Persistent Homology of Products and Gromov-Hausdorff Distances Between Hypercubes and Spheres

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Abstract

An exploration in the first half of this dissertation of the relationships among spectral sequences, persistent homology, and products of simplices, including the development of a new concept in categorical product filtration, is followed in the second half by new determinations of a) lower bounds for the Gromov-Hausdorff distance between n-spheres and $(n+1)$-hypercubes equipped with the geodesic metric and of b) new lower bounds for the coindexes of the Vietoris-Rips complexes of hypercubes equipped with the Hamming metric. In their paper,"Spectral Sequences, Exact Couples, and Persistent Homology of Filtrations" [5], Basu and Parida worked on building an n-derived exact couple from an increasing filtration X of simplicial complexes, $\mathcal{C}^{(n)}(X) = \{D^{(n)}(X), E^{(n)}(X), i^{(n)}, j^{(n)}, \partial^{(n)}\}$. The terms $E^{(n)}_{*,*}(X)$ are the bigraded vector spaces of a spectral sequence that has differentials $d^{(r)}(X)$, and the terms $D^{(n)}_{*,*}(X)$ are the persistent homology groups $H^{*,*}_*(X)$. They proved that there exists a long exact sequence whose groups are $H^{*,*}_*(X)$ and whose bigraded vector spaces are $(E^*_{*,*}(X), d^*(X))$. We establish in Section 3 of this dissertation a new, similar theorem

in the case of the categorical product filtration $X \times Y$ that states that there exists a long exact sequence consisting of $\bigoplus\limits_{l+j=n} H_l^{*,*}(X) \otimes H_j^{*,*}(Y)$ and of the bigraded vector spaces $E^*_{*,*}(X \times Y)$ of $(E^*_{*,*}(X \times Y), d^*(X \times Y))$, and prove it in part using Künneth formulas on homology. The emphasis on product spaces continues in Section 5, where we establish new lower bounds for the Gromov-Hausdorff distance between n-spheres and $(n+1)$-hypercubes, I^{n+1}, when both are equipped with the geodesic distance. From these lower bounds, we conjecture new lower bounds for the coindices of the Vietoris-Rips complexes of hypercubes when equipped with the Hamming metric. We then determine new lower bounds for the coindices of the Vietoris-Rips complexes of hypercubes, a) by producing a map between spheres and the geometric realizations of Vietoris-Rips complexes of hypercubes using abstract convex combination and balanced sets, and b) by decomposing hollow n-cubes (homotopically equivalent to the above-mentioned spheres) into simplices of smaller dimension and smaller diameter.

Contents

1 Introduction

Homology can determine geometric features such as the number of non-connected components, holes, and voids in a topological space. Furthermore, *persistent* homology describes how long these topological features persist in a filtration of spaces. This dissertation project aims to illuminate the links between persistent homology, spectral sequences arising from exact couples, $\mathbb{R}^+$-filtered simplicial sets (X, l), metric spaces, product spaces, hypercubes I^n, and Gromov-Hausdorff distances between spaces. My original dissertation work consists of two parts: one, proving the existence of a long exact sequence linking spectral sequences with the tensor products of persistent homologies of two different simplicial complexes, and two, constructing theoretical lower bounds for the Gromov-Hausdorff distance between n-dimensional spheres and $(n+1)$-dimensional hypercubes, when the hypercube is endowed with the geodesic metric.

A spectral sequence, denoted $(E^*_{*,*}d^*)$, is a technical tool used by topologists and algebraic geometers. We can describe a spectral sequence as a bookkeeping device that consists of an infinite number of pages, starting in some cases with the 0-th page, with each page containing an infinite number of chain or cochain complexes. Among others, spectral sequences can arise from filtered differential graded modules, double complexes, or exact couples. In the case of exact couples $\mathcal{C} = \{D, E, i, j, \partial\}$, from the three maps i, j, and ∂, we can define the differentials d^r and we can obtain the E^r-terms of the spectral sequence, as described in section 2.1.2 "Exact Couples."

In Section 2.1.3, we study the connection between an increasing filtration of simplicial complexes with a spectral sequence arising from an exact couple when the ring is a field F, as described by Basu and Parida in their paper "Spectral Sequences, Exact Couples, and Persistent Homology of Filtrations"[5]. The increasing filtration, namely X, leads to a long exact sequence (LES) in homology. We use the maps of this LES to construct bigraded maps d^1. Then, we take the direct sum of the homology and the relative homology over $n = p + q$ to build groups $D^1(X)$ and $E^1(X)$, and to form the exact couple from which the spectral sequence $(E^{(r)}(X), d^{(r)})$ is derived. Moreover, we present the relation between $D^{(r)}(X)$ and $i^{(r)}_{*,*}$ with the persistent homology corresponding to the filtration X. This is what ultimately gives us the desired result, which is an exact sequence (2.8) that relates the persistent homology $H^{*,*}_*(X)$ with $E^*_{*,*}(X)$, via the maps $i^{(r)}, j^{(r)}$, and $\partial^{(r)}$.

The next important concept is the use of Künneth formulas with persistent homology. Section 2.2 presents these Künneth formulas in which each group of the short exact sequence is a module over the ring $k[\mathbb{R}^+]$. Theorem 2.20 deals with the persistent homology of an $\mathbb{R}^+$-filtered simplicial set (X, l). A filtered simplicial set (X, l) is defined using a contravariant functor $X : \triangle \to \mathbb{R}^+$-sSet and a map on n-simplices, $l \colon X_n \to \mathbb{R}^+$. These $\mathbb{R}^+$-filtered simplicial sets play an essential role because the groups $F_*(X, l)$ depend on the n-simplices of X and also on the simplices that are born at $l(\sigma)$, where $\sigma \in X_n$. These groups $F_*(X, l)$ determine the chain groups that define the homology of (X, l) and therefore the persistent homology of (X, l).

In Section 2.3 the map l is defined in terms of the maximum pairwise distance in a subset of a metric space, and with this definition, we can define the persistent homology of a metric space. Then, following the presentation in Carlsson and Fileppenko's paper "Persistent Homology of the Sum Metric" [6], we provide Künneth formulas describing the relationship between two metric spaces, such as the one that appears in Theorem 2.23.

In Section 2.4 we study persistent homology of hypercubes I^k for $k \geq 1$. In some sense, hypercubes are the simplest of all product spaces. The concepts and theorems of three subjects — persistent homology of (X, l), the modules $F_*(X, l)$, and persistent homology of metric spaces — were developed in order to study the persistent homology of hypercubes up to dimension 2 [6]. The Künneth formulas of metric spaces and the definition of coordinate inclusions $\phi : I^k \to I^r$, where $k \leq r$, are needed to show the isomorphism types of the persistent homology of hypercubes. The i-dimensional persistent homology group $PH_i(I^k)$ is identified as the direct sums of bars, which is another way to represent a persistent homology. We see that by equipping I^k with the Hamming metric, the persistent homology of dimension 0 and of dimension 1 are non-trivial for $k \geq 1$ while the 2-dimensional persistent homology is zero [6]. Also, we provide another important statement from the same paper that relates coordinate inclusions to $PH_1(I^k)$ for $k > 0$. This 1-dimensional persistent homology is generated by the union of $\phi_*(PH_1(I^2))$ over the coordinate inclusions. This result is needed in order to complete the proof that the 2-dimensional persistent homology of I^k is zero when $k > 0$ [6].

The concept of the LES as defined by Basu and Parida in their paper "Spectral Sequences, Exact Couples, and Persistent Homology of Filtrations" [5] inspired us in Section 3 to begin our own individual work in establishing a relation between the persistent homology of two increasing filtrations of simplicial complexes X and Y, and of the $E^r(X \times Y)$ terms of the spectral sequence, where the increasing filtration $X \times Y$ is known as the categorical product. We developed Theorem 3.2 in order to refine those results for the categorical product: the theorem states that there exists an exact sequence whose groups are $\bigoplus_{l+j=n} H_l^{*,*}(X) \otimes H_j^{*,*}(Y)$ and $E^r(X \times Y)$. At first, we were not sure if the sequence in Theorem 3.2 was exact, but later we employed various maneuvers such as applying Künneth formulas on homology and creating a functorial commutative diagram in order to be able to complete the proof of this theorem.

In Section 4.1, we introduce the Hausdorff distance and use it to define the Gromov-Hausdorff distance. We also provide a different expression for the Gromov-Hausdorff distance in terms of the distortion of correspondences [7]. Furthermore, we discuss in this section the definition of the Vietoris-Rips complex, which will be used later in the dissertation to produce not only continuous maps from discontinuous functions, but also to discover topological obstructions. In this same section, we apply the Vietoris-Rips complex to lower bound the

Gromov-Hausdorff distance between specific metric spaces.

We turn our attention in Section 4.2 to the determination of bounds, particularly lower bounds, for the Gromov-Hausdorff distance between spheres S^n using the geodesic metric. We begin by studying the concepts in the paper "The Gromov–Hausdorff Distance Between Spheres" [7], by Lim, Memoli, and Smith, in which they found lower bounds using different tools such as distortion, the distortion-preserving lemma, Dubins-Schwarz's Theorem, and the Borsuk-Ulam Theorem. First, they applied the concepts of persistent homology and bottleneck distance in order to provide weak lower bounds for $d_{GH}(S^m, S^n)$, where $0 \leq m \leq n \leq \infty$. Later, they were able to construct better lower bounds by applying the Dubins-Schwarz Theorem 4.8 and the Distortion-Preserving Lemma 4.9, which is a generalization of the Borsuk–Ulam theorem for possibly discontinuous functions. Furthermore, in the paper "Gromov-Hausdorff Distances, Borsuk-Ulam Theorems, and Vietoris-Rips Complexes" by Adams et. al.([11], p. 14), I collaborated with multiple other authors to develop a theorem, presented herein in the same Section 4.2, in which we added the concepts of coindices and Vietoris-Rips complexes of spaces to get much stronger lower bounds for $d_{GH}(S^m, S^n)$ when $0 \leq m \leq n$.

We resume our original dissertation work in Section 5. Here, we develop new lower bounds for the Gromov-Hausdorff distance between the n-sphere and the $(n+1)$-hypercube when the latter is supplied with the geodesic metric. We begin by working with the Gromov-Hausdorff distance between spheres when both are equipped with the geodesic metric, which was the subject of the collaboration paper by Adams, Mémoli, and Frick, et.al., called "Gromov-Hausdorff Distances, Borsuk-Ulam Theorems, and Vietoris-Rips Complexes" in which I participated [11]. In Subsection 5.1, we independently develop a new tool, Theorem 5.1, to determine lower bounds for the Gromov-Hausdorff distance when hypercubes and spheres are provided with the geodesic metric, using the Bursuk-Ulam Theorem and the Vietoris-Rips complex at different scale parameters. Using this theorem, in Subsection 5.2

we conjecture a lower bound for the coindex of the Vietoris-Rips complex of hypercubes using the geodesic metric, and we restate the coindex using the Hamming metric. Next, in order to verify the conjecture, we develop in Subsection 5.3 odd maps between spheres and the Vietoris-Rips complexes of hypercubes, using balanced sets of vectors, abstract convex combination, and triangulation of n-cubes into cubes of smaller dimensions. Towards the end of Subsection 5.3, we develop our principal Theorem 5.14 establishing a better lower bound for the coindex of the Vietoris-Rips complex of hypercubes using the Hamming metric, the desired result.

Finally, in Section 6, we present several topics of interest for future research: hypercubes form a common bridge between these possible topics. First, in Section 6.1 we propose the future possibility of connecting persistent homology of hypercubes with spectral sequences and investigating what Theorem 3.2 implies from the point of view of hypercubes. In Section 6.2, we indicate an interest not only in possibly applying Künneth formulas to hypercubes I^k, but also in computing the persistent homology of the homotopy types of Vietoris–Rips complexes of I^k at scale parameter 3. Finally, in Section 6.3 we discuss the possibility in the future of determining lower bounds and upper bounds for the Gromov–Hausdorff distances in other cases that also include hypercubes and spheres.

2 Background Material

2.1 Initial Information

2.1.1 Persistent Homology

The precise definition of Persistent Homology is:

Definition 2.1. *Suppose we have an increasing filtration of a simplicial complex*

$$X: \quad \emptyset = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{N-1} \subseteq X_N.$$

Let $i^{s,t}$ be the inclusion map

$$i^{s,t}\colon X_s \hookrightarrow X_t$$

where $s \leq t$. Then the image of the map induced by inclusion,

$$i_n^{s,t}(X)\colon H_n(X_s) \to H_n(X_t),$$

*is the n-dimensional **persistent homology;** in other words, $H_n^{s,t}(X) = \mathrm{Im}\ i_n^{s,t}(X)$ is the n-dimensional persistent homology, as s and t vary over $0 \leq s \leq t \leq N$.*

Moreover, $b_n^{s,t}(X) = \mathrm{rank}(H_n^{s,t}(X))$ is called the persistent Betti number.

From the previous definition we notice that when $s = t$, $H_n^{s,s}(X) = H_n(X_s)$.

We also define the persistent multiplicities of a filtration X as follows:

Definition 2.2. *Let X be a filtration. If $i < j$ then the persistent multiplicities of X is given by*

$$\mu_n^{i,j}(X) = b_n^{i,j-1}(X) - b_n^{i,j}(X) - b_n^{i-1,j-1}(X) + b_n^{i-1,j}(X).$$

In Definition 2.2, $\mu_n^{i,j}$ determines the numbers of bars which are born at time i and die at time j.

2.1.2 Exact Couples

Definition 2.3. *Let R be a commutative ring with unity. Suppose that D and E are R-modules and that $i : D \to D$, $j : D \to E$, and $\partial : E \to D$ are R-module homomorphisms. We say that*

$$\begin{tikzcd} D \arrow[r, "i"] & D \arrow[d, "j"] \\ & E \arrow[ul, "\partial"] \end{tikzcd}$$

is an exact couple *if* $\mathrm{Im}\, i = \ker j$, $\mathrm{Im}\, j = \ker \partial$, *and* $\mathrm{Im}\, \partial = \ker i$. *We denote this exact couple as $\mathcal{C} = \{D, E, i, j, \partial\}$.*

From an exact couple we can define $d : E \to E$, where $d = j \circ \partial$. Since $\mathrm{Im}\, i = \ker \partial$, we have

$$d \circ d = j \circ \partial \circ j \circ \partial = j \circ (\partial \circ j) \circ \partial = j \circ 0 \circ \partial = 0.$$

So, it makes sense to talk about the homology group $H(E, d)$.

Remark 2.4. *Let $\mathcal{C} = \{D, E, i, j, \partial\}$ be an exact couple. Let us define $D' = \mathrm{Im}\, i$ and $E' = H(E, d) = \ker d / \mathrm{Im}\, d = \ker(j \circ \partial)/\mathrm{Im}\, (j \circ \partial)$. The map i' is defined as the restriction of the map i, namely $i' = i|_{i(D)} : D' \to D'$. Now, we define $j' : D' \to E'$ by $j'(i(x)) = j(x) + d(E)$ and $\partial' : E' \to D'$ by $\partial'(e + dE) = \partial(e)$. It can be shown that these maps are well-defined and that $\mathcal{C}' = \{D', E', i', j', \partial'\}$ is an exact couple.*

The exact couple $\mathcal{C}' = \{D', E', i', j', \partial'\}$ is known as the **first derived couple.**

We can define the exact couple $\mathcal{C}'' = \{D'', E'', i'', j'', \partial''\}$ in the same way we defined the exact couple $\mathcal{C}'$.

If we iterate this process n times, we get the ***n*-th derived couple** $\mathcal{C}^{(n)} = \{D^{(n)}, E^{(n)}, i^{(n)}, j^{(n)}, \partial^{(n)}\}$. The maps of the n-th derived couple are defined as follows: $i^{(n)}$ is the inclusion $i^{(n)} = i^{(n-1)}|_{i^{(n-1)}(D)} : D^{(n)} \to D^{(n)}$, the map $j^{(n)} : D^{(n)} \to E^{(n)}$ is defined by $j^{(n)}(i^{(n-1)}(x)) = j^{(n-1)}(x) + d(E)$, and $\partial^{(n)} : E^{(n)} \to D^{(n)}$ is defined by $\partial^{(n)}(e + dE) = \partial^{(n-1)}(e)$.

An exact couple leads us to a homological spectral sequence. To see this, consider the exact couple below

$$\begin{tikzcd} D \arrow[r, "i"] & D \arrow[d, "j"] \\ & E \arrow[ul, "\partial"] \end{tikzcd},$$

where the bidegrees of i, j, and ∂ are $(1,-1), (0,0)$, and $(-1,0)$; respectively. We define a differential $d^1 : E \to E$, where $E = E^1 = \bigoplus_{p,q} E^1_{p,q}$ and $d^1 = j \circ \partial$. The sum of the bidegree of j and ∂ is the bidegree of $d^1_{p,q}$. So, $d^1_{p,q}$ has bidegree $(0,0) + (-1,0) = (-1,0)$.

Since $d^1 \circ d^1 = 0$, we write $E^2_{p,q} = \ker d^1_{p,q}/\mathrm{Im}\, d^1_{p+1,q}$. Now, we set $E^2 = E'$ and let $D' = \mathrm{Im}\, i$. The map i' is restricted to $\mathrm{Im}\, i$, which is a subset of D, so that indicates the bidegree of each i' equals $(1,-1)$.

The bidegree of j' is $(-1,1)$ by the way we defined it in Remark 2.4 above. To see this more clearly, let $i(x) = y$. Since $y \in \mathrm{Im}\, i$, we have $i^{-1}(y) = x$, which allows us to write the map j' as

$$j'(y) = j(i^{-1}(y)) + d(E).$$

The map i^{-1} has bidegree $(-1,1)$, so when adding the bidegrees of j and i^{-1}, we get the bidegree of j'. The bidegree of j' equals $(-1,1) + (0,0) = (-1,1)$.

The map ∂' is defined in terms of ∂, according to Remark 2.4 above. Therefore, this implies that the bidegree of ∂' is $(-1,0)$.

Next, we define $d^2 : E^2 \to E^2$ by $d^2 = j' \circ \partial'$, and thus its bidegree is $(-1,1) + (-1,0) = (-2,1)$.

If we iterate this process, we obtain the r-th derived couple with maps $i^{(r-1)}, j^{(r-1)}$ and $\partial^{(r-1)}$ whose bidegrees are $(1,-1), (1-r, r-1)$, and $(-1,0)$, respectively. Then we define

$d^r : E^r \to E^r$ by $d^r = j^{(r-1)} \circ \partial^{(r-1)}$. The bidegree of d^r is thus $(1-r, r-1) + (-1, 0) = (-r, r-1)$.

It can be proven in a straightforward fashion that $d^r \circ d^r = 0$. Thence, we have a spectral sequence, (E^r, d^r), associated with this exact couple.

In the upcoming Section 2.1.3, we will see how an exact couple, and a resulting spectral sequence, arises from a filtration of simplicial complexes.

2.1.3 Spectral Sequences and Persistent Homology of a Filtration of Spaces

This section provides information and details of what Basu and Parida accomplished in their paper "Spectral Sequences, Exact Couples, and Persistent Homology of Filtrations"[5], which is the inspiration for a portion of our current work.

In their paper [5], Basu and Parida prove that from a filtration of simplicial sets one can derive a spectral sequence of bigraded vector spaces. The dimension of these vector spaces, the E-terms of the spectral sequence, can be expressed in terms of the ranks of persistent homology in dimensions n and $n-1$.

Theorem 2.5. *Let*

$$X: \quad \emptyset = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{N-1} \subseteq X_N$$

be an increasing filtration of simplicial sets, where $X_i = \emptyset$ if $i < 0$ and $X_i = X_N$ if $i > N$. Then for every integer r, p, q with $r \geq 1$,

$$\operatorname{rank}(E^r_{p,q}(X)) = b_n^{p,p+r-1}(X) - b_n^{p-1,p+r-1}(X) + b_{n-1}^{p-r,p-1}(X) - b_{n-1}^{p-r,p}(X),$$

where $p + q = n$, $b_n^{s,t} = \operatorname{rank}(H_n^{s,t}(X))$, and each $H_n^{s,t}(X)$ is a finitely generated vector space.

Basu and Parida begin building the proof by creating a homology of pairs of the form (X_m, X_{m-1}). Starting with the filtration X, they build a short exact sequence of complexes for the pair (X_p, X_{p-1}):

$$0 \to C_*(X_{p-1}) \xrightarrow{s} C_*(X_p) \xrightarrow{\pi} C_*(X_{p-1}, X_p) \to 0$$

where s and π represent the inclusion and the canonical map, respectively. We know that from a short exact sequence of complexes, we obtain a long exact sequence in homology,

$$\cdots \to H_n(X_{p-1}) \xrightarrow{i} H_n(X_p) \xrightarrow{j} H_n(X_p, X_{p-1}) \xrightarrow{\partial} H_{n-1}(X_{p-1}) \cdots ,$$

where i is induced by inclusion, j is induced by π, and ∂ is the connecting homomorphism. Then, we proceed by writing the LES as a staircase:

$$\begin{tikzcd}
{} \arrow[r] & H_{n+1}(X_p) \arrow[r, "j"] & H_{n+1}(X_p, X_{p-1}) \arrow[r, "\partial"] & H_n(X_{p-1}) \arrow[d, "i"] & & & \\
& & & H_n(X_p) \arrow[r, "j"] & H_n(X_p, X_{p-1}) \arrow[r, "\partial"] & H_{n-1}(X_{p-1}) \arrow[d, "i"] & \\
& & & & & H_{n-1}(X_p) \arrow[r] & {}
\end{tikzcd}$$

.

Similarly, for the pair (X_{p+1}, X_p) we write

$$\begin{tikzcd}
{} \arrow[r] & H_{n+1}(X_{p+1}) \arrow[r, "j"] & H_{n+1}(X_{p+1}, X_p) \arrow[r, "\partial"] & H_n(X_p) \arrow[d, "i"] & & & \\
& & & H_n(X_{p+1}) \arrow[r, "j"] & H_n(X_{p+1}, X_p) \arrow[r, "\partial"] & H_{n-1}(X_p) \arrow[d, "i"] & \\
& & & & & H_{n-1}(X_{p+1}) \arrow[r] & {}
\end{tikzcd}$$

We repeat the same process for pairs $(X_{p+2}, X_{p+1}), (X_{p+3}, X_{p+2}), \ldots$, and then we connect all these long exact sequences into one diagram:

$$
\begin{tikzcd}
 & {} \arrow[d, "i"] & & & & & \\
{} \arrow[r] & H_{n+1}(X_p) \arrow[r, "j"] \arrow[d, "i"] & H_{n+1}(X_p, X_{p-1}) \arrow[r, "\partial"] & H_n(X_{p-1}) \arrow[d, "i"] & & & \\
{} \arrow[r] & H_{n+1}(X_{p+1}) \arrow[r, "j"] \arrow[d, "i"] & H_{n+1}(X_{p+1}, X_p) \arrow[r, "\partial"] & H_n(X_p) \arrow[r, "j"] \arrow[d, "i"] & H_n(X_p, X_{p-1}) \arrow[r, "\partial"] & H_{n-1}(X_{p-1}) \arrow[r] \arrow[d, "i"] & {} \\
{} \arrow[r] & H_{n+1}(X_{p+2}) \arrow[r, "j"] \arrow[d, "i"] & H_{n+1}(X_{p+2}, X_{p+1}) \arrow[r, "\partial"] & H_n(X_{p+1}) \arrow[r, "j"] \arrow[d, "i"] & H_n(X_{p+1}, X_p) \arrow[r, "\partial"] & H_{n-1}(X_p) \arrow[r] \arrow[d, "i"] & {} \\
 & \vdots & & H_n(X_{p+2}) \arrow[r, "j"] \arrow[d, "i"] & H_n(X_{p+2}, X_{p+1}) \arrow[r, "\partial"] & H_{n-1}(X_{p+1}) \arrow[r] \arrow[d, "i"] & {} \\
 & & & \vdots & & \vdots &
\end{tikzcd}
\tag{2.6}
$$

We wish i, j, and ∂ to posses bidegree $(1,-1), (0,0)$, and $(-1,0)$, respectively. To fulfill this goal, we first use $n = p + q$ and then write diagram 2.6 as follows:

$$
\begin{tikzcd}
 & {} \arrow[d, "i"] & & & & & \\
{} \arrow[r] & H_{p,q+1}(X_p) \arrow[r, "j"] \arrow[d, "i"] & H_{p,q+1}(X_p, X_{p-1}) \arrow[r, "\partial"] & H_{p-1,q+1}(X_{p-1}) \arrow[d, "i"] & & & \\
{} \arrow[r] & H_{p+1,q}(X_{p+1}) \arrow[r, "j"] \arrow[d, "i"] & H_{p+1,q}(X_{p+1}, X_p) \arrow[r, "\partial"] & H_{p,q}(X_p) \arrow[r, "j"] \arrow[d, "i"] & H_{p,q}(X_p, X_{p-1}) \arrow[r, "\partial"] & H_{p-1,q}(X_{p-1}) \arrow[r] \arrow[d, "i"] & {} \\
{} \arrow[r] & H_{p+2,q-1}(X_{p+2}) \arrow[r, "j"] \arrow[d, "i"] & H_{p+2,q-1}(X_{p+2}, X_{p+1}) \arrow[r, "\partial"] & H_{p+1,q-1}(X_{p+1}) \arrow[r, "j"] \arrow[d, "i"] & H_{p+1,q-1}(X_{p+1}, X_p) \arrow[r, "\partial"] & H_{p,q-1}(X_p) \arrow[r] \arrow[d, "i"] & {} \\
 & \vdots & & H_{p+2,q-2}(X_{p+2}) \arrow[r, "j"] \arrow[d, "i"] & H_{p+2,q-2}(X_{p+2}, X_{p+1}) \arrow[r, "\partial"] & H_{p+1,q-2}(X_{p+1}) \arrow[r] \arrow[d, "i"] & {} \\
 & & & \vdots & & \vdots &
\end{tikzcd}
\tag{2.7}
$$

We set $E^1(X) = \bigoplus_{n=p+q} H_{p,q}(X_p, X_{p-1})$ and $D^1(X) = \bigoplus_{n=p+q} H_{p,q}(X_p)$, where $E^1_{p,q}(X) = H_{p,q}(X_p, X_{p-1})$ and $D^1_{p,q}(X) = H_{p,q}(X_p)$.

Then, looking at the diagram in Equation (2.7), $i : D^1(X) \to D^1(X)$ has bidegree $(1,-1)$, $j : D^1(X) \to E^1(X)$ has bidegree $(0,0)$, and $\partial : E^1(X) \to D^1(X)$ has bidegree $(-1,0)$, achieving the exact couple $\mathcal{C}(X) = \{D^1(X), E^1(X), i, j, \partial\}$.

In Section 2.2 right after Remark 2.4, we stated that we can get the r-th derived couple

$$\mathcal{C}^{(r)}(X) = \{D^{(r)}(X), E^{(r)}(X), i^{(r)}, j^{(r)}, \partial^{(r)}\},$$

and therefore a spectral sequence $(E^{(r)}(X), d^{(r)})$ arises, where each differential $d^{(r)}$ has bidegree equal to $(-r, r-1)$.

Once Basu and Parida construct the spectral sequence, they then prove by induction that for an integer $r > 0$,

$$D^{(r)}_{p-1,q+1}(X) = \mathrm{Im}\,(i^{p-r,p-1}_{p+q}(X)) = H^{p-r,p-1}_{n}(X)$$

and

$$i^{(r)}_{p-1,q+1} = i^{p-1,p}_{n}(X) \mid_{D^{(r)}_{p-1,q+1}(X)},$$

where $i^{p-1,p}_{n}$ is found in [5, Definition 2.1].

They apply this last result and unravel the r-th derived couple $\mathcal{C}^{(r)}(X)$ to show that

$$\cdots \to H^{p,p+r-1}_{n}(X) \stackrel{j^{(r)}_{p+r-1,q-r+1}}{\longrightarrow} E^{(r)}_{p,q}(X) \stackrel{\partial^{(r)}_{p,q}}{\to} H^{p-r,p-1}_{n-1}(X) \stackrel{i^{(r)}_{p-1,q}}{\to} H^{p-r+1,p}_{n-1}(X) \to \cdots \tag{2.8}$$

is an exact sequence in which $\mathrm{Im}\,(i^{(r)}_{p+r-1,q-r+1}) = H^{p,p+r}_{n}(X)$.

Basu and Parida determine the dimension of each $E^{(r)}$ term by applying the next theorem to the exact sequence given in equation (2.8).

Theorem 2.9. *Let the V_i's be finite dimensional vector spaces. If*

$$V_4 \stackrel{f_4}{\to} V_3 \stackrel{f_3}{\to} V_2 \stackrel{f_2}{\to} V_1 \stackrel{f_1}{\to} V_0$$

is an exact sequence, then $\dim(V_2) = \dim(V_3) - \dim(\mathrm{Im}\,(f_4)) - \dim(V_1) - \dim(\mathrm{Im}\,(f_1))$.

Now, we can conclude that

$$\dim(E_{p,q}^{(r)}(X)) = b_n^{p,p+r-1}(X) - b_n^{p-1,p+r-1}(X) + b_{n-1}^{p-r,p-1} - b_{n-1}^{p-r,p}(X),$$

the desired result.

Another important result is Corollary 2.10. This corollary states that the sum of the dimension of the $E_{*,*}^{*}$-terms can be represented in terms of the sum of the persistent multiplicities of the filtration X.

Corollary 2.10. *For all $n \geq 0$,*

$$\sum_{p+q=n} \dim(E_{p,q}^{(r)}(X)) = \sum_{j-i\geq r} \left(\mu_n^{i,j}(X) + \mu_{n-1}^{i,j}(X)\right) + b_n(X)$$

where $b_n(X)$ is a sum depending on persistent ranks.

2.2 Persistent Homology of (X, l)

In Section 3, we will encounter the Künneth formula that states: if k is a field and X and Y are increasing filtration of simplicial complexes, then there exists an exact sequence

$$0 \to \bigoplus_{l+j=n} H_l^{p,p+r-1}(X) \otimes H_j^{p,p+r-1}(Y) \to H_n^{p,p+r-1}(X \times Y) \to \bigoplus_{l+j=n-1} \mathrm{Tor}_l(H_l^{p,p+r-1}(X), H_j^{p,p+r-1}(Y)) \to 0.$$

We consider versions of Künneth formula for filtrations built from metric spaces $(X, d_X), (Y, d_Y)$, and $(X \times Y, d_X + d_Y)$. We first discuss various definitions such as $k[\mathbb{R}^+]$-modules (where k is a field), $\mathbb{R}^+$-graded simplicial modules, the modules $F_*(X, l)$, and the $\mathbb{R}^+$-filtered simplicial set (X, l). Gunnar Carlsson and Benjamin Filippenko [6] developed this terminology to prove theorems that involve persistent homology of metric spaces.

2.2.1 $\boldsymbol{R}^+$-Filtered Simplicial Set $(\boldsymbol{X}, \boldsymbol{l})$

Definition 2.11. *Let $k[\mathbb{R}^+]$ be the ring of polynomials with coefficients in k and with exponents in $\mathbb{R}^+$. Then $k[\mathbb{R}^+]$-sMod is the category that has $\mathbb{R}^+$-graded $k[\mathbb{R}^+]$-modules as its objects and graded module homomorphisms as its morphisms.*

Definition 2.12. *Let $\triangle$ be the simplex category. An $\mathbb{R}^+$ simplicial module $k[\mathbb{R}^+]$-module M is a contravariant functor $M \colon \triangle \to k[\mathbb{R}^+]$-sMod.*

Remark 2.13. *Explicitly, Definition 2.12 can be explained as follows. Let $[m]$ and $[n]$ be objects in the simplex category $\triangle$. Here, $[m]$ and $[n]$ are an m-simplex and an n-simplex. Then, for each arrow $\phi \colon [m] \to [n]$, we get a morphism $M_\phi = M(\phi) \colon M_n \to M_m$, where $M_n = M([n])$ and M_m are objects of the category of $\mathbb{R}^+$-graded $k[\mathbb{R}^+] - modules$ $k[\mathbb{R}^+]$-sMod.*

Definition 2.14. *(**$\mathbb{R}^+$-filtered simplicial set.**) Let X be a simplicial set, where X_n is the set of n-simplices in X. A $\mathbb{R}^+$-filtered simplicial set (X, l) is a simplicial set together with maps $l : X_n \to \mathbb{R}^+$, namely, the filtration function. These maps satisfy the property $l(X_\phi(\sigma)) \leq l(\sigma)$ for all $n \geq 0$, $\sigma \in X_n$, and all arrows $\phi : [m] \to [n]$.*

The set $\mathbb{R}^+$-sSet represents the category that has objects $\mathbb{R}^+$-filtered simplicial sets (X, l_X) and has arrows $f \colon (X, l_X) \to (Y, l_Y)$ given by $f \colon X \to Y$ satisfying $l_Y(f(\sigma)) \leq l_X(\sigma)$ for all $\sigma \in X$ ([6], *p.* 12). The contravariant functor $X \colon \triangle \to \mathbb{R}^+$-sSet can be described as follows. Consider $\phi \colon [m] \to [n]$, then $X_\phi \colon X_n \to X_m$. So, for each n-simplex σ in X_n we have that $X_\phi(\sigma) \in X_m$. Therefore, $l(X_\phi(\sigma)) \in \mathbb{R}^+$, and the inequality $l(X_\phi(\sigma)) \leq l(\sigma)$ makes sense.

2.2.2 Bars, and the Modules $\boldsymbol{F_*(X, l)}$.

We wish to define the persistent homology of a $\mathbb{R}^+$-filtered simplicial set (X, l), and the module $F_n(X, l)$ will allow us to do that. But first, we need to define what we call *bars* because the module is expressed in terms of these bars.

Definition 2.15. ***(Bars)*** *Let T be a variable in $k[\mathbb{R}^+]$. By Gabriel's Theorem, any finitely presented persistent $k[\mathbb{R}^+]$-module can be read as a direct sum of bars of the following form:*

- $(a,b) = \Gamma^a k[\mathbb{R}^+]/T^{b-a}$. *This represents a topological feature that is born at a and dies at b.*

- $(a,\infty) = \Gamma^a k[\mathbb{R}^+]/T^{\infty-a} := \Gamma^a k[\mathbb{R}^+]$. *This is a topological feature that is born at a but never dies.*

The bars (a,b) are persistent modules of the form

$$0 \to 0 \to \cdots \to k \to k \to k \to 0 \to 0 \to \cdots .$$

The first k shows up at time a, and the last k shows up just before time b.

Remark 2.16. *There exist the following isomorphisms:*

a.

$$(a,b) \otimes_{k[\mathbb{R}^+]} (c,d) \cong (a+c, \min\{a+d, b+c\})$$

b.

$$\mathrm{Tor}_1((a,b),(c,d)) \cong (\max\{a+d, b+c\}, b+d)$$

Now, we are ready to talk about the module $F_n(X,l)$, and we will define it using infinite bars.

Definition 2.17. *Given a $\mathbb{R}^+$-filtered simplicial set (X,l), $F(X,l)$ is a simplicial object whose set of n-simplices is the free module*

$$F_n(X,l) := \bigoplus_{\sigma \in X_n} \Gamma^{l(\sigma)} k[\mathbb{R}^+]\langle\sigma\rangle.$$

The value of the filtration function $l(\sigma)$ is a non-negative real number. So, $F_n(X, l)$ is written as a direct sum of infinite bars $(l(\sigma), \infty)$. The notation $\Gamma^{l(\sigma)} k[\mathbb{R}^+]\langle\sigma\rangle$ implies that the n-dimensional topological feature generated by the n-simplex σ is born at time $l(\sigma)$ but never dies.

According to Remark 2.13, the contravariant functor $F : \mathbb{R}^+\text{-sSet} \to k[\mathbb{R}^+]\text{-module}$ could be similarly described. Given $\phi\colon [m] \to [n]$, we have the map $F(X, l)_\phi\colon F_n(X, l) \to F_m(X, l)$. Thus, for each n-simplex σ in $F_n(X, l)$, $F(X, l)_\phi(\sigma)$ belongs to $F_m(X, l)$.

If ϕ has domain $[n-1]$ and codomain $[n]$, then we write $d_n\colon [n-1] \to [n]$. It follows that $F(X, l)_{d_n}\colon F_n(X, l) \to F_{n-1}(X, l)$. We now write the map $F(X, l)_{d_n}$ concretely: let σ be in $F_n(X, l)$ then $d_n(\sigma) \in F_{n-1}(X, l)$, so our map is

$$F(X, l)_{d_n}(\sigma) = T^{l(\sigma)-l(d_n(\sigma))} d_n(\sigma).$$

If we denote $F(X, l)_{d_n}$ as d_n as well, then this map above turns into

$$d_n(\sigma) = T^{l(\sigma)-l(d_n(\sigma))} d_n(\sigma),$$

where we must remember that the d_n on the right side is different from d_n on the left side. We take linear combinations of the n-simplices in $F_n(X, l)$ with coefficients on $k[\mathbb{R}^+]$, and these will be the elements of the chain group $C_n(F(X, l))$. We build the map $\partial_*\colon C_*(F(X, l)) \to C_{*-1}(F(X, l))$ by

$$\partial_*\left(\sum_\sigma s_\sigma \sigma\right) = \sum_\sigma s_\sigma d_*(\sigma),$$

where $s_\sigma \in k[\mathbb{R}^+]$.

Note that we obtain the chain complex $(C_*(F(X, l), \partial_*)$. Next, we define the chain complex a little bit different and more formally.

Definition 2.18. *Given (X, l), we call $PC_*(X, l)$ the persistent chain complex of (X, l). We* *define it*

$$PC_*(X,l) = C_*(F(X,l)).$$

Definition 2.19. *The persistent homology of* (X,l) *is the homology of its persistence chain complex, so we write*

$$PH_*(X,l) = H_*(PC_*(X,l)).$$

An $\mathbb{R}^+$-filtered simplicial set (X,l) is essential because later we will see that filtration maps will depend on a metric whose distances live in $\mathbb{R}^+$, and the persistent homology of metric spaces is what we want to study.

We provide our first Künneth Theorem for persistent homology of $\mathbb{R}^+$-filtered simplicial sets.

Theorem 2.20 ([6])**.** *Consider the* $\mathbb{R}^+$*-filtered simplicial sets* (X,l_X) *and* (Y,l_Y) *with the condition that the* X_n *and* Y_n *are finite sets for all* $n \geq 0$*, then there exists a short exact sequence*

$$0 \to \bigoplus_{l+j=n} PH_l(X,l_X) \otimes_{k[\mathbb{R}^+]} PH_j(Y,l_Y) \to PH_n(X \times Y, l_X + l_Y) \to \bigoplus_{l+j=n-1} \mathrm{Tor}_1(PH_l(X,l_X), PH_j(Y,l_Y)) \to 0.$$

From now on, we denote the nth persistent homology of $(X \times Y, l_X + l_Y)$ as follows: $PH_n(X \times Y, l_X + l_Y) = PH(X,Y)$.

We can also construct the nth persistent homology of $(X \times Y, l_{X \times Y})$, and that is denoted $PH_n(X \times Y) = PH_n(X \times Y, l_{X \times Y})$.

2.3 Persistent Homology of Metric Spaces.

Let us start this section by defining a filtration function, namely, the max-length map:

$$l_X \colon X_n \to \mathbb{R}^+,$$

where

$$l_X(x_0, x_1, ..., x_n) = \max\{d_X(x_i, x_j) \mid 0 \leq i, j \leq n\}. \tag{2.21}$$

Note that that l_X relies on a metric, and this helps us understand why the persistent homology is defined as follows.

Definition 2.22. *Given a metric space (X, d_X), its persistent chain complex is $PC_*(X) = PC_*(X, l_X)$ and its persistent homology is $PH_*(X) = PH_*(X, l_X)$.*

The filtration map satisfies the property $l_{X\times Y}(\sigma, \tau) \leq l_X(\sigma)+l_Y(\tau)$ for all $(\sigma, \tau) \in X_n \times Y_n$, so the equality does not always hold. As a consequence, we cannot say the persistent homology of the metric space $(X \times Y, d_X + d_Y)$ is $PH_*(X, Y)$. However, the next theorem provides a relation between persistent homology of metric spaces (X, d_X) and (Y, d_Y) and $PH_*(X, Y)$.

Theorem 2.23. *([6], p. 16). Given two finite metric spaces (X, d_X) and (Y, d_Y), for $n \geq 0$ there is a short exact sequence*

$$0 \to \bigoplus_{l+j=n} PH_l(X) \otimes PH_j(Y) \to PH_n(X, Y) \to \bigoplus_{l+j=n-1} \mathrm{Tor}_1(PH_l(X), PH_j(Y)) \to 0.$$

which is natural with respect to the maps $(X, d_X) \to (X', d_{X'})$ and $(Y, d_Y) \to (Y', d_{Y'})$. Moreover, the sequence splits.

If we replace $PH_n(X, Y)$ by $PH_n(X \times Y)$, Theorem (2.23) will hold for just $n = 0, 1$.

Without Theorem (2.23) we are not able to show that $PH_2(I^k)$ has zero bars when $k > 0$: this is a claim that we prove in the next section, Section (2.4).

Next, we define the homology $\overline{PH}_*(X, Y)$ which is also a necessary tool for proving $PH_2(I^k) = 0$ when $k > 0$.

Let us define the the following persistent chain groups:

$$PC_n(X,Y)\colon = \bigoplus_{(\sigma,\tau)\in X_n\times Y_n} \Gamma^{l_X(\sigma)+l_Y(\tau)} k[\mathbb{R}^+]\langle\sigma,\tau\rangle$$

and

$$PC_n(X\times Y)\colon = \bigoplus_{(\sigma,\tau)\in X_n\times Y_n} \Gamma^{l_{X\times Y}(\sigma,\tau)} k[\mathbb{R}^+]\langle\sigma,\tau\rangle$$

It can be shown that

$$i_n\colon PC_n(X,Y)\to PC_n(X\times Y)$$

given by

$$i_n(\sigma,\tau) = T^{l_X(\sigma)+l_Y(\tau)-l_{X\times Y}(\sigma,\tau)}\langle\sigma,\tau\rangle$$

is an $\mathbb{R}^+$-graded $k[\mathbb{R}^+]$-linear embedding. Based on the inclusion i_n, we can define the relative chain complex whose chain groups are

$$\overline{PC}_n(X,Y) \coloneqq PC_n(X\times Y)/i_n(PC_n(X,Y)).$$

This relative chain complex has homology groups, namely, $\overline{PH}_*(X,Y)$.

Since we have a map of chain complexes

$$(PC_*(X,Y),\partial_*) \xrightarrow{i} (PC_*(X\times Y,\partial_*) \xrightarrow{\pi} (\overline{PC}_*(X,Y),\overline{\partial}_*),$$

then there is a long exact sequence

$$\cdots \xrightarrow{\delta_{n+1}} PH_n(X,Y) \xrightarrow{(i_n)_*} PH_n(X\times Y) \xrightarrow{(\pi_n)_*} \overline{PH}_n(X,Y) \xrightarrow{\delta_n} \cdots . \tag{2.24}$$

The exact sequence (2.24) is natural ([6], $p.$ 17).

The next theorem provides information about what happens with the map $(i_n)_*$ when $n = 0, 1, 2$.

Theorem 2.25. *([6], p. 17) Consider metric spaces (X, d_X) and (Y, d_Y), and also the product metric space $(X \times Y, d_X + d_Y)$. Then the map $(i_n)_*$ from* (2.24) *is:*

i. an isomorphism when $n = 0$

ii. an isomorphism when $n = 1$

iii. a surjection when $n = 2$

2.4 Low-dimension Persistent Homology of the Hypercube I^k

In this section, we study the two main theorems of Carlsson and Fileppenko's paper "Persistent Homology of the Sum Metric" [6]. Both theorems provide information about the low-dimensional persistent homology of hypercubes I^k. Here, we consider hypercubes equipped with the Hamming metric.

Definition 2.26. *The hypercube metric space I^n is the set of all binary strings of length n, equipped with the Hamming metric.*

The Hamming metric computes the number of positions in which two binary strings of equal length differ.

Example 2.27. *Consider the hypercube I^6. The Hamming metric $d(011010, 111000) = 2$. Note that we have two strings of length 6. The first and fifth positions of the strings are different, meaning that the distance is equal to 2.*

We stress that a hypercube I^k can be viewed as a product space, for example as a k-fold product of the cube I^1. The Hamming metric on I^k is then formed by taking the "ℓ_1 product" of the metric on I^1.

Now, we introduce the concept of coordinate inclusions from I^r into I^k. This is a useful terminology that helps us identify the 1-dimensional persistent homology of hypercubes.

Definition 2.28. *([6], p. 22) Given $0 \leq r \leq k$, a point $\xi \in I^{k-r}$, and any choice of r coordinates $1 \leq i_1 < i_2 < \ldots < i_r \leq k$ of I^k, there is an isometric embedding $\phi : I^r \to I^k$, where for $x \in I^r$ the r chosen coordinates of $\phi(x)$ are equal to x, in other words,*

$$x = (x_1, x_2, \ldots, x_r) = (\phi(x)_{i_1}, \phi(x)_{i_2}, \ldots, \phi(x)_{i_r}),$$

and the other $k - r$ coordinates of $\phi(x)$ are given by the point ξ. These maps ϕ are called coordinate inclusion of I^r into I^k. The set of coordinate inclusions of I^r into I^k is denoted $C(r, k)$.

Theorem 2.29. *([6], p. 23) For $k > 0$, we have*

i. $PH_0(I^k) \cong (0,1)^{2^k - 1} \oplus (0, \infty)$

ii. $PH_1(I^k) \cong (1,2)^{k2^{k-1} - (2^k - 1)}$.

Furthermore, $PH_1(I^k) = \left\langle \bigcup_{\phi \in C(2,k)} \phi_*(PH_1(I^2)) \right\rangle$, *where each ϕ_* is induced by coordinate inclusions ϕ.*

Proof. First, let us start saying that the Vietoris–Rips complex of I^k at scale 0, $\mathrm{VR}(I^k, 0)$, is 2^k disconnected points.

The proof of (i) proceeds by induction on k. Let $k = 1$, then for $r = 0$ we get $\mathrm{VR}(I, 0)$ which only contains two different non-connected points. When $r = 1$, then we have $\mathrm{VR}(I, 1) = [0, 1]$. This means that one 0-dimensional persistent homology point is born at time 0 and dies at time 1. The other point never dies. So, $PH_0(I) = (0,1) \oplus (0, \infty)$. Assume that $PH_0(I^{k-1}) \cong (0,1)^{2^{k-1}-1} \oplus (0, \infty)$. If we apply Theorem (2.23) in the case of

$PH_0(X \times Y)$, then we get

$$PH_0(I^k) = PH_0(I \times I^{k-1}) \cong PH_0(I) \otimes PH_0(I^{k-1}).$$

We apply Remark (2.16) to compute the tensor product of two bars. It is straightforward to get to $PH_0(I^k) \cong (0,1)^{2^k-1} \oplus (0,\infty)$.

We now proceed to prove (ii). If we now apply Theorem (2.23) in the case of $PH_1(X \times Y)$, we obtain

$$PH_1(I^k) = PH_1(I \times I^{k-1}) \cong \bigoplus_{l+j=1} [PH_l(I) \otimes PH_j(I^{k-1})] \oplus \mathrm{Tor}_1(PH_0(I), PH_0(I^{k-1})). \quad (2.30)$$

We proceed by induction. For $k = 1$, $PH_1(I) = 0$, because we do not have any holes. This simplifies the direct sum of (2.30) as $\bigoplus_{l+j=1} [PH_l(I) \otimes PH_j(I^{k-1})] = PH_0(I) \otimes PH_1(I^{k-1})$. Then, we write

$$PH_1(I^k) \cong PH_0(I) \otimes PH_1(I^{k-1}) \oplus \mathrm{Tor}_1(PH_0(I), PH_0(I^{k-1})).$$

Now, we assume by induction that $PH_1(I^{k-1}) = (1,2)^{(k-1)2^{k-2}-(2^{k-1}-1)}$.

To compute the Tor bars, we need to apply the property of direct sum of Tor and the property *b.* of Remark (2.16). Finally, (2.30) gives us

$$PH_1(I^k) \cong (1,2)^{k2^{k-1}-(2^k-1)}.$$

We do not provide any details of the proof of the last part of this Theorem (2.29).

□

Theorem 2.31. *([6], p. 24) For $k > 0$, $PH_2(I^k) = 0$.*

This theorem says that at homological dimension $n = 2$ there are no bars.

Proof. Our first goal is to prove that the image of the map $(i_2)_*\colon PH_2(I, I^{k-1}) \to PH_2(I^k)$ is zero.

Let us prove this theorem by induction. For $k = 1, 2, 3$, we see by inspection that $PH_2(I^k) = 0$. Assume that $PH_2(I^{k-1}) = 0$ for $k > 3$. When we apply Theorem (2.23) we obtain

$$PH_2(I, I^{k-1}) \cong \bigoplus_{l+j=2} PH_l(I) \otimes PH_j(I^{k-1}) \oplus \bigoplus_{l+j=1} \operatorname{Tor}_1(PH_l(I), PH_j(I^{k-1})).$$

Since $PH_2(I) = 0$ and $PH_2(I^{k-1}) = 0$ for $k > 3$, $PH_2(I, I^{k-1}) \cong \bigoplus_{l+j=1} \operatorname{Tor}_1(PH_l(I), PH_j(I^{k-1}))$.

If we use $PH_1(I) = 0$, then $PH_2(I, I^{k-1}) \cong \operatorname{Tor}_1(PH_0(I), PH_1(I^{k-1}))$. We can verify in a similar manner that $PH_2(I, I^2) \cong \operatorname{Tor}_1(PH_0(I), PH_1(I^2))$.

Now, consider any coordinate inclusion $\phi\colon I^2 \to I^{k-1}$. This map induces the following map in homology:

$$id_* \otimes \phi_*\colon \operatorname{Tor}_1(PH_0(I), PH_1(I^2)) \to \operatorname{Tor}_1(PH_0(I), PH_1(I^{k-1})).$$

Since the SES in Theorem (2.23) and the LES in (2.24) are natural, each square of the diagram

$$\begin{array}{ccccc}
\operatorname{Tor}_1(PH_0(I), PH_1(I^2)) & \xrightarrow{\cong} & PH_2(I, I^2) & \xrightarrow{(i_2)_*} & PH_2(I^3) \\
\downarrow{\scriptstyle id_*\otimes\phi_*} & & \downarrow & & \downarrow \\
\operatorname{Tor}_1(PH_0(I), PH_1(I^{k-1})) & \xrightarrow{\cong} & PH_2(I, I^{k-1}) & \xrightarrow{(i_2)_*} & PH_2(I^k)
\end{array}$$

commutes. Therefore the diagram commutes.

By part b. of Remark 2.16 and by the property of the direct sums of Tor, we get

$$\mathrm{Tor}_1(PH_0(I), PH_1(I^{k-1})) = \mathrm{Tor}_1((0,1) \oplus (0,\infty), PH_1(I^{k-1})) \cong \mathrm{Tor}_1((0,1), PH_1(I^{k-1})).$$

Now, let us form a projective resolution of the bar $(0,1)$:

$$0 \to (1,\infty) \to (0,\infty) \to (0,1) \to 0.$$

if we tensor this resolution with the bar $(1,2)$ and then use part a. of Remark 2.16 to obtain the chain complex

$$0 \to (2,3) \xrightarrow{m} (1,2) \to 0.$$

Using part a. of Remark 2.16, we have that $(1,\infty) \otimes (1,2) \cong (2,3)$. The map m has kernel equal to $(2,3)$, so $\mathrm{Tor}_1((0,1),(1,2)) \cong (2,3)$. Therefore, the Tor group and the tensor group are isomorphic when we use this resolution, and when we tensor it with $(1,2)$. By Theorem 2.29, $PH_1(I^{k-1}) \cong (1,2)^{(k-1)2^{k-2}-(2^{k-1}-1)}$. This implies that

$$\mathrm{Tor}((0,1), PH_1(I^{k-1})) = (1,\infty) \otimes PH_1(I^{k-1}).$$

Thus,

$$\mathrm{Tor}(PH_0(I), PH_1(I^{k-1})) = (1,\infty) \otimes PH_1(I^{k-1}).$$

We replace the Tor groups with this last tensor product of bars to get the commutative diagram

$$\begin{array}{ccccc} (1,\infty) \otimes PH_1(I^2) & \xrightarrow{\cong} & PH_2(I, I^2) & \xrightarrow{(i_2)_*} & PH_2(I^3) \\ \downarrow{\scriptstyle id_{(1,\infty)} \otimes \phi_*} & & \downarrow & & \downarrow \\ (1,\infty) \otimes PH_1(I^{k-1}) & \xrightarrow{\cong} & PH_2(I, I^{k-1}) & \xrightarrow{(i_2)_*} & PH_2(I^k). \end{array}$$

Observe that by Theorem 2.29 the map $id_{(1,\infty)} \otimes \phi_*$ is a surjection.

Each map in the commutative diagram is surjective. Also, the persistent homology $PH_2(I^3)$ is equal to zero. As a consequence, $PH_2(I^k) = 0$, and this ends the proof.

$\square$

3 Analysis of Basu and Parida's Argument for a Categorical Product $X \times Y$.

In this section, our goal is to take Basu and Parida's main result, Theorem 2.5, and refine it for a filtration of a simplicial complex of the form $(X \times Y)_*$. To achieve this, we must try to prove Theorem 3.2. But first, let us begin by defining the categorical product filtration.

Definition 3.1. ***Categorical product filtration***. *Given two filtrations*

$$X : \emptyset = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{N-1} \subseteq X_N = X_{N+1} = X_{N+2} = \cdots$$

and

$$Y : \emptyset = Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_{N-1} \subseteq Y_N = Y_{N+1} = Y_{N+2} = \cdots ,$$

we define the categorical product filtration $X \times Y$ *:*

$$(X \times Y)_0 \subseteq (X \times Y)_1 \subseteq \cdots \subseteq (X \times Y)_{N-1} \subseteq (X \times Y)_N = (X \times Y)_{N+1} = (X \times Y)_{N+2} = \cdots ,$$

where $(X \times Y)_p \doteq X_p \times Y_p$.

Theorem 3.2. *For each* $r \geq 1$ *and* $n = p + q$*, the groups* $E^{(*)}_{*,*}(X \times Y)$ *and the persistent homology groups* $H^{(*)}_{*,*}(X)$ *and* $H^{(*)}_{*,*}(Y)$ *are related by a long exact sequence of the following form.*

$$\begin{aligned} \ldots \to \bigoplus_{l+j=n} H_l^{p,p+r-1}(X) \otimes H_j^{p,p+r-1}(Y) \to E^{(r)}_{p,q}(X \times Y) \to \bigoplus_{l+j=n-1} H_l^{p-r,p-1}(X) \otimes H_j^{p-r,p-1}(Y) \\ \to \bigoplus_{l+j=n-1} H_l^{p-r+1,p}(X) \otimes H_j^{p-r+1,p}(Y) \to \ldots \end{aligned}$$

How do we prove Theorem 3.2? Our first idea is to construct a commutative diagram with two sequences in which one of them is exact. We write the diagram as follows:

$$
\begin{tikzcd}
\cdots \arrow[r] & \bigoplus\limits_{l+j=n} H_l^{p,p+r-1}(X) \otimes H_j^{p,p+r-1}(Y) \arrow[r] \arrow[d] & E^{(r)}_{p,q}(X \times Y) \arrow[r] \arrow[d, "id"] & \bigoplus\limits_{l+j=n-1} H_l^{p-r,p-1}(X) \otimes H_j^{p-r,p-1}(Y) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_n^{p,p+r-1}(X \times Y) \arrow[r, "j^{(r)}_{p+r-1,q-r+1}"] & E^{(r)}_{p,q}(X \times Y) \arrow[r, "\partial^{(r)}_{p,q}"] & H_{n-1}^{p-r,p-1}(X \times Y) \arrow[r, "i^{(r)}_{p-1,q}"] & \cdots
\end{tikzcd}
\tag{3.3}
$$

We wish to show that the leftmost and rightmost vertical arrows are isomorphisms. We first prove that they are isomorphisms using a specific example.

Example 3.4. *Let S^n denote the n-dimensional sphere. Consider X to be the filtration $S^0 \subset S^1$ (so $X_0 = S^0$ and $X_1 = S^1$), and Y is the same filtration $S^0 \subset S^1$.*

1.

$$f_0^{0,0}(X) \colon H_0(S^0) \to H_0(S^0) \cong \mathbb{F}^2$$

$$H_0^{0,0}(X) = \operatorname{Im} f_0^{0,0}(X) \cong \mathbb{F}^2, \; H_0^{0,0}(X) \otimes H_0^{0,0}(X) \cong \mathbb{F}^2.$$

The Künneth Theorem tells us that $H_0(S^0) \otimes H_0(S^0) \cong H_0(S^0 \times S^0)$. So,

$$H_0(S^0 \times S^0) = H_0(S^0) \otimes H_0(S^0) \cong \mathbb{F}^2 \otimes \mathbb{F}^2 \cong \mathbb{F}^4.$$

The map $g_0^{0,0}(X \times Y) \colon H_0(S^0 \times S^0) \to H_0(S^0 \times S^0)$ has $\operatorname{Im} g_0^{0,0}(X \times Y) \cong \mathbb{F}^4$. Note that $\operatorname{Im} f_0^{0,0}(X) \otimes \operatorname{Im} f_0^{0,0}(Y) \cong \mathbb{F}^4$, and therefore

$$\operatorname{Im} g_0^{0,0}(X \times Y) \cong \operatorname{Im} f_0^{0,0}(X) \otimes \operatorname{Im} f_0^{0,0}(Y).$$

2. The map $f_0^{0,1}(X) \colon H_0(S^0) \to H_0(S^1)$ has $\operatorname{Im} f_0^{0,1}(X) \cong \mathbb{F}$.

Consider $g_0^{0,1}(X \times Y)\colon H_0(S^0 \times S^0) \to H_0(S^1 \times S^1)$*. Using the Künneth Theorem, we obtain*

$$H_0(S^1 \times S^1) \cong H_0(S^1) \otimes H_0(S^1) \cong \mathbb{F} \otimes \mathbb{F} \cong \mathbb{F}.$$

Then we say $\mathrm{Im}\ g_0^{0,1}(X \times Y) \cong \mathbb{F}$*. These previous computations indicate that*

$$\mathrm{Im}\ g_0^{0,1}(X \times Y) \cong \mathrm{Im}\ f_0^{0,1}(X) \otimes \mathrm{Im}\ f_0^{0,1}(Y).$$

3. *The map* $f_0^{1,1}(X)\colon H_0(S^1) \to H_0(S^1)$ *gives us* $\mathrm{Im}\ f_0^{1,1}(X) \cong \mathbb{F}$.

   Now, we consider $g_0^{1,1}(X \times Y)\colon H_0(S^1 \times S^1) \to H_0(S^1 \times S^1)$ *and we note*

$$\mathrm{Im}\ g_0^{1,1}(X \times Y) \cong \mathrm{Im}\ f_0^{1,1}(X) \otimes \mathrm{Im}\ f_0^{1,1}(Y).$$

4. *The map* $f_1^{0,0}(X)\colon H_1(S^0) \to H_1(S^0)$ *gives us* $\mathrm{Im}\ f_1^{0,0}(X) = 0$.

   For $g_1^{0,0}(X \times Y)\colon H_1(S^0 \times S^0) \to H_1(S^0 \times S^0)$, *if we again use the Künneth Theorem we get*

$$H_1(S^0 \times S^0) \cong [H_0(S^0) \otimes H_1(S^0)] \oplus [H_1(S^0) \otimes H_0(S^0)] = 0.$$

   Consider $g_1^{0,0}(X \times Y)\colon H_1(S^0 \times S^0) \to H_1(S^0 \times S^0)$*. Then*

$$\mathrm{Im}\ g_1^{0,0}(X \times Y) = 0 = [\mathrm{Im}\ f_1^{0,0}(X) \otimes \mathrm{Im}\ f_0^{0,0}(Y)] \oplus [\mathrm{Im}\ f_0^{0,0}(X) \otimes \mathrm{Im}\ f_1^{0,0}(Y)].$$

5. *The map* $f_1^{0,1}(X)\colon H_1(S^0) \to H_1(S^1)$ *satisfies* $\mathrm{Im}\ f_1^{0,1}(X) = 0$.

   We have $g_1^{0,1}(X \times Y)\colon H_1(S^0 \times S^0) \to H_1(S^1 \times S^1)$.

Using the Künneth theorem

$$H_1(S^1 \times S^1) \cong [H_0(S^1) \otimes H_1(S^1)] \oplus [H_1(S^1) \otimes H_0(S^1)] \cong \mathbb{F}^2.$$

Now we can state that

$$\text{Im } g_1^{0,1}(X \times Y) = 0 = [\text{Im } f_1^{0,1}(X) \otimes \text{Im } f_0^{0,1}(Y)] \oplus [\text{Im } f_1^{0,1}(X) \otimes \text{Im } f_0^{0,1}(Y)].$$

6. *Consider* $f_1^{1,1}(X)\colon H_1(S^1) \to H_1(S^1)$. *Note* $\text{Im } f_1^{1,1}(X) \cong \mathbb{F}$.

   We have $g_1^{1,1}(X \times Y)\colon H_1(S^1 \times S^1) \to H_1(S^1 \times S^1)$. *Then,* $\text{Im } g_1^{1,1}(X \times Y) \cong \mathbb{F}^2$. *We can easily verify that*

$$[\text{Im } f_1^{1,1}(X) \otimes \text{Im } f_0^{1,1}(Y)] \oplus [\text{Im } f_0^{1,1}(X) \otimes \text{Im } f_1^{1,1}(Y)] \cong \mathbb{F}^2,$$

   and so

$$\text{Im } g_1^{1,1}(X \times Y) \cong [\text{Im } f_1^{1,1}(X) \otimes \text{Im } f_0^{1,1}(Y)] \oplus [\text{Im } f_0^{1,1}(X) \otimes \text{Im } f_1^{1,1}(Y)].$$

7. *Consider* $f_2^{0,0}(X)\colon H_2(S^0) \to H_2(S^0)$. *Here we have* $\text{Im } f_2^{0,0}(X) = 0$. *Note that* $H_2(S^0 \times S^0)$*=0 by using the Künneth Theorem.*

   The map $g_2^{0,0}(X \times Y) : H_2(S^0 \times S^0) \to H_2(S^0 \times S^0)$ *gives* $\text{Im } g_2^{0,1}(X \times Y) = 0$. *We can verify that*

$$\text{Im } g_2^{0,0}(X \times Y) \cong \bigoplus_{l+j=2} \text{Im } f_l^{0,1}(X) \otimes \text{Im } f_j^{0,1}(Y).$$

We can do the same with the rest of the $g_*^{*,*}$. So, it is true that for this filtration

$$H_n^{s,t}(X \times Y) \cong \bigoplus_{l+j=n} H_l^{s,t}(X) \otimes H_j^{s,t}(Y).$$

For the specific filtrations in Example 3.4, we have therefore verified that the leftmost and rightmost vertical arrows of (3.3) are isomorphisms.

We now prove for *any* arbitrary filtrations X and Y:

$$H_n^{s,t}(X \times Y) \cong \bigoplus_{l+j=n} H_l^{s,t}(X) \otimes H_j^{s,t}(Y)$$

We know that we have inclusion maps $X_p \hookrightarrow X_{p+r-1}$ and $Y_p \hookrightarrow Y_{p+r-1}$. Then, we apply the fact that the Künneth formula is natural, in other words, the inclusion maps $X_p \hookrightarrow X_{p+r-1}$ and $Y_p \hookrightarrow Y_{p+r-1}$ induce the following commutative diagram:

$$\begin{tikzcd} \bigoplus_{l+j=n} H_l^p(X) \otimes H_j^p(Y) \arrow[r] \arrow[d] & \bigoplus_{l+j=n} H_l^{p+r-1}(X) \otimes H_j^{p+r-1}(Y) \arrow[d] \\ H_n^p(X \times Y) \arrow[r] & H_n^{p+r-1}(X \times Y) \end{tikzcd} \quad , \tag{3.5}$$

in which the vertical arrows are isomorphisms.

Let us define the vertical maps of diagram (3.5) by

$$\alpha : \bigoplus_{l+j=n} H_l^p(X) \otimes H_j^p(Y) \to H_n^p(X \times Y),$$

where

$$\alpha\left(\sum_{l+j=n} [\sigma_l] \otimes [\tau_j]\right) = \sum_{l+j=n} [\sigma_l \otimes \tau_j]$$

is an isomorphism.

According to May, J.P. in the book *A Concise Course in Algebraic Topology*, ([9], p.130), there exists another version of Künneth formula with the same map α we defined above, which we use in the theorem below.

Theorem 3.6. ***(Künneth Theorem)*** *Suppose R is a PID and the chain R-modules C_i are free. Then for each n there is a natural short exact sequence*

$$0 \to \bigoplus_{i=0}^{n}(H_i(C) \otimes_R H_{n-i}(C')) \xrightarrow{\alpha} H_n(C \otimes_R C') \to \bigoplus_{i=0}^{n-1} \mathrm{Tor}_R(H_i(C), H_{n-i-1}(C')) \to 0$$

where α is defined as

$$\alpha\bigg(\sum_{i=0}^{n}[b_i] \otimes [c_{n-i}]\bigg) = \sum_{i=0}^{n}[b_i \otimes c_{n-i}].$$

The codomain of α in Theorem 3.6 is $H_n(C \otimes C')$, and our goal is to identify it with $H_n(X \times Y)$. Set $C = C_*(X)$ and $C' = C_*(Y)$, the cellular chain complexes. Then $C \otimes C^{'} = C_*(X \times Y)$ by Proposition 3B.1 ([2] *p.* 269). Hence, $H_n(C(X) \otimes C(Y)) = H_n(X \times Y)$.

This Proposition 3B.1 states the following: the boundary map in the cellular chain complex $C_*(X \times Y)$ is determined by the boundary maps in the cellular chain complexes $C_*(X)$ and $C_*(Y)$ via the formula

$$d(e^i \times e^j) = d(e^i) \times e^j + (-1)^i e^i \times d(e^j).$$

Observe that the Tor terms are zero in our case, which means that the map α given in Theorem 3.6 is an isomorphism.

We now restrict α to persistent homology,

$$\alpha : \bigoplus_{l+j=n} H_l^{p,p+r-1}(X) \otimes H_j^{p,p+r-1}(Y) \to H_n^{p,p+r-1}(X \times Y).$$

It follows this last α will be defined in the following manner: if $[z_l] \in H_l^{p,p+r-1}(X) \subseteq$

$H_l(X_{p+r-1})$ and $[w_j] \in H_j^{p,p+r-1}(Y) \subseteq H_j(Y_{p+r-1})$, then

$$\alpha\left(\sum_{l+j=n}([z_l] \otimes [w_j])\right) = \sum_{l+j=n}[z_l \otimes w_j].$$

This last α also becomes an isomorphism.

The horizontal rows of the bottom sequence in (3.3) are defined as

$$j^r(i^{r-1}([\sigma \otimes \tau])) = j^{r-1}([\sigma \otimes \tau]) + d^{r-1}(E^{r-1}), \quad \partial^{(r)}(e + d^{r-1}(E^{r-1})) = \partial^{r-1}(e),$$

$$i^{(r)}([\sigma \otimes \tau]) = i^{(r-1)} \mid_{i^{(r-1)}(D^{r-1})} ([\sigma \otimes \tau]).$$

We return to the proof of Theorem 3.2. We want each square of the diagram in equation (3.3) to commute, so the maps of the top arrows will be defined from left to right by $j^r(\alpha)$, $(\alpha^{-1} \circ \partial^{(r)})$, and $\alpha^{-1} \circ i^{(r)} \circ \alpha$. Note that the last map contains two α's in the composition but they are not the same because we call all the vertical arrows α; we must keep track of to which α's we are referring.

Next, we proceed by showing that the top row sequence of the commutative diagram in (3.3) is exact. Note that the bottom row is exact, the diagram is commutative, and the vertical arrows are isomorphisms. We conclude, by the nine lemma ([3] Exercise 1.3.2, p.11), that the top sequence of (3.3) is exact as well. This completes the proof of Theorem 3.2.

Using Theorem 2.9, we obtain the corollary that

$$dim(E_{p,q}^{(r)}(X \times Y)) = \sum_{l+j=n} b_l^{p,p+r-1}(X) b_j^{p,p+r-1}(Y) - dim(\mathrm{Im}\ (\alpha^{-1} \circ i^{(r)}_{p+r-2,q-r+2} \circ \alpha))+$$

$$\sum_{l+j=n-1} b_l^{p,p+r-1}(X) b_j^{p,p+r-1}(Y) - dim(\mathrm{Im}\ (\alpha^{-1} \circ i^{r}_{p-1,q} \circ \alpha)).$$

We now summarize the steps we followed to show that the sequence in Theorem 3.2 is exact. Subsequently, we determined a formula to compute the rank of the groups $E^*_{p,q}(X \times Y)$.

Basu and Parida's paper [5] played an important role in demonstrating these two results.

Theorem 3.2 presents a long exact sequence whose groups are $\bigoplus_{l+j=n} H_l^{*,*}(X) \otimes H_j^{*,*}(Y)$ and $E_{*,*}^{*}(X \times Y)$. We proved this sequence is exact by first creating a diagram with two sequences. The top sequence is the one written in Theorem 3.2, and the bottom sequence is the long exact sequence given in the diagram in equation (2.8), but this time we replace X by $X \times Y$. Later, we used the naturality of the topological Künneth formula to create an isomorphism α from $\bigoplus_{l+j=n} H_l^{*}(X) \otimes H_j^{*}(Y)$ to $H_n^{*}(X \times Y)$. Resources such as a functorial version of Künneth formula (Theorem 3.6) and the identification of $H_n(C(X) \otimes C(Y))$ with $H_n(X \times Y)$ serve to pave the way to restrict this isomorphism α to a direct sum of the tensor product of the persistent homology of X and Y and the persistent homology of the categorical product. This map α leads us to discover the maps we place on the top horizontal arrows to make a such diagram commute. These maps are formed by taking compositions of different maps: α, α^{-1}, and the bigraded maps $i_{*,*}^{*}, \partial_{*,*}^{*}$ and $j_{*,*}^{*}$. We proceeded to apply the nine lemma to conclude the top sequence is exact, and this completed the desired result. The result helped us to straightforwardly express $\text{rank}(E_{p,q}^{*}(X \times Y))$ in terms of the sum of the $\text{rank}(\text{Im } (\alpha^{-1} \circ i_{*,*}^{*} \circ \alpha))$ and the sum of product of ranks of persistent homology of X and Y.

4 The Gromov–Hausdorff Distance

In the next two sections, a map means a continuous function.

4.1 Basic Concepts

We first introduce the Hausdorff distance, both rigorously and graphically, and then we use the Hausdorff distance to define the Gromov-Hausdorff distance [7]. After that, we provide a different expression for the Gromov-Hausdorff distance in terms of the distortion, in order to later demonstrate a sketch of the proof of Theorem 5.1, which is fundamental to the analysis of Section 5. Furthermore, we discuss the definition of the Vietoris-Rips complex, which will be used in Section 5 to produce not only maps from discontinuous functions but also to discover topological obstructions. In that same section, we apply the Vietoris-Rips complex to lower bound the Gromov-Hausdorff distance between specific metric spaces.

Definition 4.1. *Consider a metric space Z. Let X and Y be metric spaces that are subsets of Z. The Hausdorff distance between X and Y is*

$$d_H(X,Y) = \inf\{r \geq 0 \mid X \subseteq B(Y;r) \text{ and } Y \subseteq B(X;r)\}$$

The Hausdorff distance measures the least radius r such that if we thicken Y by r it contains X and if we also thicken X by r it contains Y. The distance is computed when both metric subspaces are aligned in a common metric space. See Figure 1 below.

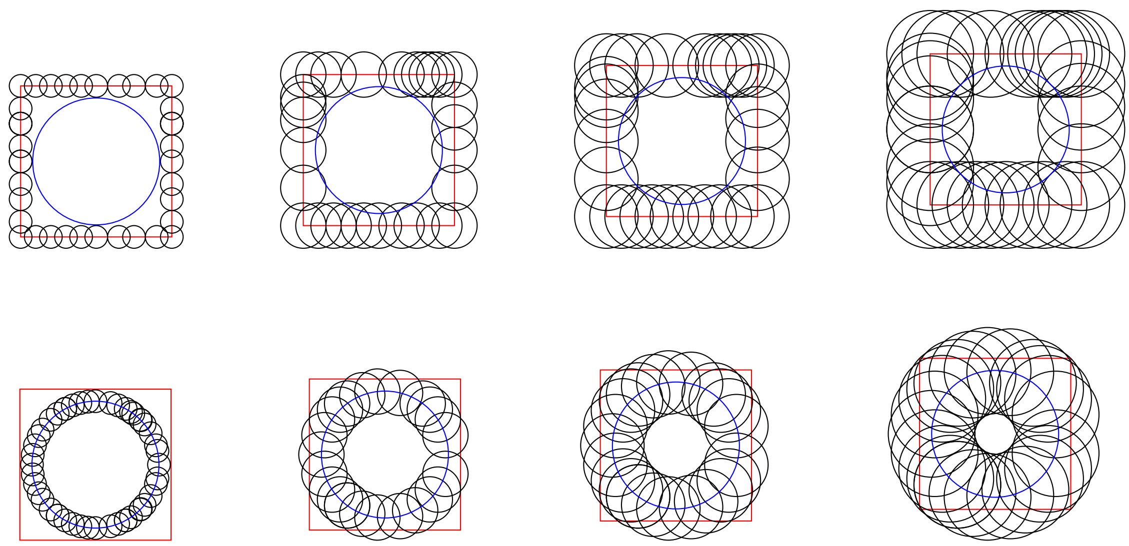

Figure 1: Let X be the rectangle and Y the circle. We have balls of radii 0.15, 0.30, 0.42, and 0.574 around X and Y. The radius are increased until the balls cover the space Y as well as X. The smallest radius such that the balls cover the spaces X and Y is the value of $d_H(X, Y)$.

The Gromov-Hausdorff distance measures the distance between two metric spaces that are not necessarily aligned in a larger metric space. Moreover, these two metric spaces may not be subsets of the same space.

Definition 4.2. *Suppose that X and Y are bounded metric spaces. The Gromov-Hausdorff distance is*

$$d_{GH}(X, Y) = \inf_{\substack{\phi\colon X \hookrightarrow Z \\ \lambda\colon Y \hookrightarrow Z}} \{d_H^Z(\phi(X), \lambda(Y))\},$$

where ϕ and λ are all possible isometric embeddings.

Definition (4.2) states that through the functions ϕ and λ, we get an alignment in the space Z that would permit us to calculate the Hausdorff distance. Figure (2) below helps to visualize the Gromov-Hausdorff distance.

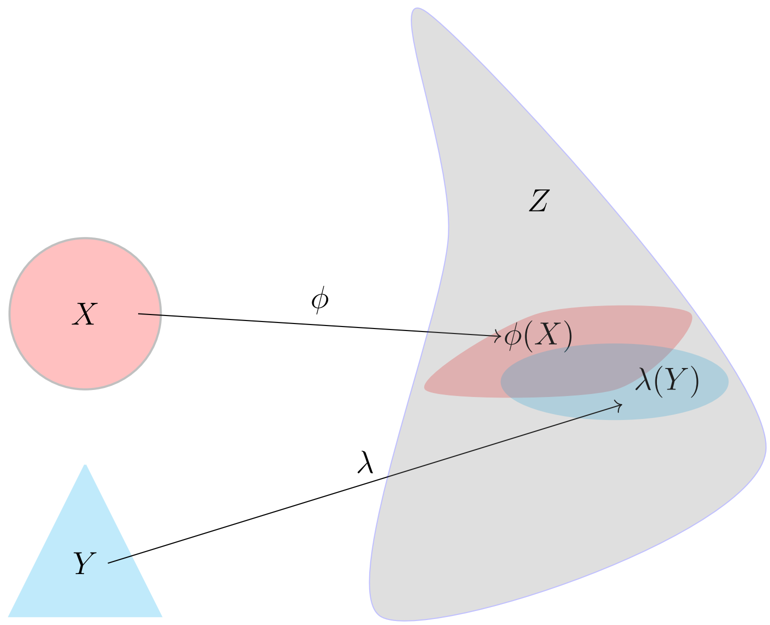

Figure 2: Graphical illustration of the Gromov-Hausdorff distance.

The Gromov-Hausdorff distance can also be expressed in terms of the distortion of correspondences.

Definition 4.3. *Correspondence.*

Let (X, d_X) and (Y, d_Y) be two arbitrary metric spaces. A correspondence is a relation $R \subseteq X \times Y$ such that $\pi_X(R) = X$ and $\pi_Y(R) = Y$. The maps π_X and π_Y are known as the canonical maps.

The set of all correspondences between X and Y is denoted $R(X, Y)$.

Definition 4.4. *For any correspondence R in $X \times Y$ ([7], p. 1) the distortion is defined as*

$$\operatorname{dis}(R) = \sup_{(x,y),(x',y') \in R} |d_X(x, x') - d_Y(y, y')|.$$

Suppose that we have a surjective function $g \colon X \to Y$, then $\operatorname{Graph}(g)$ is a correspondence.

The distortion of Graph(g) is denoted

$$\mathrm{dis}(g) = \sup_{x,x' \in R} |d_X(x,x') - d_Y(g(x), g(x'))|.$$

The next proposition provides some properties of distortion:

Proposition 4.5. *Let (X, d_X) and (Y, d_Y) be bounded metric spaces. Let R be any correspondence contained in $X \times Y$.*

1. *Consider $Y = *$, i.e, Y is a singleton. If $R = X \times *$, then* $\mathrm{dis}(X \times *) = \mathrm{diam}(X)$
2. *If $R = X \times Y$, then* $\mathrm{dis}(X \times Y) \leq \max\{\mathrm{diam}(X), \mathrm{diam}(Y)\}$

Proof. 1. By Definition 4.4

$$\mathrm{dis}(X \times *) = \sup_{x,x' \in X} |d_X(x,x') - d_Y(*,*)| = \sup_{x,x' \in X} d_X(x,x') = \mathrm{diam}(X).$$

2. By Definition 4.4

$$\mathrm{dis}(X \times Y) = \sup_{\substack{(x,y) \in X \times Y \\ (x',y') \in X \times Y}} |d_X(x,x') - d_Y(y,y')|.$$

Since $d_X(x,x') \leq \mathrm{diam}(X)$ and $d_Y(y,y') \leq \mathrm{diam}(Y)$,

$$\sup_{\substack{(x,y) \in X \times Y \\ (x',y') \in X \times Y}} |d_X(x,x') - d_Y(y,y')| \leq \sup |\,\mathrm{diam}(X) - \mathrm{diam}(Y)| \leq \max\{\mathrm{diam}(X), \mathrm{diam}(Y)\}.$$

Hence, $\mathrm{dis}(X \times Y) \leq \max\{\mathrm{diam}(X), \mathrm{diam}(Y)\}$.

$\square$

We can also approximate values of the Gromov-Hausdorff distance using the distortion:

$$d_{GH}(X,Y) = \frac{1}{2}\inf_R \operatorname{dis}(R) \tag{4.6}$$

over all correspondences R ([7], p.3).

The Gromov-Hausdorff distance is a not a metric because we could have $d_{GH}(X,Y) = 0$ when $X \neq Y$. For example, if X and Y are bounded isometric metric spaces, then $d_{GH}(X,Y) = 0$. This can be proven in a very straightforward manner using the definition of the distortion of a function and the equation 4.6.

Now, let us move on to the definition of the Vietoris-Rips complex.

Definition 4.7. *Let (X, d_X) be a metric space. Then, the Vietoris-Rips complex is*

$$\mathrm{VR}(X;r) = \{(x_0, x_1, \ldots, x_n) \mid d(x_i, x_j) \leq r,\ \forall\, 0 \leq i, j \leq n\},$$

where the vertices x_i are in X.

The Vietoris-Rips complex contains all simplices whose distance between any two of their vertices is $\leq r$. Figure 3 illustrates $\mathrm{VR}(X;r)$ at scale $r = 2$.

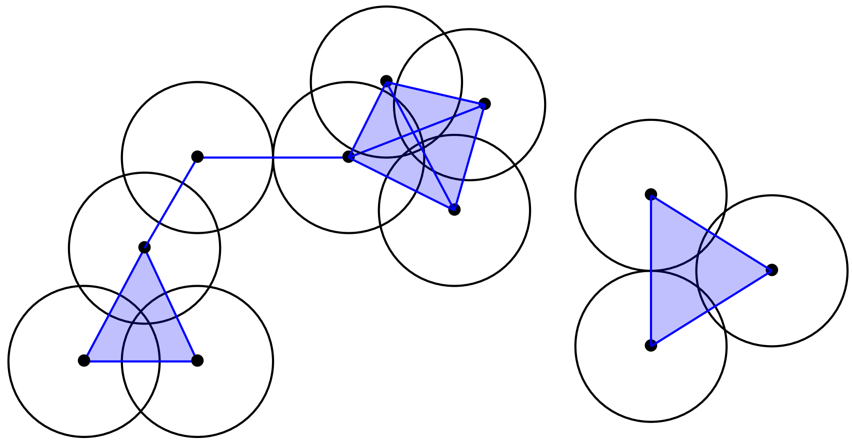

Figure 3: Each circle has a radius of 1. If two circles intersect, the distance between the circles' centers is ≤ 2, and we draw a segment between them, which is a 1-simplex. If any three circles pairwise intersect, so that their 1-simplices form a triangle, the triangle together with its interior form a 2-simplex (Above, filled in with lavender.) If any four circles pairwise intersect, so that their 1-simplices form a tetrahedron, the tetrahedron together with its interior form a 3-simplex. Similarly, if five circles pairwise intersect, a 4-simplex is created. Generally, if any n-circles pairwise intersect, an $(n-1)$-simplex is created.

4.2 The Gromov-Hausdorff Distance Between Spheres Using the Geodesic Metric.

A current area of activity in geometry and topology is the determination of bounds, particularly lower bounds, for the Gromov-Hausdorff distance between spheres using the geodesic metric. In their paper "The Gromov–Hausdorff Distance Between Spheres" [7], Lim, Memoli, and Smith demonstrate how they found lower bounds using different tools such as distortion, the distortion-preserving lemma, Dubins-Schwarz's Theorem, and the Borsuk-Ulam Theorem. They first provide weak lower bounds for $d_{GH}(S^m, S^n)$, where $0 \leq m \leq n \leq \infty$, by applying the concepts of persistent homology and bottleneck distance. This determines the following result:

$$d_{GH}(S^m, S^n) \geq \frac{1}{4}\xi_m,$$

where $\xi_m = \cos^{-1}\Bigg(-\frac{1}{m+1}\Bigg)$ and $0 < m < n < \infty$. This lower bound for these Gromov–Hausdorff distances is found by computing the geodesic distance between any two vertices of a regular $(m+1)$-simplex inscribed in S^m. For example, if we want the value of ξ_1, we must calculate the geodesic distance between any two vertices of the equilateral triangle inscribed in the circle S^1. The distance in this case is $2\pi/3$, which indicates that for $n > 1$, $d_{GH}(S^1, S^n) \geq \frac{\pi}{6}$.

Lim, Memoli, and Smith [7] were later able to construct better lower bounds: they proved that $d_{GH}(S^m, S^n) \geq \frac{1}{2}\xi_m$ when $0 < m < n < \infty$ by applying the Dubins-Schwarz Theorem 4.8 and the Distortion-Preserving Lemma 4.9, which is a generalization of the Borsuk–Ulam theorem for possibly discontinuous functions:

Theorem 4.8 (Dubins-Schwarz Theorem)**.** *For each $m > 0$, if a function $g\colon S^{m+1} \to S^m$ is odd, then* $\operatorname{dis}(g) \geq \xi_m$.

For Lim, Memoli, and Smith's proof, see [7], *p.* 46, Appendix A.

We now prove the Distortion-Preserving Lemma according to these same authors:

Lemma 4.9 (Distortion-Preserving Lemma)**.** *([7] p. 19) Let $\phi\colon C \to S^m$ be any odd function. For any non-negative m, n, let C be a non-empty set contained in S^n that satisfies $C \cap (-C) = \emptyset$. Then the function $\phi^*\colon C \cup (-C) \to S^m$, where $\phi^*(x) = \phi(x)$ when $x \in C$ and $\phi^*(-x) = -\phi(x)$, is odd. Moreover,* $\operatorname{dis}(\phi) = \operatorname{dis}(\phi^*)$.

Proof. It is pretty straightforward to show that ϕ^* is an odd function.

Now, we proceed to prove $\operatorname{dis}(\phi) = \operatorname{dis}(\phi^*)$.

Let $x, y \in C$ then

$$\begin{aligned}
|d_{S^n}(x, -y) - d_{S^m}(\phi^*(x), \phi^*(-y))| &= |\pi - d_{S^n}(x, y) - (\pi - d_{S^m}(\phi(x), \phi(y)))| \\
&= |-d_{S^n}(x, y) + d_{S^m}(\phi(x), \phi(y)))|.
\end{aligned}$$

Then, applying the definition of distortion, we get

$$|d_{S^n}(x,-y) - d_{S^m}(\phi^*(x), \phi^*(-y))| \le \operatorname{dis}(\phi).$$

Now,

$$\begin{aligned}|d_{S^n}(-x,-y) - d_{S^m}(\phi^*(-x), \phi^*(-y))| &= |\pi - d_{S^n}(-x,y) - (\pi - d_{S^m}(\phi(-x), \phi(y)))| \\ &= | - d_{S^n}(-x,y) + d_{S^m}(\phi(-x), \phi(y)))| \\ &\le \operatorname{dis}(\phi).\end{aligned}$$

So, for $x, y \in C$, $\operatorname{dis}(\phi^*) \le \operatorname{dis}(\phi)$. Also, by the way we defined ϕ^*, we have that $\operatorname{dis}(\phi) \le \operatorname{dis}(\phi^*)$.

Hence, $\operatorname{dis}(\phi) = \operatorname{dis}(\phi^*)$, and that completes the proof. □

The above results permitted them to obtain the special case when $n = 1$ and $m \ge 2$:

$$d_{GH}(S^1, S^m) \ge \frac{\pi}{3}.$$

Notice that now we have a better lower bound for the Gromov-Hausdorff distance between the circle and the m-sphere, namely $\pi/3$.

There are cases in which the implementation of Lyusternik-Schnirelmann, a theorem that is equivalent to Borsuk-Ulam's theorem ([8], p.23), is required to get a fairly good lower bound. For example, when $n > 0$ and $m < \infty$, we use this theorem to determine that $d_{GH}(S^0, S^n)$ and $d_{GH}(S^m, S^\infty)$ are both greater than or equal to $\pi/2$.

Additionally, in the same example, we can prove that the value of both $d_{GH}(S^0, S^n)$ and $d_{GH}(S^m, S^\infty)$ are exactly $\pi/2$. It can be proved that both distances are bounded above by

$\pi/2$ if we apply both part 2 of Proposition 4.5 and also the fact that the Gromov-Hausdorff distance between spheres is less than or equal to the distortion of any correspondence R.

There are other ways in which we could provide stronger lower bounds. We first discuss other basic definitions: the $\mathbb{Z}_2$-map and the coindex of a space. The symbol $\mathbb{Z}_2$ means *odd*, so a $\mathbb{Z}_2$-map refers to an *odd* map. The coindex is defined using the $\mathbb{Z}_2$-map:

Definition 4.10. *Let X be a bounded metric space. The **coindex** of X is*

$$\operatorname{coind}_{\mathbb{Z}_2}(X) = \max\{k \geq 0 \mid S^k \xrightarrow{\mathbb{Z}_2} X\},$$

where $\mathbb{Z}_2$ alludes to the existence of a $\mathbb{Z}_2$-map.

We added the concept of coindex to get much stronger lower bounds for $d_{GH}(S^m, S^n)$ when $0 \leq m \leq n$. The value $c_{m,n} \colon = \inf\{r \geq 0 \mid \operatorname{coind}_{\mathbb{Z}_2}(\mathrm{VR}(S^m; r)) \geq n\}$ is a much better lower bound for $2 \cdot d_{GH}(S^m, S^n)$ since it relies on both m and n.

In the paper "Gromov-Hausdorff Distances, Borsuk-Ulam Theorems, and Vietoris-Rips Complexes" by Adams et. al.([11], p. 14), I collaborated with multiple authors in the development of the Main Theorem of that article, which I present here.

Theorem 4.11. *The collaboration paper's "Main Theorem"([11], p.14) is as follows.*

$$\text{For } n \geq m,\ d_{GH}(S^m, S^n) \geq \frac{1}{2}c_{m,n}.$$

One of the tools used in our collaboration paper to prove Theorem (4.11) (the collaboration paper's "Main Theorem") is the fact that any odd function (it could be discontinuous) $f \colon S^n \to S^m$ has distortion $\operatorname{dis}(f) \geq \frac{1}{2}c_{m,n}$. Another useful tool is the fact that for any function $h \colon S^n \to S^m$ we always obtain an odd function $f \colon S^n \to S^m$ such that $\operatorname{dis}(h) \geq \operatorname{dis}(f)$, a statement that was determined in part using Lemma (4.9).

In Section (5), as part of our original dissertation work, we will extend Theorem (4.11) to subsets of the m-sphere, in order to help us find better lower bounds for the Gromov–Hausdorff distances between spheres and hypercubes.

5 Gromov-Hausdorff Distances Between Hypercubes and Spheres Using the Hamming Metric

In Section 4.2, we discussed the collaboration work done by the present author and other researchers on finding good lower bounds for the Gromov-Hausdorff distance between spheres of different dimensions using the geodesic metric.

In the present section, which is part of our independent, original work, we focus on finding the best lower bounds that we can for the Gromov-Hausdorff distances between n-spheres and $(n+1)$-dimensional hypercubes when the hypercubes are equipped with the geodesic metric. Later, we also lower bound the coindex of the Vietoris-Rips complexes $\mathrm{VR}(I^m; r)$ when the cube I^m is equipped with the Hamming metric, another main topic of this dissertation.

5.1 Principal Theorem and Definitions

To start the process of finding lower bounds on the Gromov-Hausdorff distance, we apply Theorem 5.1.

Theorem 5.1. *For $m, n \geq 0$ and for $Y \subseteq S^n$ with $Y = -Y$ and Y equipped with the geodesic distance, we have*

$$d_{GH}(Y, S^m) \geq \frac{1}{2} \inf\{r \geq 0 \mid \mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(Y; r)) \geq m\} =: c_m(Y).$$

We will give a sketch of the proof of Theorem (5.1), but we first present the definitions of partition of unity, ϵ-covering, $\mathbb{Z}_2$-space, and $\mathbb{Z}_2$-invariant. We will use these four concepts to guarantee the existence of an odd map between a sphere and a Vietoris-Rips complex. From this result we can determine a relation between $c_m(Y)$ and the distortion of a related map. As we know, (4.6) connects $d_{GH}(Y, S^m)$ and the distortion; hence, it will establish a connection between $d_{GH}(Y, S^m)$ and $c_m(Y)$, which will help us state that the Gromov-

Hausdorff distance is bounded below by this infimum. The following four definitions are of importance when it comes to proving Theorem (5.1).

Definition 5.2 (Partition of Unity)**.** *Let X be a topological space and let $\{U_i\}_{i\in\Lambda}$ be an open cover for X. A partition of unity subordinate to this cover is a collection of non-negative maps $\{\rho_i\}_{i\in\Lambda}$ on X such that*

i. $\operatorname{supp}(\rho_i) \subseteq U_i$

ii. For all $x \in X$ the sum of these maps is 1; that is, $\displaystyle\sum_{i\in\Lambda} \rho_i(x) = 1.$

Definition 5.3 (ϵ-covering)**.** *Let $M \subseteq X$. We say that M is an ϵ-covering if for every $x \in X$, there exists $m \in M$ such that $x \in B(m;\epsilon)$, where $B(m,\epsilon)$ is the open ball centered at m with radius ϵ.*

Definition 5.4 ($\mathbb{Z}_2$-space)**.** *A $\mathbb{Z}_2$-space is a space X together with a homeomorphism $\nu\colon X \to X$, where $\nu(x) = -x$.*

Definition 5.5 ($\mathbb{Z}_2$-invariant.)**.** *Let M be a subspace of the $\mathbb{Z}_2$-space X. We say that M is $\mathbb{Z}_2$-invariant if $M = -M$.*

Sketch of the proof of Theorem 5.1. Let $m \geq n$ and let $f\colon S^m \to Y$ be an odd map. We must select a $\mathbb{Z}_2$-invariant $(r/2)$-covering X where $X \subseteq S^m$; that is, for every point $s \in S^m$, there exists a point $x \in X$ such that $d_X(x,s) < r/2$ and $X = -X$. If we apply a $\mathbb{Z}_2$-invariant partition of unity subordinate to these balls $B(x;r/2) \subseteq S^m$, then it follows that, by Lemma 4.1 in [11] *p*.13, there exists an odd map $\phi\colon S^m \to \mathrm{VR}(X;\epsilon)$. The restriction odd map $f|_X\colon X \to Y$ induces, by Lemma 3.5 in ([11], *p*.12), the simplicial map $\bar{f}|_X\colon \mathrm{VR}(X;\epsilon) \to \mathrm{VR}(Y;\epsilon + \operatorname{dis}(f|_X))$, for any positive ϵ.

By the same Lemma 3.5 in ([11], *p*.12), the map $\bar{f}|_X$ is odd because $f|_X$ is odd. So, $\bar{f}|_X \circ \phi\colon S^m \to \mathrm{VR}(Y;\operatorname{dis}(f|_X)+\epsilon)$ is also an odd map. This indicates that $\operatorname{dis}(f|_X)+\epsilon \geq$

$\inf\{r \geq 0 | \operatorname{coind}_{\mathbb{Z}_2}(\mathrm{VR}(Y;r)) \geq m\}$ for all positive ϵ. It follows that

$$\operatorname{dis}(f|_X) \geq \{r \geq 0 | \operatorname{coind}_{\mathbb{Z}_2}(\mathrm{VR}(Y;r)) \geq m\}.$$

By definition (2) in ([7], p.3), we have

$$2 \cdot d_{GH}(Y, S^m) \geq \inf_{\substack{g\colon Y \to S^m \\ h\colon S^m \to Y}} \max\{\operatorname{dis}(g); \operatorname{dis}(h)\} \geq \inf\{\operatorname{dis}(h) | h\colon S^m \to Y\}.$$

Using Theorem 4.9, any map can be modified to get an odd map f such that $\operatorname{dis}(f) \leq \operatorname{dis}(h)$. This gives

$$d_{GH}(Y, S^m) \geq \frac{1}{2} \inf\{\operatorname{dis}(f) | f\colon S^m \to Y \text{ is odd}\} \geq \frac{1}{2} \inf\{r \geq 0 | \operatorname{coind}_{\mathbb{Z}_2}(\mathrm{VR}(Y;r)) \geq m\}.$$

□

5.2 Lower Bounding the Gromov-Hausdorff Distance Between a Sphere and a Hypercube Using the Geodesic Metric

From now on, we will write I_g^n and I_h^n when I^n is equipped with the geodesic metric and the Hamming metric, respectively.

Using the geodesic metric for I^n, we want to lower bound the Gromov-Hausdorff distance between a sphere and a hypercube as follows

$$2 \cdot d_{GH}(I_g^n, S^m) \geq \inf\{r \geq 0 \mid \operatorname{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_g^n; r)) \geq m\}.$$

We need to compute the smallest value for the scale r that allows the coindex of the Vietoris-Rips complex of I_g^n to be greater than or equal to the dimension of the sphere S^m. To be able to do this, we must determine the largest k (or the least r) so that there exists an odd map $S^k \to \mathrm{VR}(I_g^n; r)$.

Our goal is to first concentrate on building the best lower bounds for

$$d_{GH}(I_g^{n+1}, S^n)$$

when I_g^{n+1} exists inside S^n; we endow I_g^{n+1} with the spherical geodesic metric. This tells us that the vertices of I_g^{n+1} are of the form $(\pm\frac{1}{\sqrt{n+1}}, \pm\frac{1}{\sqrt{n+1}}, \ldots, \pm\frac{1}{\sqrt{n+1}})$.

To come up with general lower bounds, we first lower bound the above distance for particular values of n. We will use the homotopy types of $\mathrm{VR}(I_g^n; r)$ to determine the coindex. In cases where the homotopy type is known, the Vietoris-Rips complexes are homotopy equivalent to spheres and wedges of spheres, so we write them as such. Then, we apply the Borsuk-Ulam Theorem which states that there are no odd maps from S^n to S^{n-1}.

We will compute lower bounds for $d_{GH}(I_g^{n+1}, S^n)$ when $n = 1, 2, \cdots, 6$ based on the homotopy types of $\mathrm{VR}(I_g^n; r)$ appearing below in Table 1, which is taken from the work of Adamaszek and Adams [1].

Homotopy types of $\mathrm{VR}(I_h^n; r)$ [1]

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
$r=0$	S^0	$\bigvee^3 S^0$	$\bigvee^7 S^0$	$\bigvee^{15} S^0$	$\bigvee^{31} S^0$	$\bigvee^{63} S^0$	$\bigvee^{127} S^0$	$\bigvee^{255} S^0$	$\bigvee^{511} S^0$
$r=1$	•	S^1	$\bigvee^5 S^1$	$\bigvee^{17} S^1$	$\bigvee^{49} S^1$	$\bigvee^{129} S^1$	$\bigvee^{321} S^1$	$\bigvee^{769} S^1$	$\bigvee^{1793} S^1$
$r=2$	•	•	S^3	$\bigvee^9 S^3$	$\bigvee^{49} S^3$	$\bigvee^{209} S^3$	$\bigvee^{769} S^3$	$\bigvee^{2561} S^3$	$\bigvee^{7937} S^3$
$r=3$	•	•	•	S^7					
$r=4$	•	•	•	•	S^{15}				
$r=5$	•	•	•	•	•	S^{31}			
$r=6$	•	•	•	•	•	•	S^{63}		
$r=7$	•	•	•	•	•	•	•	S^{127}	
$r=8$	•	•	•	•	•	•	•	•	S^{255}

Table 1: The black dots indicate that $\mathrm{VR}(I_h^n; r)$ is homotopy equivalent to a point.

Because the hypercubes are discrete spaces, the Vietoris-Rips complex only changes at certain values of r, so we will evaluate the coindices only at those values.

Case $n = 0$:

$2 \cdot d_{GH}(I_g, S^0) \geq \inf\{r \geq 0 \mid \text{coind}_{\mathbb{Z}_2}(\text{VR}(I_g; r)) \geq 0\}$

For $n = 0$, the values of r where the Vietoris-Rips complex $\text{VR}(I_g^{n+1}; r)$ changes are 0 and π. We only evaluate $r = 0$ since it is the smallest r for which the coindex is ≥ 0.

$\text{coind}_{\mathbb{Z}_2}(\text{VR}(I_g; 0)) = \max\{k \geq 0 \mid S^k \to \text{VR}(I_g; 0)\} = \max\{k \geq 0 \mid S^k \overset{\mathbb{Z}_2}{\to} S^0\} = 0,$

So, $2 \cdot d_{GH}(I_g, S^0) \geq 0$.

Case $n = 1$:

$2 \cdot d_{GH}(I_g^2, S^1) \geq \inf\{r \geq 0 \mid \text{coind}_{\mathbb{Z}_2}(\text{VR}(I_g^2; r)) \geq 1\}$

For $n = 1$, the values of r where the Vietoris-Rips complex $\text{VR}(I_g^{n+1}; r)$ changes are 0, $\pi/2$, and π. We only present the computations when $r = 0, \pi/2$ since the smallest r that makes the coindex ≥ 1 is $\pi/2$.

$\text{coind}_{\mathbb{Z}_2}(\text{VR}(I_g^2; 0)) = \max\{k \geq 0 \mid S^k \to \text{VR}(I_g^2; 0)\} = \max\{k \geq 0 \mid S^k \overset{\mathbb{Z}_2}{\to} \bigvee^3 S^0\} = 0$

$\text{coind}_{\mathbb{Z}_2}(\text{VR}(I_g^2; \pi/2)) = \max\{k \geq 0 \mid S^k \to \text{VR}(I_g^2; \pi/2)\} = \max\{k \geq 0 \mid S^k \overset{\mathbb{Z}_2}{\to} S^1\} = 1$

So, $2 \cdot d_{GH}(I_g^2, S^1) \geq \pi/2$.

Case $n = 2$:

$2 \cdot d_{GH}(I_g^3, S^2) \geq \inf\{r \geq 0 \mid \text{coind}_{\mathbb{Z}_2}(\text{VR}(I_g^3, r)) \geq 2\}$

For $n = 2$, the values of r where the Vietoris-Rips complex $\text{VR}(I_g^{n+1}; r)$ changes are 0, $\cos^{-1}(1/3)$, $\cos^{-1}(-1/3)$, and π. However, we just go all the way until $r = \cos^{-1}(-1/3)$. This latter r value is the smallest one that provides us with a coindex ≥ 2.

$\text{coind}_{\mathbb{Z}_2}(\text{VR}(I_g^3; 0)) = \max\{k \geq 0 \mid S^k \overset{\mathbb{Z}_2}{\to} \bigvee^7 S^0\} = 0$

$\text{coind}_{\mathbb{Z}_2}(\text{VR}(I_g^3; \cos^{-1}(1/3))) = \max\{k \geq 0 \mid S^k \overset{\mathbb{Z}_2}{\to} \bigvee^5 S^1\} = 1$

$\text{coind}_{\mathbb{Z}_2}(\text{VR}(I_g^3; \cos^{-1}(-1/3))) = \max\{k \geq 0 \mid S^k \overset{\mathbb{Z}_2}{\to} S^3\} = 3$

So, $2 \cdot d_{GH}(I_g^3, S^2) \geq \cos^{-1}(-1/3)$.

Case $n = 3$:

$2 \cdot d_{GH}(I_g^4, S^3) \geq \inf\{r \geq 0 \mid \mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_g^4; r)) \geq 3\}$

For $n = 3$, the values of r where the Vietoris-Rips complex $\mathrm{VR}(I_g^{n+1}; r)$ changes are $r = 0$, $\cos^{-1}(1/2)$, $\pi/2$, $\cos^{-1}(-1/2)$, and π. We only calculate the coindex when $r = 0, \cos^{-1}(1/2), \pi/2$ since the smallest value of r that gives coindex ≥ 3 is $\pi/2$.

$\mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_g^4; 0)) = \max\{k \geq 0 \mid S^k \xrightarrow{\mathbb{Z}_2} \bigvee^{15} S^0\} = 0$

$\mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_g^4; \cos^{-1}(1/2))) = \max\{k \geq 0 \mid S^k \xrightarrow{\mathbb{Z}_2} \bigvee^{17} S^1\} = 1$

$\mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_g^4; \pi/2)) = \max\{k \geq 0 \mid S^k \xrightarrow{\mathbb{Z}_2} \bigvee^{9} S^3\} = 3$

So, $2 \cdot d_{GH}(I_g^4, S^3) \geq \pi/2$.

Case $n = 4$:

$2 \cdot d_{GH}(I_g^5, S^4) \geq \inf\{r \geq 0 \mid \mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_g^5; r) \geq 4\}$

For $n = 4$, the values of r where the Vietoris-Rips complex $\mathrm{VR}(I_g^{n+1}; r)$ changes are 0, $\cos^{-1}(3/5)$, $\cos^{-1}(1/5)$, $\cos^{-1}(-1/5)$, $\cos^{-1}(-3/5)$, and π. We evaluate just the r values from 0 through $\cos^{-1}(-1/5)$ to get the smallest r that makes the coindex ≥ 4.

$\mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_g^5; 0)) = 0$

$\mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_g^5; \cos^{-1}(3/5))) = \max\{k \geq 0 \mid S^k \xrightarrow{\mathbb{Z}_2} \bigvee^{49} S^1\} = 1$

$\mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_g^5; \cos^{-1}(1/5))) = \max\{k \geq 0 \mid S^k \xrightarrow{\mathbb{Z}_2} \bigvee^{49} S^3\} = 3$

Isomorphism types of $H_i(\mathrm{VR}(I_h^n; 3), M)$ with $M = \mathbb{Z}, \mathbb{Z}_2$

$H_i(\mathrm{VR}(I_h^n; 3); M)$	$i = 4$	$i = 7$	$1 \leq i \leq 7$, $i \neq 4, 7$
$H_i(\mathrm{VR}(I_h^5; 3); \mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}^{10}$	0
$H_i(\mathrm{VR}(I_h^6; 3); \mathbb{Z})$	$\mathbb{Z}^{11}$	$\mathbb{Z}^{60}$	0
$H_i(\mathrm{VR}(I_h^7; 3); \mathbb{Z})$	$\mathbb{Z}^{71}$	$\mathbb{Z}^{280}$	0
$H_i(\mathrm{VR}(I_h^8; 3); \mathbb{Z}_2)$	$\mathbb{Z}_2^{351}$	$\mathbb{Z}_2^{1120}$	0
$H_i(\mathrm{VR}(I_h^9; 3); \mathbb{Z}_2)$	$\mathbb{Z}_2^{1471}$	$\mathbb{Z}_2^{4032}$	0

Table 2: From Adamaszek and Adams [1]

The computational evidence in Table (2), from Adamaszek and Adams [1], demonstrates that $H_4(\mathrm{VR}(I_g^5;3),\ \mathbb{Z}) \cong \mathbb{Z}$, and $H_7(\mathrm{VR}(I_g^5;3), \mathbb{Z}) \cong \mathbb{Z}^{10}$; hence, we conjecture that, $\mathrm{VR}(I_h^5;3)$ is $\bigvee^{10} S^7 \vee S^4$ up to homotopy. The Vietoris-Rips complex $\mathrm{VR}(I_h^5;3)$ is equivalent to $\mathrm{VR}(I_g^5;\cos^{-1}(-1/5))$, implying that the latter is also conjecturally homotopy equivalent to $\bigvee^{10} S^7 \vee S^4$. Consequently, we conjecture that $\mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_g^5;\cos^{-1}(-1/5))) = 7$.

So, we conjecture that $2 \cdot d_{GH}(I_g^5, S^4) \geq \cos^{-1}(-1/5)$.

Case $n = 5$:

$2 \cdot d_{GH}(I_g^6, S^5) \geq \inf\{r \geq 0 |\, \mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_g^6; r)) \geq 5\}$

For $n = 5$, the values of r where the Vietoris-Rips complex $\mathrm{VR}(I_g^{n+1}; r)$ changes are 0, $\cos^{-1}(2/3)$, $\cos^{-1}(1/3)$, $\pi/2$, $\cos^{-1}(-1/3)$, $\cos^{-1}(-2/3)$, and π. Here, we present the values of the coindices at $r = 0, \cos^{-1}(2/3), \cos^{-1}(1/3), \pi/2$. The coindex starts to be ≥ 5 once $r = \pi/2$ is reached.

$\mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_g^6; 0)) = 0$

$\mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_g^6; \cos^{-1}(2/3))) = 1$

$\mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_g^6; \cos^{-1}(1/3))) = 3$

$\mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_g^6; \pi/2))) = 7$

Again, using Table (2), $\mathrm{VR}(I_h^6; 3)$ is conjectured to be homotopy to a wedge sum of multiple copies of S^4 and S^7. Therefore, as before, the equivalent simplicial complex, $\mathrm{VR}(I_g^6; \pi/2)$, is also conjecturally homotopy equivalent to a wedge sum of 4-spheres and 7-spheres.

So, we conjecture $2 \cdot d_{GH}(I_g^6, S^5) \geq \pi/2$.

Case $n = 6$:

$2 \cdot d_{GH}(I_g^7, S^6) \geq \inf\{r \geq 0 \mid \text{coind}_{\mathbb{Z}_2}(\text{VR}(I_g^7; r) \geq 6\}$

For $n = 6$, the values of r when the Vietoris-Rips complex $\text{VR}(I_g^{n+1}; r)$ changes are 0, $\cos^{-1}(5/7)$, $\cos^{-1}(3/7)$, $\cos^{-1}(1/7)$, $\cos^{-1}(-1/7)$, $\cos^{-1}(-3/7)$, $\cos^{-1}(-5/7)$, and π. We evaluate from $r = 0$ through $r = \cos^{-1}(1/7)$ since the smallest r that gives the coindex ≥ 6 is in this list.

$\text{coind}_{\mathbb{Z}_2}(\text{VR}(I_g^7; 0)) = 0$

$\text{coind}_{\mathbb{Z}_2}(\text{VR}(I_g^7; \cos^{-1}(5/7))) = 1$

$\text{coind}_{\mathbb{Z}_2}(\text{VR}(I_g^7; \cos^{-1}(3/7))) = 3$

Using the same logic, we conjecture that $\text{coind}_{\mathbb{Z}_2}(\text{VR}(I_g^7; \cos^{-1}(1/7))) = 7$ because the simplicial complex $\text{VR}(I_g^7; 3)$, which is equivalent to $\text{VR}(I_g^7; \cos^{-1}(1/7))$, is conjectured to be wedge sums of S^4 and S^7 up to homotopy.

Here, we conjecture $2 \cdot d_{GH}(I_g^7, S^6) \geq \cos^{-1}(1/7)$.

This list leads us to conjecture that

$$\text{coind}_{\mathbb{Z}_2}(I_g^{n+1}; \cos^{-1}(\tfrac{n+1-2i}{n+1})) = 2^i - 1, \tag{5.6}$$

where i is a non-negative integer.

It follows that $2 \cdot d_{GH}(I_g^{n+1}, S^n)$ has the following sub-optimal lower bound:

$$\begin{aligned}
2 \cdot d_{GH}(I_g^{n+1}, S^n) &\geq \inf\{r \geq 0 \mid \operatorname{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_g^{n+1}; r)) \geq n\} \\
&\geq \sup\{r \geq 0 \mid \operatorname{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_g^{n+1}; r)) < n\} \\
&\geq \sup\{\cos^{-1}(\tfrac{n+1-2i}{n+1}) \mid 2^i - 1 < n\} \\
&\geq \sup\{\cos^{-1}(\tfrac{n+1-2i}{n+1}) \mid 2^i < n+1\} \\
&\geq \sup\{\cos^{-1}(\tfrac{n+1-2i}{n+1}) \mid i < \log_2(n+1)\} \\
&= \cos^{-1}(\tfrac{n+1-2\cdot\log_2(n+1)}{n+1}).
\end{aligned}$$

Now, we rephrase Equation 5.6 using the more standard Hamming metric on I^n, denoted I_h^n, in which case the equation becomes

$$\operatorname{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_h^n; r)) = 2^r - 1, \tag{5.7}$$

where r is a non-negative integer.

5.3 The Smallest r That Guarantees $\operatorname{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_h^n; r)) \geq n - 1$

If we move far enough *to the right* (with n increasing) in Table 1, we realize that the conjectured lower bound in 5.7 is not as good as the value $n - 1$. However, if we move far enough *down* in the aforementioned table (with r increasing), $2^r - 1$ is a better lower bound for $\operatorname{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_h^n; r))$.

We now move on to find the smallest value of r which guarantees that $\operatorname{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_h^n; r)) \geq n - 1$. We first try to do it by considering specific values such as $n = 1, 2$ and $r = 1, 2$. Later, by means of Theorem (5.14), we are going to determine a more general r satisfying $\operatorname{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_h^n; r)) \geq n - 1$ using balanced sets and triangulations. We wish to understand which values of r make this coindex exceed $n - 1$.

To verify that $\operatorname{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_h^n; r)) \geq n-1$, we will first try to build odd maps

$$S^{n-1} \mapsto \mathrm{VR}(I_h^n; r)$$

for the smallest possible value of r that we can find. We will learn later in this section that it suffices to take $r \geq t(n-1)$, where t is a function that will also be defined later.

It is not trivial to come up with maps for the general case, so let us first find maps for two *specific* cases. Let us study the cases when $\operatorname{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_h^n; r)) \geq n-1$ and see if we can find some patterns that allow us to construct the maps we desire. To construct the maps

$$S^1 \mapsto \mathrm{VR}(I_h^2; 1) \tag{5.8}$$

and

$$S^2 \mapsto \mathrm{VR}(I_h^3; 2), \tag{5.9}$$

we first define odd maps f and ϕ from the circle to a hollow square and from the sphere to a hollow cube.

The square is defined $\{(x, y) \mid \max\{|x|, |y|\} = 1\}$.

Then, we define $f\colon S^1 \to \{(x, y) \mid \max\{|x|, |y|\} = 1\}$ via

$$f(x, y) = \frac{(x, y)}{\max\{|x|, |y|\}},$$

which is odd and continuous.

Similarly, the cube is defined $\{(x, y, z) \mid \max\{|x|, |y|, |z|\} = 1\}$.

Then, the second odd map

$$\phi\colon S^2 \to \{(x, y, z) | \max\{|x|, |y|, |z|\} = 1\}$$

is defined as

$$\phi(x,y,z) = \frac{(x,y,z)}{\max\{|x|,|y|,|z|\}}.$$

The next step is to construct maps from a square to $\mathrm{VR}(I_h^2;1)$, and from a cube to $\mathrm{VR}(I_h^3;2)$; however, we will skip the case of the square because the case of the cube is more interesting. We construct a map from the cube to the $\mathrm{VR}(I_h^3;2)$, by constructing a map from each face of the cube to each tetrahedra in $\mathrm{VR}(I_h^3;2)$.

Let us map the face $\{(x,y,z) \mid x=-1, |y|\le 1, |z|\le 1\}$ to a tetrahedron of $\mathrm{VR}(I_h^3;2)$ in the following way:

$$(y,z) \mapsto \tfrac{(z+1)(1+y)}{4}(-1,-1,-1)+\tfrac{(1-z)(1+y)}{4}(-1,1,-1)+\tfrac{(z+1)(1-y)}{4}(-1,-1,1)+\tfrac{(1-z)(1-y)}{4}(-1,1,1).$$

Let us map the face $\{(x,y,z) \mid x=1, |y|\le 1, |z|\le 1\}$ to a tetrahedron in the following way:

$$(y,z) \mapsto \tfrac{(z+1)(1+y)}{4}(1,-1,-1)+\tfrac{(1-z)(1+y)}{4}(1,1,-1)+\tfrac{(z+1)(1-y)}{4}(1,-1,1)+\tfrac{(1-z)(1-y)}{4}(1,1,1).$$

Let us map the face $\{(x,y,z) \mid y=-1, |x|\le 1, |z|\le 1\}$ to a tetrahedron in the following way:

$$(x,z) \mapsto \tfrac{(z+1)(1+x)}{4}(-1,-1,-1)+\tfrac{(1-z)(1+x)}{4}(1,-1,-1)+\tfrac{(z+1)(1-x)}{4}(-1,-1,1)+\tfrac{(1-z)(1-x)}{4}(1,-1,1).$$

Finally, let us map the face $\{(x,y,z) \mid y=1, |y|\le 1, |z|\le 1\}$ to a tetrahedron in the following way:

$$(x,z) \mapsto \tfrac{(z+1)(1+x)}{4}(-1,1,-1)+\tfrac{(1-z)(1+x)}{4}(1,1,-1)+\tfrac{(z+1)(1-x)}{4}(-1,1,1)+\tfrac{(1-z)(1-x)}{4}(1,1,1).$$

We do the same thing for the cases $z=1$ and $z=-1$. It is pretty straightforward to show that the six functions pasted together form an odd map. It follows that the composition of ϕ and the piecewise function above is the odd map $S^2 \to \mathrm{VR}(I_h^3; 2)$ as we described in Equation 5.9.

More generaally, using the same idea we describe an odd map

$$S^{n-1} \mapsto \mathrm{VR}(I_h^n; n-1) \tag{5.10}$$

as follows:

A map from an $(n-1)$-sphere to the hollow n-cube $\{(x_1, x_2, \cdots, x_n) \mid \max\{|x_1|, |x_2|, \cdots, |x_n|\} = 1\}$ is given by

$$\phi_n(x_1, \cdots, x_n) = \frac{(x_1, x_2, \cdots, x_n)}{\max\{|x_1|, |x_2|, \cdots, |x_n|\}}.$$

Now let us build a map Γ from the n-cube to $\mathrm{VR}(I_h^n; n-1)$ by sending each face of the n-cube to an $(n-1)$-simplex:

$$(-1, x_2, \cdots, x_n) \overset{\Gamma}{\longmapsto} \sum_{(y_2, \cdots, y_n) \in \{-1,1\}^{n-1}} \frac{(1 - y_2 x_2)(1 - y_3 x_3) \cdots (1 - y_n x_n)}{2^{n-1}} (-1, y_2, \cdots, y_n).$$

$$(1, x_2, \cdots, x_n) \overset{\Gamma}{\longmapsto} \sum_{(y_2, \cdots, y_n) \in \{-1,1\}^{n-1}} \frac{(1 - y_2 x_2)(1 - y_3 x_3) \cdots (1 - y_n x_n)}{2^{n-1}} (1, y_2, \cdots, y_n).$$

Note that we mapped just two faces of the n-cube, that is when $x_1 = 1, -1$. Similarly, we map the other faces by fixing the other coordinates x_i to be either -1 or 1. Finally, the composition $\Gamma \circ \phi_n$ is the desired odd map from S^{n-1} to $\mathrm{VR}(I_h^n; n-1)$.

We had already built a map for 5.10; however, $n-1$ is not necessarily the value of the infimum for some n greater than 3. There exist values of n that satisfy $\inf\{r \geq 0 \mid \mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_h^n; r)) \geq n-1\} < n-1$. The least n for which this holds is $n = 4$.

We ask ourselves which properties r should satisfy in order to obtain the largest lower bound for $d_{GH}(I^{n+1}, S^n)$. To answer this, we must try to create an odd map

$$S^{n-1} \mapsto \mathrm{VR}(I_h^n; r),$$

where r is "as small as possible."

Let us first consider working on determining a map $S^3 \to \mathrm{VR}(I_h^4; 2)$, since this is the first candidate for which we may have $\inf\{r \geq 0 \mid \mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I^4; r)) \geq 3\} = 2 < 3$. We will analyze the case when $n = 4$ to see if this permits us to build a similar map for any $n > 4$.

To build this map, we will apply three concepts: balanced sets of vectors, abstract convex combination, and decomposition of n-cubes into cubes of smaller dimensions.

Now, let us first define a balanced set of vectors.

Definition 5.11 (Balanced sets). *Let B be a subset of the hypercube I_h^n. We say that B is a balanced set if the average of all the vectors in B equals* $(1/2, 1/2, \cdots, 1/2)$.

Let us now define an odd map from S^3 to $|\mathrm{VR}(I_h^4; 2)|$. The idea consists of mapping each face of $\partial[0,1]^4$ into each side of $\mathrm{VR}(I_h^4; 2)$. Each side of $\partial[0,1]^4$ is a copy of $[0,1]^3$, and each side of $\mathrm{VR}(I_h^4; 2)$ is a copy of $\mathrm{VR}(I_h^3; 2)$. This tells us that we need to build a map $h : [0,1]^3 \to |\mathrm{VR}(I_h^3; 2)|$. To do this, the first step is to triangulate of $[0,1]^3$ into five tetrahedra, each of diameter at most 2:

$$\tau_1 = \{(0,1,1), (1,1,0), (1,0,1), (0,0,0)\}$$

$$\tau_2 = \{(0,0,0), (1,0,0), (1,1,0), (1,0,1)\}$$

$$\tau_3 = \{(0,1,1), (1,1,0), (0,0,0), (0,1,0)\}$$

$$\tau_4 = \{(1,1,1), (0,1,1), (1,1,0), (1,0,1)\}$$

$$\tau_5 = \{(0,0,0),(0,1,1),(0,0,1),(1,0,1)\}.$$

Note that, by Definition 5.11, τ_1 is a balanced tetrahedron. There are various way of triangulating the 3D cube into tetrahedra, but we are selecting a triangulation containing a balanced tetrahedron. In Figure 4, we can clearly see how $[0,1]^3$ is decomposed into five tetrahedra.

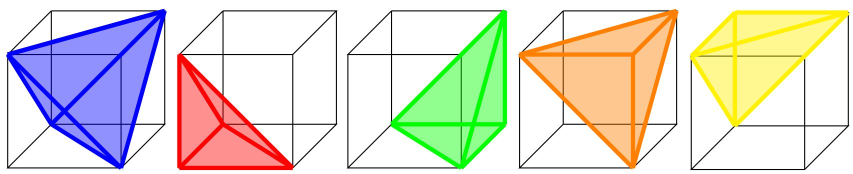

Figure 4: Triangulation of the cube into five tetrahedra. Here, we have the tetrahedra $\tau_1, \tau_2, \tau_3, \tau_4$, and τ_5, respectively. The lavender tetrahedron, τ_1, is the balanced tetrahedron.

Theorem 5.12. *Let τ_i, for $i = 1, \cdots, 5$, be the five tetrahedra described above. Then, there exists a map $h\colon [0,1]^3 \to |\operatorname{VR}(I_h^3; 2)|$ defined by*

$$h(x) = \sum_{v \in \tau_i} \lambda_v v,$$

where τ_i is the tetrahedron whose geometric realization contains x, and the right side of h is an abstract convex combination.

Proof. The solid cube can be written $\bigcup_{i=1}^{5} \|\tau_i\| = [0,1]^3$, where τ_1 is the balanced tetrahedron. Each $x \in [0,1]^3$ can be written as a convex combination of vertices of some tetrahedron τ_i; that is, $x = \sum_{v\in\tau_i} \lambda_v v$ with $\lambda_v \geq 0$ and $\sum_{v\in\tau_i} \lambda_v = 1$. For example, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = \sum_{v\in\tau_1} \frac{1}{4} v$.

The map h can be defined in the following manner:

$$h\colon [0,1]^3 \to |\operatorname{VR}(I_h^3; 2)|,$$

where

$$h(x) = \sum_{v \in \tau_i} \lambda_v v.$$

The right side of h now refers to an abstract convex combination.

It remains to be verified that h is a well-defined map. Assume, without loss of generality, that $x \in [0,1]^3$ with $x \in \|\tau_1\|$ and $x \in \|\tau_2\|$. For example, consider $\tau_1 = \{(0,1,1),(1,1,0),(1,0,1),(0,0,0)\}$ and $\tau_2 = \{(0,0,0),(1,0,0),(1,1,0),(1,0,1)\}$; the other cases will end up being analogous. It follows that $x = \sum_{v \in \tau_1} \lambda_v v$ and $x = \sum_{v \in \tau_2} \lambda_v v$. Since x is in the geometric realization of both, τ_1 and τ_2, $x \in \|\tau_1 \cap \tau_2\| = \|\{(0,0,0),(1,0,1),(1,1,0)\}\|$. Since x belongs to a geometric realization of three vertices, x is in the triangle that is the intersection of τ_1 and τ_2. This means $\lambda_{(0,1,1)} = \lambda_{(1,0,0)} = 0$. Since τ_1 and τ_2 have the other vertices in common, $\sum_{v \in \tau_1} \lambda_v v = \sum_{v \in \tau_2} \lambda_v v$. Therefore, we have proven that h is well-defined. $\square$

We completed the case I^3 at scale 2 with balanced set τ_1. If we move to hypercubes of higher dimensions, we can obtain a scale r less than $n-1$ when we decompose the n-cube into cubes of smaller dimensions. We will use the fact that any n-cube can be decomposed into n-simplices.

Indeed, $[0,1]^4 = [0,1]^3 \times [0,1] = (\bigcup_{i=1}^{5} \|\tau_i\|) \times (\|\tau\|) = \bigcup \|\tau_i \times \tau\|$, where $\tau = \{0,1\}$ is an edge triangulating $[0,1]$. The simplices τ_i triangulated $[0,1]^3$ possess a diameter of at most 2 and τ is of diameter 1. This indicates $[0,1]^4$ can be decomposed into 4-dimensional simplices with diameter at most 3. The 4-cube can be also broken up into $[0,1]^2 \times [0,1]^2$. In this case, however, we get simplices with diameter at most 4 which is not as small. Diameter 3 is better than 4 because we mentioned above that we desire a scale r smaller than $n-1$.

Hypercubes of higher dimension can be decomposed in various manners, but only a subset of decompositions provides the best diameter.

If we write $[0,1]^5 = [0,1]^4 \times [0,1]$, then we get that $[0,1]^5$ is the union of 5-simplices with diameter at most 4.

If we decompose $[0,1]^6$ as $[0,1]^6 = [0,1]^3 \times [0,1]^3$ we get that this 6-cube is the union of 6-simplices with diameter at most 4.

We can decompose $[0,1]^7, [0,1]^8$, $[0,1]^9$, $[0,1]^{10}$, and $[0,1]^{11}$ as follows:

1. $[0,1]^7 = [0,1]^6 \times [0,1]$ which gives the union of 7-simplices with diameter at most $4+1=5$.

2. $[0,1]^8 = [0,1]^6 \times [0,1]^2$ which gives the union of 8-simplices with diameter at most $4+2=6$.

3. $[0,1]^9 = [0,1]^6 \times [0,1]^3$, and this gives the union of 9-simplices with diameter at most $4+2=6$.

4. $[0,1]^{10} = [0,1] \times [0,1]^9$ provides the union of 10-simplices with diameter at most $1+6 = 7$.

5. $[0,1]^{11} = [0,1]^2 \times [0,1]^9$ provides the union of 11-simplices with diameter at most $2+6=8$.

These examples provided above will be useful when establishing a proof for Theorem (5.14), which is the general case.

Definition 5.13. *Let $t(n)$ be the smallest scale parameter such that we can divide I_h^n into n-dimensional simplices of diameter at most $t(n)$.*

Table 3 shows a list of known values of $t(n)$ for different values of n:

Property 1. If $[0,1]^n$ can be divided into n-simplices of diameter at most r and $[0,1]^m$ can be divided into m-simplices of diameter at most r', then $[0,1]^{n+m}$ can be divided into $(n+m)$-simplices of diameter at most $r+r'$.

n	1	2	3	4	5	6	7	8	9
$t(n)$	1	2	2	≤ 3	≤ 4	≤ 4	≤ 5	≤ 6	≤ 6

Table 3

Proof. Let $\{\sigma_i\}$ be n-simplices of diameter at most r subdividing $[0,1]^n$, and let $\{\tau_j\}$ be m-simplices of diameter at most r' subdividing $[0,1]^m$. Note that each $(n+m)$-*cell* $\sigma_i \times \tau_j$ has diameter at most $r+r'$. So, after dividing each such cell $\sigma_i \times \tau_j$ arbitrarily into $(n+m)$-simplices, we get a subdivision of $[0,1]^{n+m}$ into simplices of diameter at most $r+r'$. □

This property allows us to establish that $t(n+m) \leq t(n)+t(m)$.

In Theorem 5.14 we will show that the coindex $\mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_h^n; r))$ is at least $n-1$ for $r \geq t(n-1)$. This theorem will later be shown for the general case, but first let us see how it works for the special case $n=4$.

In Theorem 5.12, we carefully defined a map h from the 3-cube to the $|\mathrm{VR}(I_h^3; 2)|$ using a triangulation and an abstract convex combination. Now, let us produce an odd function $\phi\colon S^3 = \partial([0,1]^4) \to |\mathrm{VR}(I_h^4; 2)|$. The 4-cube has eight sides that are 3-cubes, and we can decompose each side as we did in Figure 4. As a result, we can build eight functions in which one of them is the h from Theorem 5.12. Let us name h as h_1. The other seven functions can be denoted $h_2, h_2 \cdots, h_8$.

To determine each h_*, we need to first express the other 3-cubes $[0,1]_j^3$ as $\bigcup_{i=1}^{5} ||\sigma_{i,j}||$, where $\sigma_{1,j}$ is in this case the balanced tetrahedron. Then,

$$h_*(x) = \sum_{v \in \sigma_{i,j}} \lambda_v v.$$

So, ϕ is the function

$$\phi = h_1, h_2 \cdots, h_8.$$

Using the Pasting Lemma, we know that ϕ is a piecewise map in which each h_* maps one of the eight faces of the 4-cube.

The function ϕ can be made to be odd if we first send v to $-v$, and then, through h_*, we map each face of $\mathrm{VR}(I_h^4; 2)$ to its opposite face in an odd way. In other words, we mandate that $h_*(-v) = -h_*(v)$.

It follows we have produced an odd map ϕ from S^3 to $|\,\mathrm{VR}(I_h^4; 2)|$, indicating that $\mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_h^4; 2)) \geq 3$. Also, we used five 3-simplices of diameter 2 to decompose each 3-cube, realizing the value of $t(3) = 2$. Therefore, we have completed this particular case.

Theorem 5.14. *Consider the space I^n equipped with the Hamming metric, then $\mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I_h^n; r)) \geq n - 1$ for $r \geq t(n-1)$.*

Proof. Our goal is to produce an odd map $\phi\colon \partial([0,1]^n) \to |\,\mathrm{VR}(I_h^n; r)|$.

Suppose that we have a triangulation of one of the faces of the n-cube into simplices of dimension $n-1$ and of diameter at most $t(n-1)$.

The map ϕ will be produced by piecing together $2n$ maps (namely $h_1, h_2, \cdots, h_{2n}$) from each of the $2n$ faces of $[0,1]^n$, to $|\,\mathrm{VR}(I_h^n; r)|$.

By Definition (5.13), let $\tau_{i,j}$ be $(n-1)$-simplices of diameter at most $t(n-1) \leq r$ in the Vietoris-Rips complex $\mathrm{VR}(I_h^n; r)$ with $[0,1]_j^{n-1} = \bigcup_{i=1}^{m} \|\tau_{i,j}\|$. Here, the $(n-1)$-dimensional cubes $[0,1]_j^{n-1}$, where $j = 1, 2, \cdots, 2n$, refers to each of the faces of $[0,1]^n$.

Now, let us define the function h_* in the following manner:

$$h_*\colon [0,1]^{n-1} \to |\,\mathrm{VR}(I_h^n; r)|,$$

where

$$h_*\Big(\sum_{v \in \tau_i} \lambda_v v\Big) = \sum_{v \in \tau_i} \lambda_v v.$$

Next, we define the function

$$\phi\colon \partial([0,1]^n) \to |\operatorname{VR}(I_h^n; r)|$$

by

$$\phi = h_1, h_2, \cdots, h_{2n}$$

The function ϕ could be odd if we first send v to its antipode, i.e, $v \to -v$. Then, the antipode will be mapped to the opposite face of $|\operatorname{VR}(I_h^n; r)|$ through the map h_*, meaning that $h_*(-v) = -h_*(v)$. Finally, by the Pasting Lemma, ϕ becomes an odd map.

□

Corollary 5.15. *For $n \geq 3$ and $k \geq 3$ we have the following:*

i. $t(n) \geq n - 1$

ii. $t(3k) \leq 2k$

iii. $t(3k+1) \leq 2k+1$

iv. $t(3k+2) \leq 2k+2$

Proof. The proof of each case is by induction. For $n = 3$, we can see below that $t(3) \leq 2 = 3-1$. For the inductive step, assume $t(n) \leq n-1$. Then $t(n+1) \leq t(n)+t(1) \leq (n-1)+1 = n$.

Let us prove *ii*. When $k = 3$, we have $t(9) \leq 6 = 2(3)$. For the inductive step, assume that $t(3k) \leq 2k$. It follows $t(3(k+1)) = t(3k+3) \leq t(3k) + t(3) \leq 2k+2 = 2(k+1)$, and we are done.

Now, let us prove *iii*. When $k = 3$, we get $t(9) \leq 6 < 2(3)+1$. For the inductive step, let us assume that $t(3k+1) \leq 2k+1$. Then, $t(3(k+1)+1) = t(3k+1+3) \leq t(3k+1)+t(3) \leq 2k+1+2 = 2(k+1)+1$.

Finally, we prove *iv*. Let $k = 3$, then $t(3(3)+1) = t(9) + t(1) \leq 7 \leq 2(3) + 2$. By the inductive step, we assume that $t(3k+2) \leq 2k+2$. Therefore,

$$t(3(k+1)+2) = t(3k+2+3) \leq t(3k+2) + t(3) \leq 2k+2+2 = 2(k+1)+2,$$

and this completes the proof.

$\square$

Thus, we have established a better lower bound for the coindex of the Vietoris-Rips complex of hypercubes using the Hamming metric (5.14), the desired result.

6 Possibilities for Future Investigation

The work of this dissertation has given us some new ideas for possible future investigations.

6.1 Relationships Between Spectral Sequences and Groups $\boldsymbol{PH_l(I^k) \otimes PH_j(I^k)}$ in an Exact Sequence.

One possible area of future investigation would be to apply Theorem 3.2 to a filtration arising from the hypercube metric space I^k, where $k \geq 1$. In this case, the theorem would state the following:

Suppose that M and N are the same filtrations of hypercubes; i.e,

$$M = N\colon\ \mathrm{VR}(I;r) \subseteq \mathrm{VR}(I^2;r) \subseteq \cdots \subseteq \mathrm{VR}(I^{s-1};r) \subseteq \mathrm{VR}(I^s;r) \subseteq \cdots,$$

where $M_p = N_p = I^p$ for any $p > 0$, then

$$\ldots \to \bigoplus_{l+j=n} PH_l(M_p) \otimes PH_j(N_p) \to E^{(r)}_{p,q}(M_p \times N_p) \to \bigoplus_{l+j=n-1} PH_l(M_p) \otimes PH_j(N_p) \to \ldots$$

is a long exact sequence.

Here, $PH_*(M_*)$ is the persistent homology of the metric space (I^k_h, d_{HM}) associated to I^k_h. We let "HM" stand for the Hamming Metric.

In the present dissertation paper, we structured the proof of Theorem 3.2 for the categorical product, but it is important to note that the persistent homology of hypercubes with the Hamming metric is not precisely a categorical product. Another issue to keep in mind is that the ring we are considering is $k[\mathbb{R}^+]$. This ring is not a field, and that implies that the tor terms in the exact sequence

$$0 \to \bigoplus_{l+j=n} PH_l(M_p) \otimes_{k[\mathbb{R}+]} PH_j(N_p) \to PH_n(M_p, N_p) \to \bigoplus_{l+j=n-1} \mathrm{Tor}_1(PH_l(M_p), PH_j(N_p)) \to 0$$

do not always become 0. Non-zero tor terms will not allow us to construct an isomorphic map from $\bigoplus_{l+j=n} PH_l(M_*) \otimes_{k[\mathbb{R}+]} PH_j(N_*)$ to $PH_n(M_*, N_*)$. Nevertheless, one idea for future development consists of verifying for which values of n do we have $PH_n(M_*) = 0$.

Carlsson and Filippenko ([6], *p.* 5) displayed a Table 1, reproduced herein as Table 4, with different rows in which they computed $PH_n(I^k)$ for values of n ranging between 1 and 7 and for $k \geq 1$.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$PH_0(I^k)$	2	4	8	16	32	64	128
$PH_1(I^k)$	0	1	5	17	49	129	321
$PH_2(I^k)$	0	0	0	0	0	0	0
$PH_3(I^k)$	0	0	1	9	49	209	769
$PH_4(I^k)$	0	0	0	0	1	11	71
$PH_5(I^k)$	0	0	0	0	0	Unkn.	Unkn.
$PH_6(I^k)$	0	0	0	0	0	Unkn.	Unkn.
$PH_7(I^k)$	0	0	0	1	10	Unkn.	Unkn.

Table 4: Number of bars in $PH_n(I^k)$ ([6], *p.* 5)

Five rows on that table exhibit numbers different from zero; for example, we get 0 bars when computing $PH_n(I^k)$ for $n = 0, 1, 3, 4, 7$ with certain values of k. On the other hand, for $n = 5$ and 6, $PH_n(I^k) = 0$ when $1 \leq k \leq 5$. For $k \geq 5$, the number of bars in $PH_5(I^k)$ and $PH_6(I^k)$ are unknown. We could study the persistent homology of $PH_n(I^k)$ for $n = 5$ and 6 with $k > 5$ and see if they exhibit no bars in these cases. Furthermore, we could study if we get zero bars for the persistent homologies of much larger dimension. Perhaps computing some of these persistent homologies by hand, for instance for $n = 5, 6$ and for various values of k, will give us an idea of when $PH_n(I^k) = 0$.

Another alternative to finding an isomorphism would be to use a different metric in place of the Hamming metric. Creating a metric space (I^k, d) with a new metric d will perhaps solve the problem regarding deviating from a categorical product filtration, and make the tor terms equal to zero for any $PH_n(I^k)$. The sup metric, that is, $l_\infty((x_1, x_2, ..., x_k), (y_1, y_2, ..., y_k)) = max\{d_i(x_i, y_i) \colon i = 1, 2, \ldots, k\}$ can be deemed as a possible metric on I^k, if we write the hypercube as products of smaller hypercubes.

6.2 Künneth formulas for the persistent homology of hypercubes

Another possibility for future research would be to study Vietoris–Rips complexes of hypercubes from the perspective of Künneth formulas.

We could consider studying the short exact sequence

$$0 \to \bigoplus_{l+j=n} PH_l(I^{k-1}) \otimes_{k[\mathbb{R}^+]} PH_j(I) \to PH_n(I^{k-1}, I) \to \bigoplus_{l+j=n-1} \mathrm{Tor}_1(PH_j(I^{k-1}), PH_l(I)) \to 0$$

and identify each of its persistent homology groups. These groups could be identified by looking for patterns, and that also could be done by hand.

An additional option would be to examine the persistent homology of the homotopy types of the Vietoris–Rips complex of hypercubes $\mathrm{VR}(I^n; 3)$ when $n > 4$. Table 1 of these homotopy types which is repeated from earlier and is found in [1], is displayed below for values of $n = 1, 2, \ldots, 9$ at different scales r.

At scale $r = 0$, the hypercube I^n includes just non-connected vertices denoted by strings of zeroes and ones; therefore, $\mathrm{VR}(I^n, 0)$ is homotopy equivalent to a wedge-sum of 0-spheres.

At scale parameter $r = 1$, $\mathrm{VR}(I^n, 1)$ represents the hypercube graphs for $n > 1$, which are homotopy equivalent to wedges of circles. In the case of $n = 2$, we get one circle since

Homotopy types of $\mathrm{VR}(I^n; r)$

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
$r=0$	S^0	$\bigvee^3 S^0$	$\bigvee^7 S^0$	$\bigvee^{15} S^0$	$\bigvee^{31} S^0$	$\bigvee^{63} S^0$	$\bigvee^{127} S^0$	$\bigvee^{255} S^0$	$\bigvee^{511} S^0$
$r=1$	$\bullet$	S^1	$\bigvee^5 S^1$	$\bigvee^{17} S^1$	$\bigvee^{49} S^1$	$\bigvee^{129} S^1$	$\bigvee^{321} S^1$	$\bigvee^{769} S^1$	$\bigvee^{1793} S^1$
$r=2$	$\bullet$	$\bullet$	S^3	$\bigvee^9 S^3$	$\bigvee^{49} S^3$	$\bigvee^{209} S^3$	$\bigvee^{769} S^3$	$\bigvee^{2561} S^3$	$\bigvee^{7937} S^3$
$r=3$	$\bullet$	$\bullet$	$\bullet$	S^7					
$r=4$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	S^{15}				
$r=5$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	S^{31}			
$r=6$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	S^{63}		
$r=7$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	S^{127}	
$r=8$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	S^{255}

Table 1, repeated from earlier. [1]
The black dots indicate homotopy equivalent to a point.

$\mathrm{VR}(I^2, 1)$ is homotopy equivalent to a hollow square. When $n = 3$, the Vietoris–Rips complex $\mathrm{VR}(I^3, 1)$ is homotopy equivalent to a wedge of five circles because it has six squares and one of them is a linear combination of the other five.

When the parameter r is 2, the complexes $\mathrm{VR}(I^n, 2)$ are all determined up to homotopy. But as soon as we reach the scale parameter 3, the homotopy types of $\mathrm{VR}(I^n; 3)$ are not determined for large values of n. We could conjecture that $\mathrm{VR}(I^n; 3)$ are up to homotopy equal to wedges of two different dimensions of spheres. Identifying the homology type of $\mathrm{VR}(I^n; 3)$ could give us an idea what kind of spheres and how many spheres would form part of these wedge sums. Table 2 below, repeated from earlier and found in ([1] *p.* 3), exhibits a list of isomorphism types of homology groups of Vietoris–Rips complex at scale 3, up to the value of $n = 9$.

Isomorphism types of $H_i(\mathrm{VR}(I^n; 3), M)$ with $M = \mathbb{Z}, \mathbb{Z}_2$

$H_i(\mathrm{VR}(I^n; 3); M)$	$i = 4$	$i = 7$	$1 \le i \le 7$, $i \ne 4, 7$
$H_i(\mathrm{VR}(I^5; 3); \mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}^{10}$	0
$H_i(\mathrm{VR}(I^6; 3); \mathbb{Z})$	$\mathbb{Z}^{11}$	$\mathbb{Z}^{60}$	0
$H_i(\mathrm{VR}(I^7; 3); \mathbb{Z})$	$\mathbb{Z}^{71}$	$\mathbb{Z}^{280}$	0
$H_i(\mathrm{VR}(I^8; 3); \mathbb{Z}_2)$	$\mathbb{Z}_2^{351}$	$\mathbb{Z}_2^{1120}$	0
$H_i(\mathrm{VR}(I^9; 3); \mathbb{Z}_2)$	$\mathbb{Z}_2^{1471}$	$\mathbb{Z}_2^{4032}$	0

Table 2 - Repeated from earlier ([1] *p.* 3)

Observe that for $n = 5$ the fourth and seventh dimensional homology of $\mathrm{VR}(I^5; 3)$ are isomorphic to $\mathbb{Z}$ and $\mathbb{Z}^{10}$. This could lead us to conjecture that $\mathrm{VR}(I^5; 3)$ is homotopy to a wedge sum of one copy of S^4 and ten copies of S^7. Similarly, for $n = 6$ the homology of dimension four and of dimension seven are the groups $\mathbb{Z}^{11}$ and $\mathbb{Z}^{60}$, respectively, implying that $\mathrm{VR}(I^6; 3)$ may consist of a wedge sum of eleven copies of S^4 and sixty copies of S^7 up to homotopy.

In my research I used the conjecture that the number of copies of $\mathbb{Z}$ in the homology of $\mathrm{VR}(I^n; 3)$ reveals the number of copies of S^4 and S^7; in the future, it would be a worthwhile project to prove this conjecture.

It is also would be worth exploring the unknown persistent homology barcodes of I^k using the metric l_∞.

6.3 The Gromov-Hausdorff Distance–Future Investigation Topics

In Section 5, we were able to find strong lower bounds for $d_{GH}(I^{n+1}, S^n)$, and proved it by finding the best possible lower bounds for $\mathrm{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I^n_h; r))$. In the future, we could look for the best possible lower bounds for $d_{GH}(I^n, I^{n+1})$ when both hypercubes are equipped with the Hamming metric. We would like to see if the same concepts employed in Section 5 can be applied to find tight lower bounds for $d_{GH}(I^n, I^{n+1})$. In addition, we could try to prove Theorem 5.1 in the case of two hypercubes using the geodesic metric; however, it is uncertain if all the steps used to prove Theorem 5.1 will work. We are not sure if we would be able to obtain an odd map $\beta\colon I^n \to I^m$ for any map $\alpha\colon I^n \to I^m$ such that $\mathrm{dis}(\alpha) \geq \mathrm{dis}(\beta)$.

Another topic for future research could consist of upper bounding $d_{GH}(I^{n+1}, S^n)$. Additionally, we could consider doing research on computing effective lower bounds for the Gromov-Hausdorff distance in a more general case; that is, $d_{GH}(I^n, S^m)$ when I^n is endowed with either the geodesic or the Hamming metric.

7 Conclusion

In this dissertation, we first provided background in two different areas. Through Carlsson and Filippenko's paper [6], we learned the fundamental properties of modules $F_*(X, l)$, persistent chain complexes of $F_*(X, l)$, persistent homologies of $\mathbb{R}^+$-filtered simplicial sets and metric spaces (and the relation between them,) and Künneth formulas for metric spaces: they focused on investigating the isomorphism types of the persistent homology of hypercubes up to dimension 2. Basu-Parida's paper [5] concentrated on proving the existence of an exact sequence whose terms are $H_*^{*,*}(X)$ and $E^*_{*,*}(X)$, where X is an increasing filtration of simplicial complexes. This paper was the inspiration for our original work refining this exact sequence from the perspective of the categorical product $X \times Y$—we demonstrated this result in part by applying two versions of Künneth formulas.

We also presented how researchers have recently applied different technical theorems involving persistent homology, the bottleneck distance, and distortion of correspondences to find new lower bounds for the Gromov-Hausdorff distance between m-spheres and n-spheres when both are equipped with the geodesic metric. In our collaboration paper (with multiple authors) [11] it was shown that for $n \geq m$, $d_{GH}(S^m, S^n)$ is bounded below by $\inf\{r \geq 0 | \operatorname{coind}_{\mathbb{Z}_2}(\mathrm{VR}(S^m; r)) \geq n\}$. In our own original dissertation work, we refined this result for $d_{GH}(Y, S^n)$ when Y is deemed as living inside the m-sphere. We then determined new lower bounds for $d_{GH}(I^{n+1}, S^n)$, conjectured a new lower bound for $\operatorname{coind}_{\mathbb{Z}_2}(\mathrm{VR}(I^n; r))$, and later created maps between $S^n \mapsto |\mathrm{VR}(I^n_h; r)|$ to verify if the coindex was indeed lower bounded by the conjectured values. Finally, we determined much stronger lower bounds by defining a map from the faces of the n-cube to $|\mathrm{VR}(I^n_h; r)|$, by triangulating the sides of the n-cubes into m number of tetrahedra and by decomposing each face, $(n-1)$-cubes, into smaller diameter simplices.

In Section 6, we suggested many possible topics for future investigation. In Section

6.1, "Relation Between Spectral Sequences and Groups $PH_l(I^k) \otimes PH_j(I^k)$ in an Exact Sequence", we proposed possibly applying Theorem 3.2 to a filtration arising from the hypercube metric space I^k (where $k \geq 1$,) perhaps verifying for which values of n is $PH_n(M_*) = 0$, maybe studying the persistent homology of $PH_n(I^k)$ for $n = 5$ and 6 and other cases, and perchance finding an isomorphism using a different metric in place of the Hamming metric. In Section 6.2, "Künneth Formulas for the Persistent Homology of Hypercubes," we proposed the possibility of studying Vietoris–Rips complexes of hypercubes from the perspective of Künneth formulas, perhaps examining the persistent homology of the homotopy types of the Vietoris–Rips complex of hypercubes $\mathrm{VR}(I^n; 3)$ when $n > 4$, maybe proving the conjecture that the number of copies of $\mathbb{Z}$ in the homology of $\mathrm{VR}(I^n; 3)$ reveals the number of copies of S^4 and S^7, or perchance exploring the unknown persistent homology barcodes of I^k using the metric l_∞. Finally, in Section 6.3, "Future Investigation Topics Regarding the Gromov-Hausdorff Distance,", we proposed the possibility in the future of looking for the best possible lower bounds for $d_{GH}(I^n, I^{n+1})$ when both hypercubes are equipped with the Hamming metric, maybe considering if the same concepts employed in Sections (5.2) and (5.3) can be applied to find tight lower bounds for $d_{GH}(I^n, I^{n+1})$, or perchance trying to prove Theorem (5.1) in the case of two hypercubes using the geodesic metric. Two final topics that were mentioned in that same section for future research were maybe trying to upper bound $d_{GH}(I^{n+1}, S^n)$, or possibly computing effective lower bounds for the Gromov-Hausdorff distance in a more general case; that is, $d_{GH}(I^n, S^m)$ when I^n is endowed with the Hamming metric.

Thus, as you can see, the preparation of this dissertation has been an interesting and satisfying journey through various topics involving persistent homology, spectral sequences, Künneth formulas, Gromov-Hausdorff distances, spheres and hypercubes, and geodesic and Hamming metrics. I hope in the future to be able to be able to continue advancing the development of many of these and other topics.

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