r)}_{p,q}(X \times Y) & \longrightarrow & \bigoplus_{l+j=n-\mathbf{1}} H_l^{p-r,p-\mathbf{1}}(X) \otimes H_j^{p-r,p-\mathbf{1}}(Y) \cdots \\
\downarrow & & \downarrow{\scriptstyle id} & & \downarrow \\
\cdots H_n^{p,p+r-\mathbf{1}}(X \times Y) & \xrightarrow{j^{(r)}_{p+r-\mathbf{1},q-r+\mathbf{1}}} & E^{(r)}_{p,q}(X \times Y) & \xrightarrow{\partial^{(r)}_{p,q}} & H_{n-\mathbf{1}}^{p-r,p-\mathbf{1}}(X \times Y) \overset{i^{(r)}_{p-\mathbf{1},q}}{\to} \cdots .
\end{array}$$

- The top arrows are $j^r(\alpha)$, $(\alpha^{-1} \circ \partial^{(r)})$, and $\alpha^{-1} \circ i^{(r)} \circ \alpha$.
- The nine Lemma states the LES is exact, therefore;

$$dim(E^{(r)}_{p,q}(X \times Y)) = \sum_{l+j=n} b_l^{p,p+r-1}(X) b_j^{p,p+r-1}(Y) - dim(im(\alpha^{-1} \circ i^{(r)}_{p+r-2,q-r+2} \circ \alpha)) +$$

$$\sum_{l+j=n-1} b_l^{p,p+r-1}(X) b_j^{p,p+r-1}(Y) - dim(im(\alpha^{-1} \circ i^{r}_{p-1,q} \circ \alpha)).$$

Second Part of the Dissertation: Gromov-Hausdorff Distances Between Hypercubes and Spheres

- Hausdorff Distance and Gromov-Hausdorff Distance
- The Geodesic Metric, Hypercubes, and the Coindex, and Borsuk-Ulam Theorem.
- Lower Bounds for $d_{GH}(I_g^{n+1}, S^n)$.
- The Main Theorem.

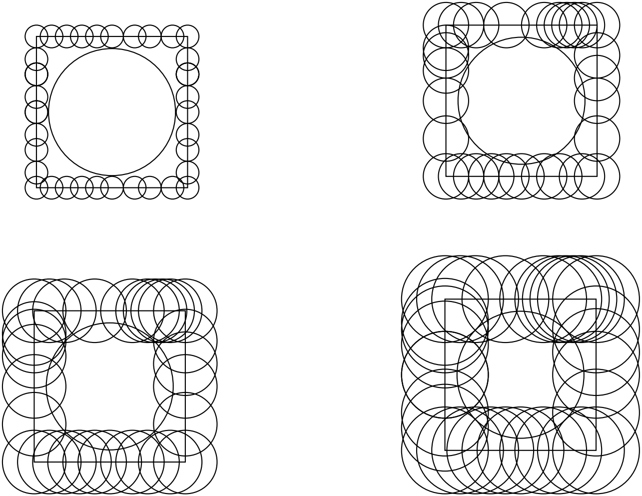
The Hausdorff Distance

Definition

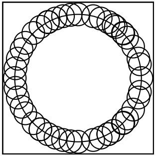
Let Z be a metric space. Let X and Y be metric spaces contained in Z. Then the Hausdorff Distance is

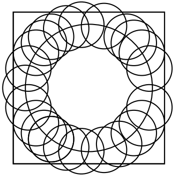
$$d_H(X, Y) = \inf\{r \geq 0 | X \subseteq B(Y; r) \text{ and } Y \subseteq B(X, r)\}.$$

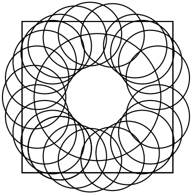
The Hausdorff Distance

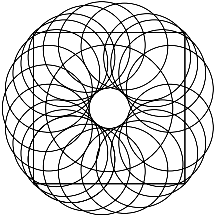


The Hausdorff Distance









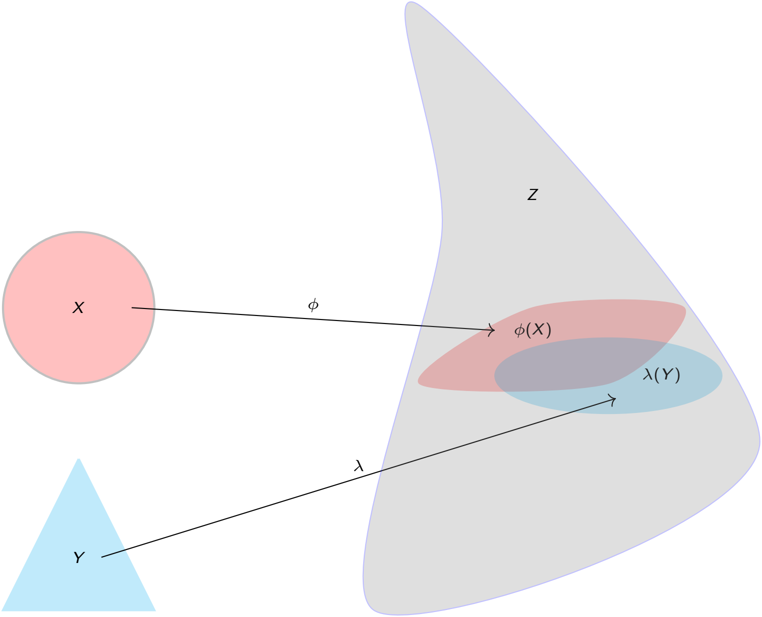
Gromov-Hausdorff Distance

Definition

Suppose that X and Y are bounded metric spaces. The Gromov-Hausdorff distance is

$$d_{GH}(X, Y) = \inf_{\substack{\phi\colon X \hookrightarrow Z \\ \lambda\colon Y \hookrightarrow Z}} \{d_H^Z(\phi(X), \lambda(Y))\},$$

where ϕ and λ are all possible isometric embeddings.



Z
X
ϕ
$\phi(X)$
$\lambda(Y)$
λ
Y

The Geodesic Metric, Hypercubes, and the Coindex

Definition (Geodesic metric)

$$d_{S^n}(x, x') = \arccos\left(\langle x, x' \rangle\right) = 2 \arcsin\left(\frac{\|x - x'\|}{2}\right)$$

Example

Consider I^4. Compute the Hamming metric between 0110 and 0011.
$d(0110, 0011) = 2$

Definition (Coindex of a space)

Let X be a $\mathbb{Z}_2$-space, then $\mathrm{coind}_{\mathbb{Z}_2}(X) = \max\{k \geq 0 | S^k \xrightarrow{\text{odd}} X\}$

The Borsuk-Ulam Theorem and the Lower Bounds for $d_{GH}(Y, S^m)$

Theorem (Borsuk-Ulam Theorem)

There is no odd map $S^n \to S^{n-1}$

Theorem

For $m, n \geq 0$ and for $Y \subseteq S^n$ with $Y = -Y$ and Y equipped with the geodesic distance, we have

$$2 \cdot d_{GH}(Y, S^m) \geq \inf\{r \geq 0 \mid \mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(Y; r)) \geq m\} =: c_m(Y).$$

- $2 \cdot d_{GH}(I_g^{n+1}, S^n) \geq \inf\{r \geq 0 \mid \operatorname{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_g^{n+1}; r)) \geq n\} =: c_m(I_g^{n+1})$
- We consider I_g^{n+1} living in S^n.
- Vertices of I_g^{n+1} are $(\pm\frac{1}{\sqrt{n+1}}, \pm\frac{1}{\sqrt{n+1}}, \ldots, \pm\frac{1}{\sqrt{n+1}})$.

Homotopy types of $\mathrm{VR}(I_h^n; r)$

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
$r=0$	S^0	$\bigvee^3 S^0$	$\bigvee^7 S^0$	$\bigvee^{15} S^0$	$\bigvee^{31} S^0$	$\bigvee^{63} S^0$	$\bigvee^{127} S^0$	$\bigvee^{255} S^0$	$\bigvee^{511} S^0$
$r=1$	$\bullet$	S^1	$\bigvee^5 S^1$	$\bigvee^{17} S^1$	$\bigvee^{49} S^1$	$\bigvee^{129} S^1$	$\bigvee^{321} S^1$	$\bigvee^{769} S^1$	$\bigvee^{1793} S^1$
$r=2$	$\bullet$	$\bullet$	S^3	$\bigvee^9 S^3$	$\bigvee^{49} S^3$	$\bigvee^{209} S^3$	$\bigvee^{769} S^3$	$\bigvee^{2561} S^3$	$\bigvee^{7937} S^3$
$r=3$	$\bullet$	$\bullet$	$\bullet$	S^7					
$r=4$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	S^{15}				
$r=5$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	S^{31}			
$r=6$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	S^{63}		
$r=7$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	S^{127}	
$r=8$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	S^{255}

The Gromov-Hausdorff Distance for Particular Values of n

i. $2 \cdot d_{GH}(I_g, S^0) \geq \inf\{r \geq 0 | \operatorname{coind}_{\mathbb{Z}_2}(\operatorname{VR}(I_g; r)) \geq 0\}$
$2 \cdot d_{GH}(I_g, S^0) \geq 0$.

ii. $2 \cdot d_{GH}(I_g^2, S^1) \geq \inf\{r \geq 0 | \operatorname{coind}_{\mathbb{Z}_2}(\operatorname{VR}(I_g^2; r)) \geq 1\}$
When $n = 1$, $\operatorname{VR}(I_g^{n+1}; r)$ changes at $r = 0, \pi/2, \pi$.
$2 \cdot d_{GH}(I_g^2, S^1) \geq \pi/2$.

iii. $2 \cdot d_{GH}(I_g^3, S^2) \geq \inf\{r \geq 0 | \operatorname{coind}_{\mathbb{Z}_2}(\operatorname{VR}(I_g^3; r)) \geq 2\}$
$r = 0$, $\cos^{-1}(1/3)$, $\cos^{-1}(-1/3)$, and π.
$2 \cdot d_{GH}(I_g^3, S^2) \geq \cos^{-1}(-1/3)$.

iv. When $n = 3$; $r = 0, \cos^{-1}(1/2), \pi/2, \cos^{-1}(-1/2), \pi$.
$2 \cdot d_{GH}(I_g^4, S^3) \geq \pi/2$.

v. When $n = 4$;
$r = 0, \cos^{-1}(3/5), \cos^{-1}(1/5), \cos^{-1}(-1/5), \cos^{-1}(-3/5), \pi$.
$2 \cdot d_{GH}(I_g^5, S^4) \geq \cos^{-1}(-1/5)$.

vi. When $n = 5$:
$2 \cdot d_{GH}(I_g^6, S^5) \geq \pi/2$.

vii. When $n = 6$:
$2 \cdot d_{GH}(I^7, S^6) \geq \cos^{-1}(1/7)$.

Conjecture:

$$\mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_g^{n+1}; \cos^{-1}(\tfrac{n+1-2i}{n+1}))) \geq 2^i - 1.$$

$$\mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_h^n; r)) \geq 2^r - 1.$$

$t(n)$ and the Main Theorem

Definition

Let $t(n)$ be the smallest scale parameter such that we can divide I_h^n into n-dimensional simplices of diameter at most $t(n)$.

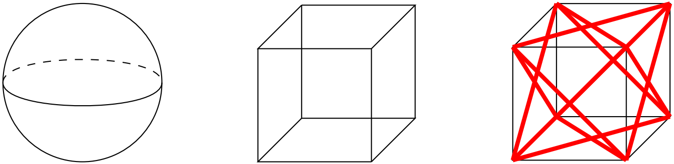
n	1	2	3	4	5	6	7	8	9
$t(n)$	1	2	2	≤ 3	≤ 4	≤ 4	≤ 5	≤ 6	≤ 6

Theorem

Consider the space I_h^n, then $\operatorname{coind}_{\mathbb{Z}_2}(\operatorname{VR}(I_h^n; r)) \geq n-1$ for $r \geq t(n-1)$

- Far enough to the right in the table: $\operatorname{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_h^n; r)) \geq n-1$.
- Far enough down in the table: $\operatorname{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_h^n; r)) \geq 2^r - 1$.
- To verify that the coindex $\geq n-1$, we need $S^{n-1} \to \mathrm{VR}(I_h^n; r)$.
- Start with $S^2 \to \mathrm{VR}(I_h^3; 2)$.
  $\phi_3 : S^2 \to$ 3-cube, $\phi_3(x, y, z) = \frac{(x,y,z)}{\max\{|x|,|y|,|z|\}}$.
  f : 3-cube $\to \mathrm{VR}(I_h^2; 2)$, Take face
  $\{(x, y, z) | x = -1, |y| \leq 1, |z| \leq 1\}$,

$$(y, z) \to \frac{(z+1)(1+y)}{4}(-1,-1,-1) + \frac{(1-z)(1+y)}{4}(-1,1,-1)$$
$$+\frac{(z+1)(1-y)}{4}(-1,-1,1) + \frac{(1-z)(1-y)}{4}(1,1,1).$$



The Best Lower Bound r Smaller Than $n-1$.

- Construct a map $S^3 = \partial([0,1]^4) \to \mathrm{VR}(I_h^4; 2)$ mapping each $[0,1]^3$ to each $\mathrm{VR}(I_h^3; 2)$.
- Step 1:
- Triangulate $[0,1]^3$ into tetrahedra.

$\tau_1 = \{(0,1,1),(1,1,0),(1,0,1),(0,0,0)\},$
$\tau_2 = \{(0,0,0),(1,0,0),(1,1,0),(1,0,1)\},$
$\tau_3 = \{(0,1,1),(1,1,0),(0,0,0),(0,1,0)\},$
$\tau_4 = \{(1,1,1),(0,1,1),(1,1,0)(1,0,1)\},$
$\tau_5 = \{(0,0,0),(0,1,1),(0,0,1)(1,0,1)\}.$

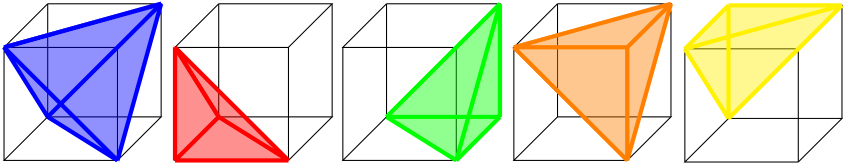


Figure: Triangulation of the cube into five tetrahedra. Here, we have the tetrahedra $\tau_1, \tau_2, \tau_3, \tau_4$, and τ_5, respectively. The lavender tetrahedron, τ_1, is the balanced tetrahedron.

- Step 2:
  - Map each tetrahedron of diameter 2.
  - $h_*\colon [0,1]^3 \to \mathrm{VR}(I_h^3; 2)$, where $h_*(x) = \sum_{v \in \tau_i} \lambda_v v$.
- The scale $r < n - 1$ when moving to higher dimensions.
- $[0,1]^4 = [0,1]^3 \times [0,1] = (\bigcup_{i=1}^{5} \|\tau_i\|) \times (\|\tau\|) = \bigcup \|\tau_i \times \tau\|$, diam ≤ 3.

  $[0,1]^4 = [0,1]^2 \times [0,1]^2$, diam ≤ 4.

Definition

Let $t(n)$ be the smallest scale parameter such that we can divide I_h^n into n-dimensional simplices of diameter at most $t(n)$.

n	1	2	3	4	5	6	7	8	9
$t(n)$	1	2	2	≤ 3	≤ 4	≤ 4	≤ 5	≤ 6	≤ 6

Theorem

Consider the space I_h^n, *then* $\operatorname{coind}_{\mathbb{Z}_2}(\operatorname{VR}(I_h^n; r)) \geq n-1$ *for* $r \geq t(n-1)$

Proof of the Main Theorem

- The goal is to produce an odd map $\phi\colon \partial([0,1]^n) \to \mathrm{VR}(I_h^n; r)$.
- Triangulate the faces of the n-cube into $(n-1)$-simplices of diameter at most $t(n-1)$.
- ϕ from $2n$ faces.
- $\tau_{i,j}$ are the $(n-1)$-simplices of diameter at most $t(n-1) \le r$.
- $[0,1]_j^{n-1} = \bigcup_{i=1}^{m} \|\tau_{i,j}\|$.
- $[0,1]^{n-1} \to \mathrm{VR}(I_h^n; r)$ defined by $\sum_{v\in\tau_i} \lambda_v v \to \sum_{v\in\tau_i} \lambda_v v$.
- $v \to -v$, then the antipode is sent to the opposite face of $\mathrm{VR}(I_h^n; r)$.

Possible Future Research

- Suppose that M and N are the same filtrations of Vietoris-Rips complexes of hypercubes; i.e,

$$M = N\colon\ VR(I^k;1) \subseteq VR(I^k;2) \subseteq \cdots \subseteq VR(I^k;s-1) \subseteq VR(I^k;s) \subseteq \cdots$$

then

$$\ldots \to \bigoplus_{l+j=n} PH_l(M_p) \otimes PH_j(N_p) \to E^{(r)}_{p,q}(M_p \times N_p) \to \bigoplus_{l+j=n-1} PH_l(M_p) \otimes PH_j(N_p) \to \ldots$$

is a long exact sequence.

- What about the sup metric $l_\infty((x_1, x_2, ..., x_k), (y_1, y_2, ..., y_k)) = max\{d_i(x_i, y_i) \colon i = 1, 2, \ldots, k\}$?
- ? $d_{GH}(I^n, I^m) \geq \inf\{r \geq 0 | \operatorname{coind}_{\mathbb{Z}_2}(\operatorname{VR}(I^n; r)) \geq m\}$
- ? $d_{GH}(I^n, S^m) \geq \inf\{r \geq 0 | \operatorname{coind}_{\mathbb{Z}_2}(\operatorname{VR}(I^n; r)) \geq m\}$

Thank you!

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