

Fair Division

Henry Adams

UF Reading Group, Fall 2023

Sources

"Rental Harmony: Sperner's Lemma in Fair Division"
Francis Su, MAA Monthly, 1999

"To Divide the Rent, Start With a Triangle"
Albert Sun, New York Times, 2014

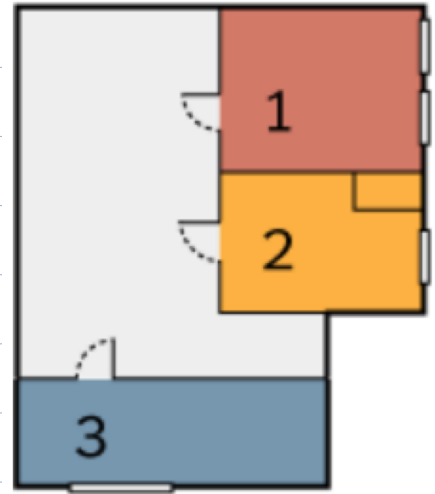
Problem

3 roommates: A, B, C

3 rooms: 1, 2, 3

Rent: \$3,000

How do you divide the rent
in an envy-free fashion?



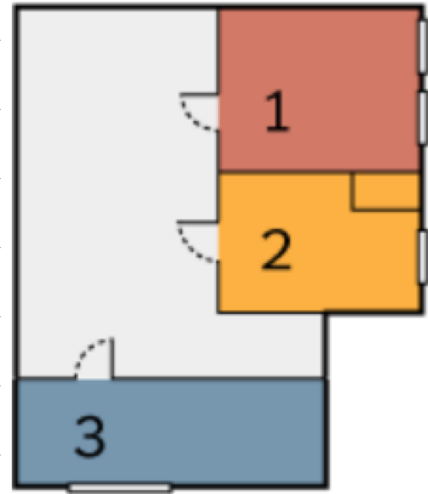
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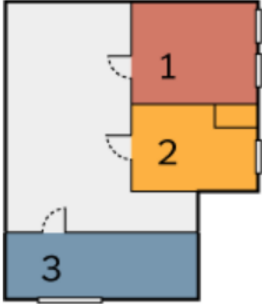


Remark Can be done with any number
of rooms and roommates.

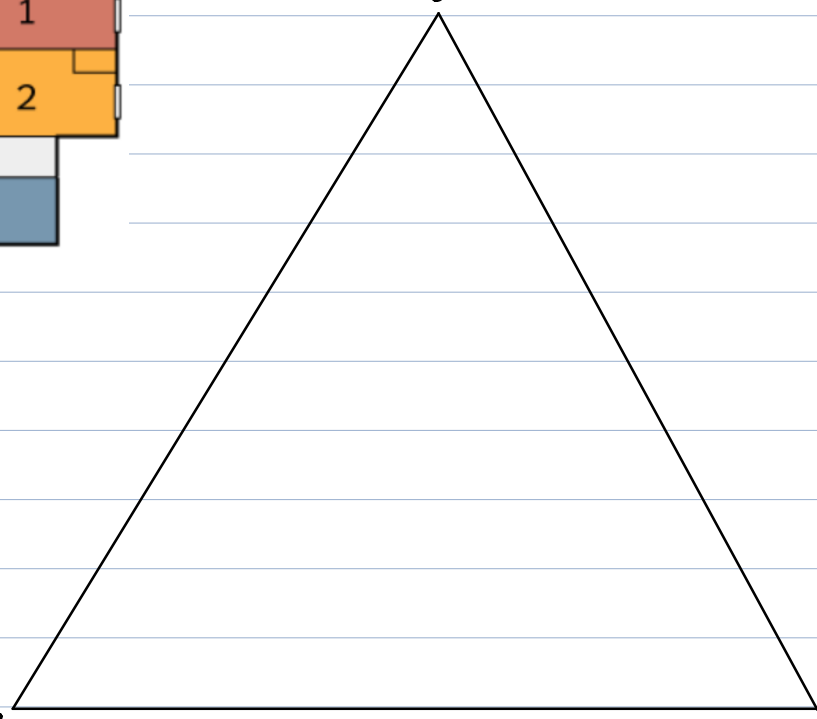
Assumption that will be relaxed

Assume that each roommate prefers a free room
over a non-free room.

Algorithm • Draw triangle of rent divisions.



1: 0
2: 0
3: 3000

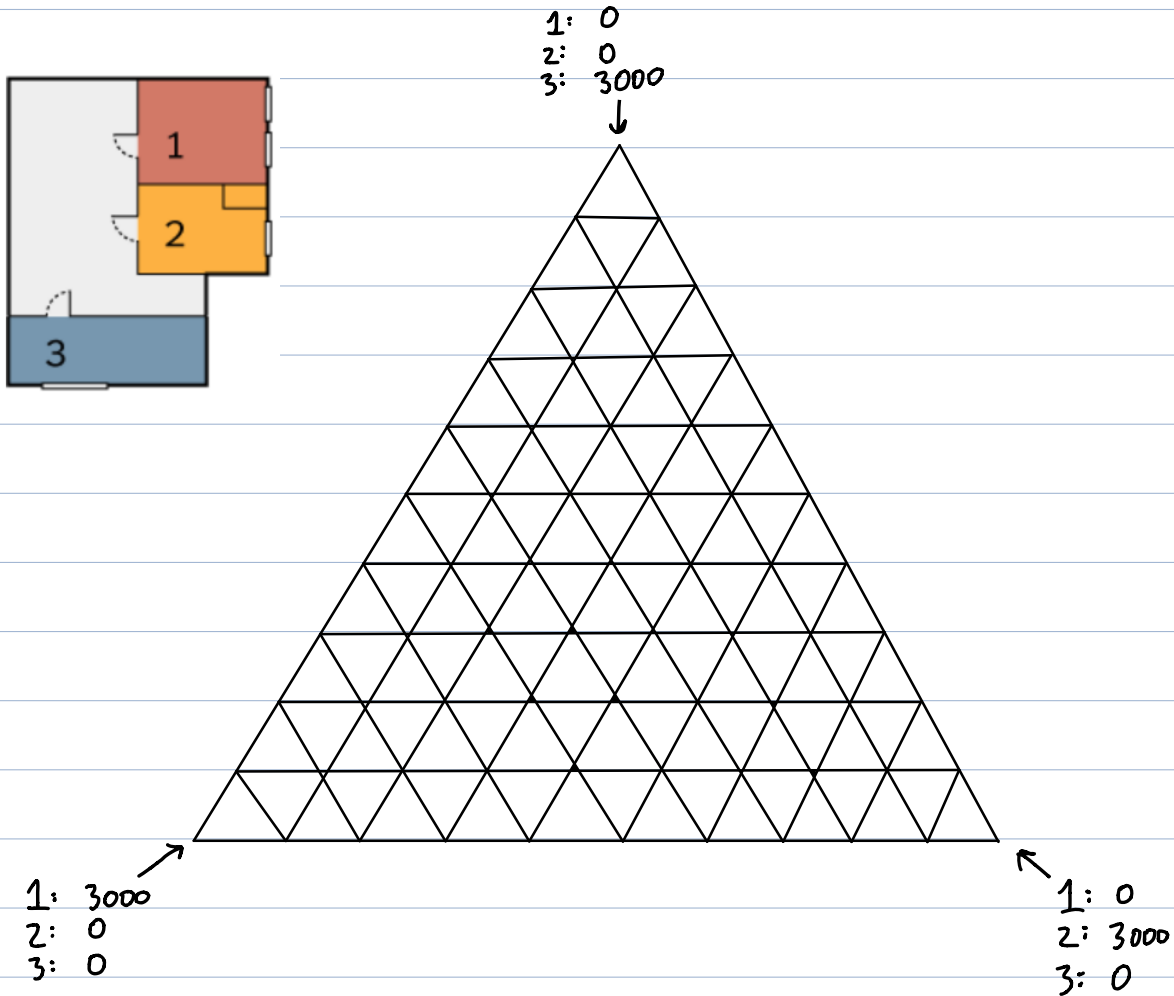


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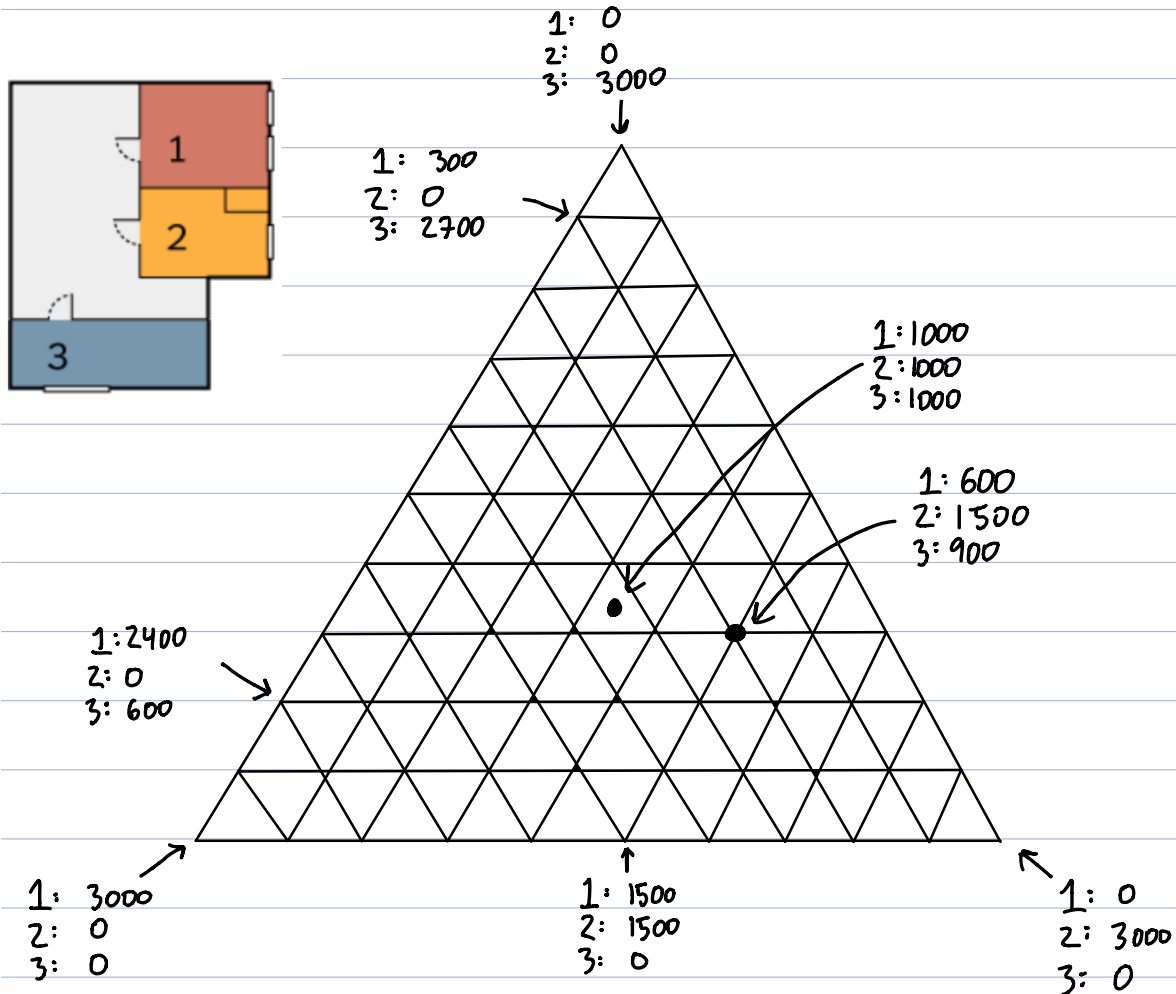
Algorithm

- Draw triangle of rent divisions.
- Subdivide to acceptable level.



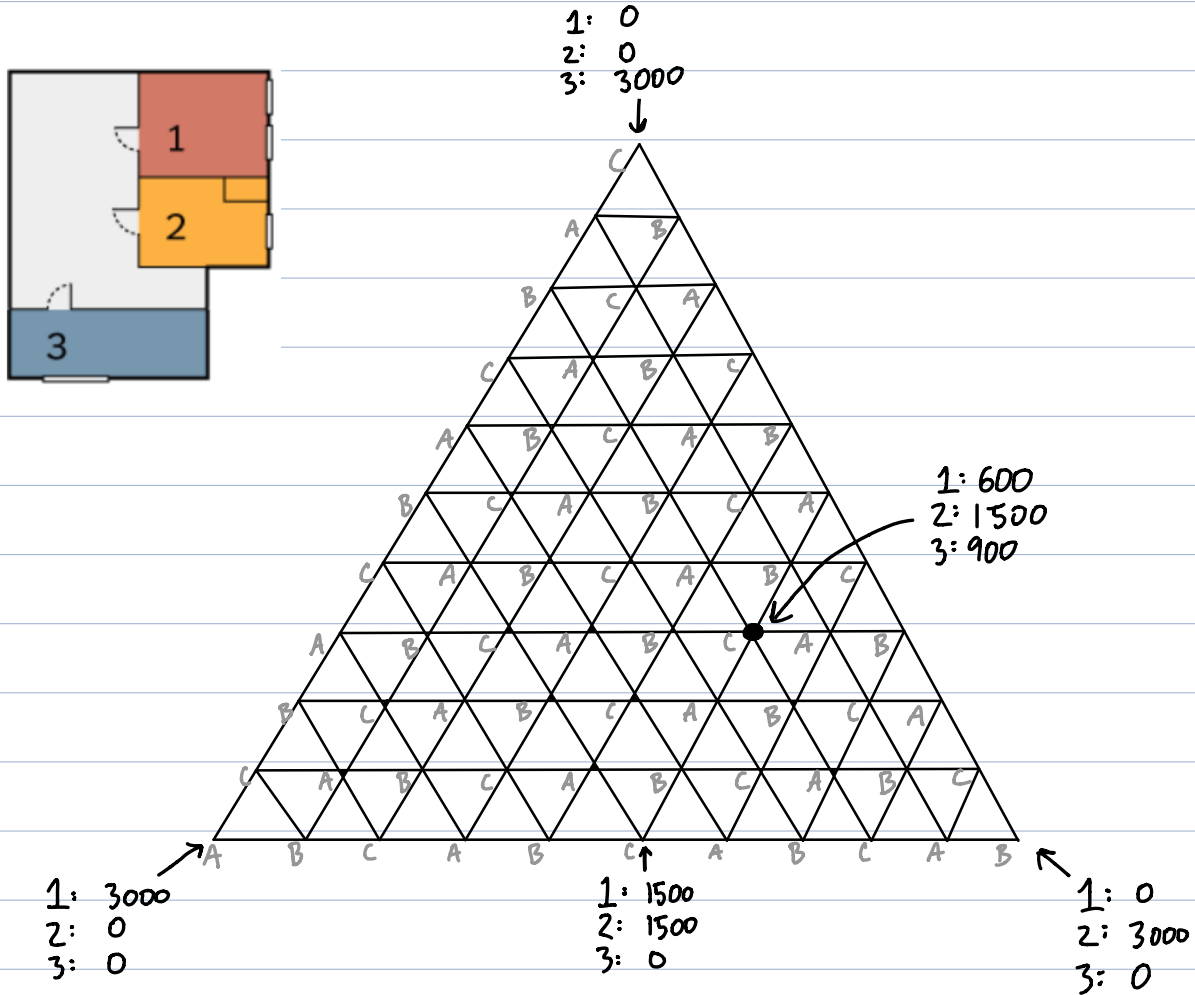
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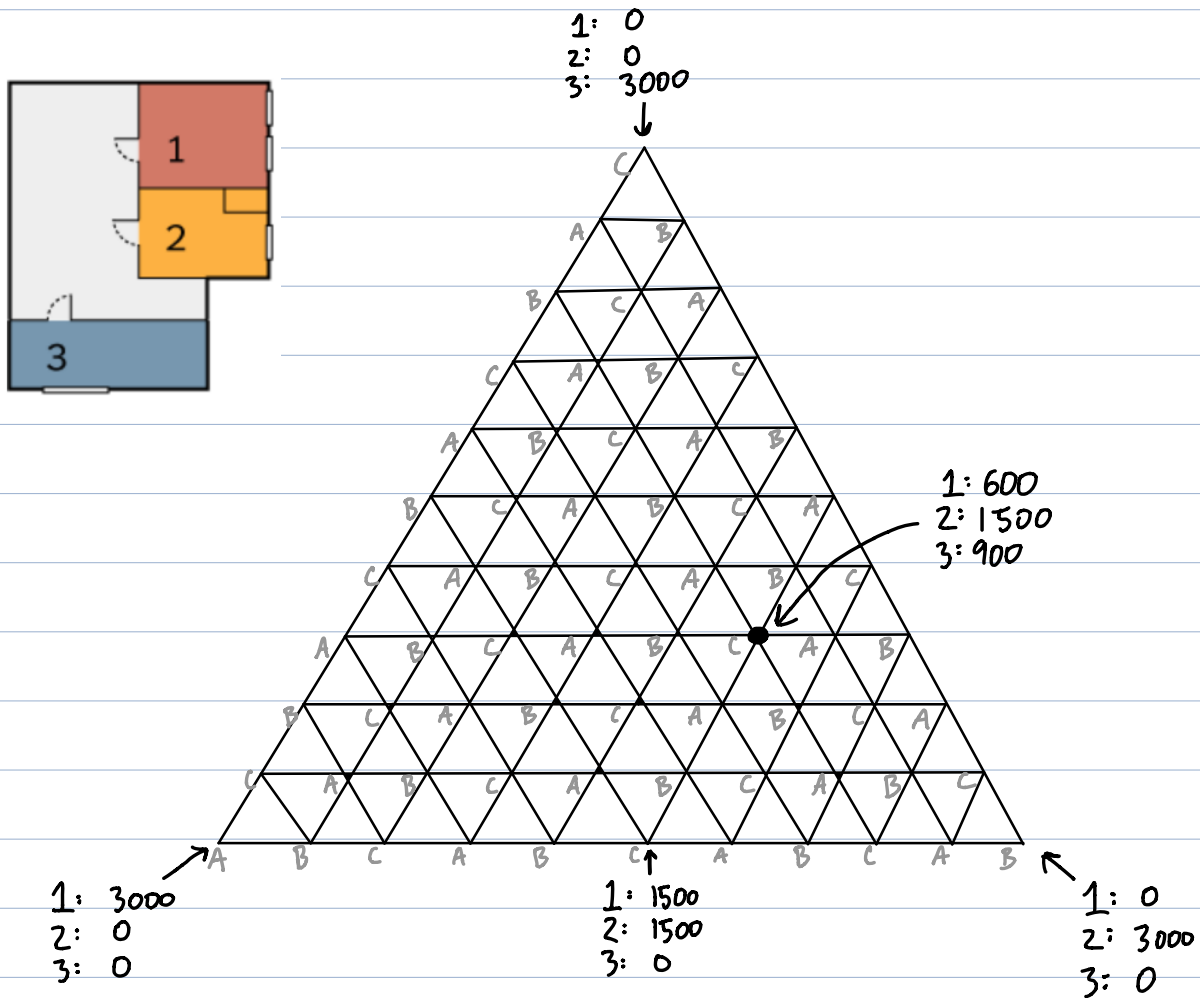
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- Draw triangle of rent divisions.
- Subdivide to acceptable level.
- Alternate vertex labels A, B, C.



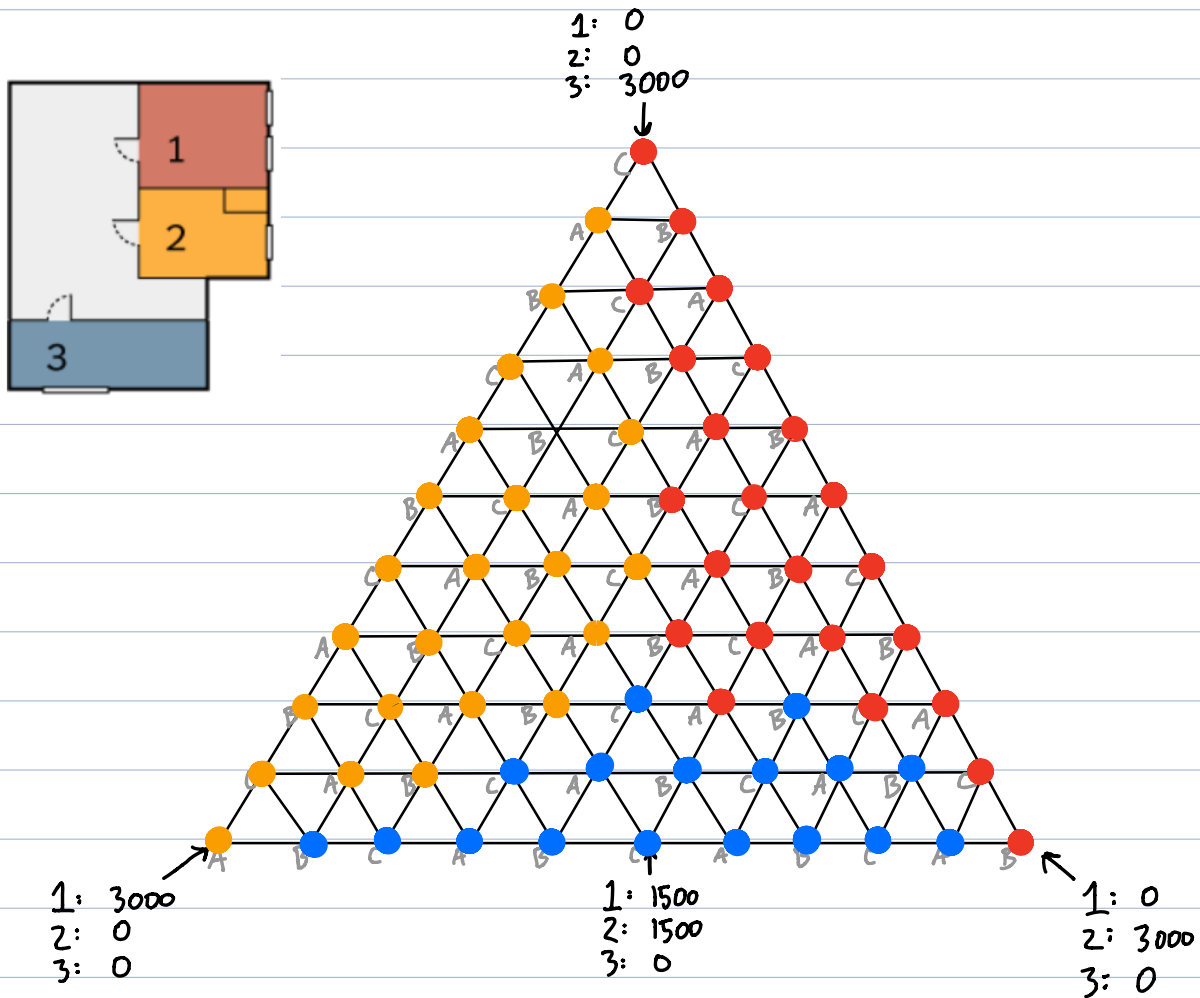
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- Poll roommates at their vertices.



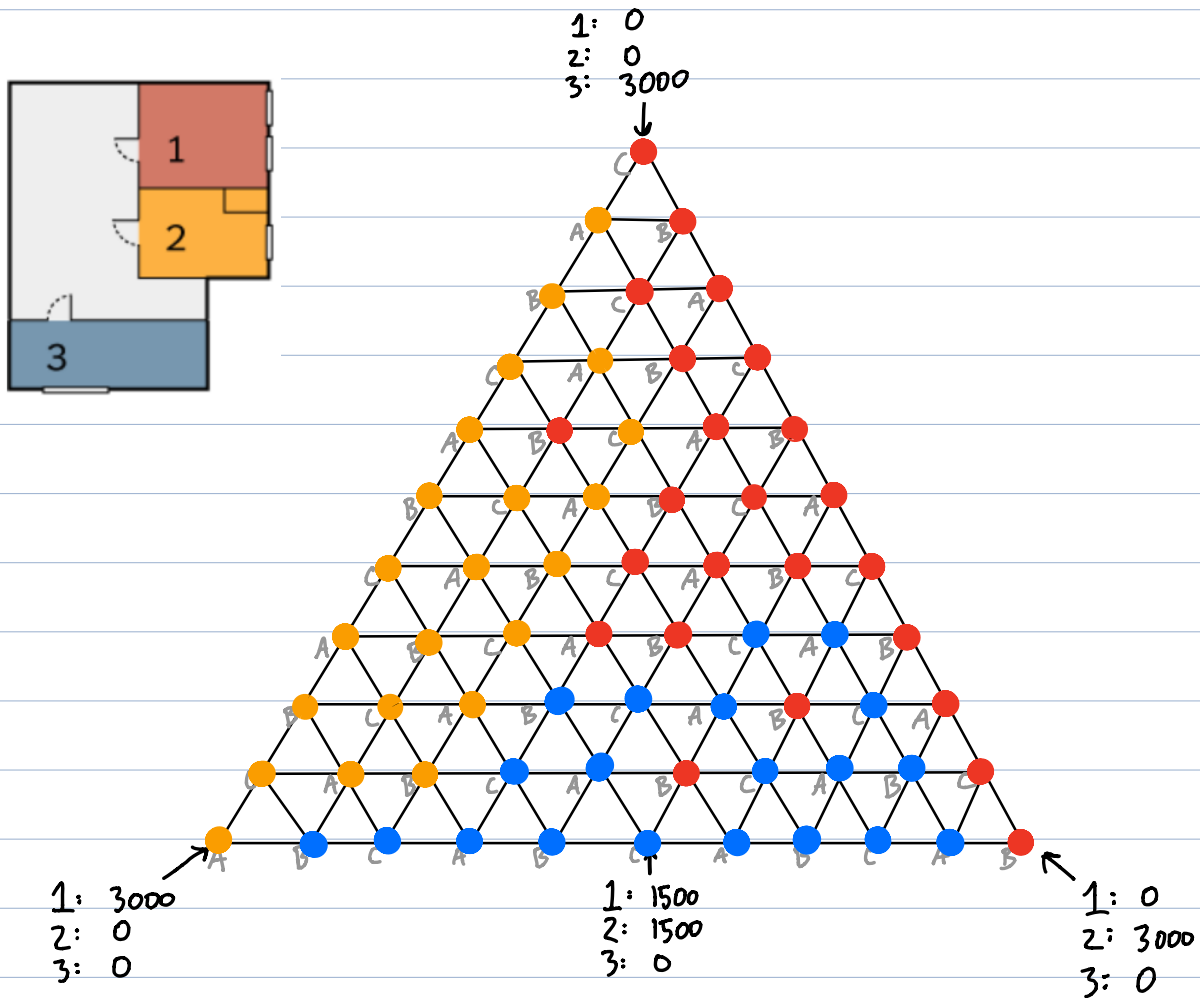
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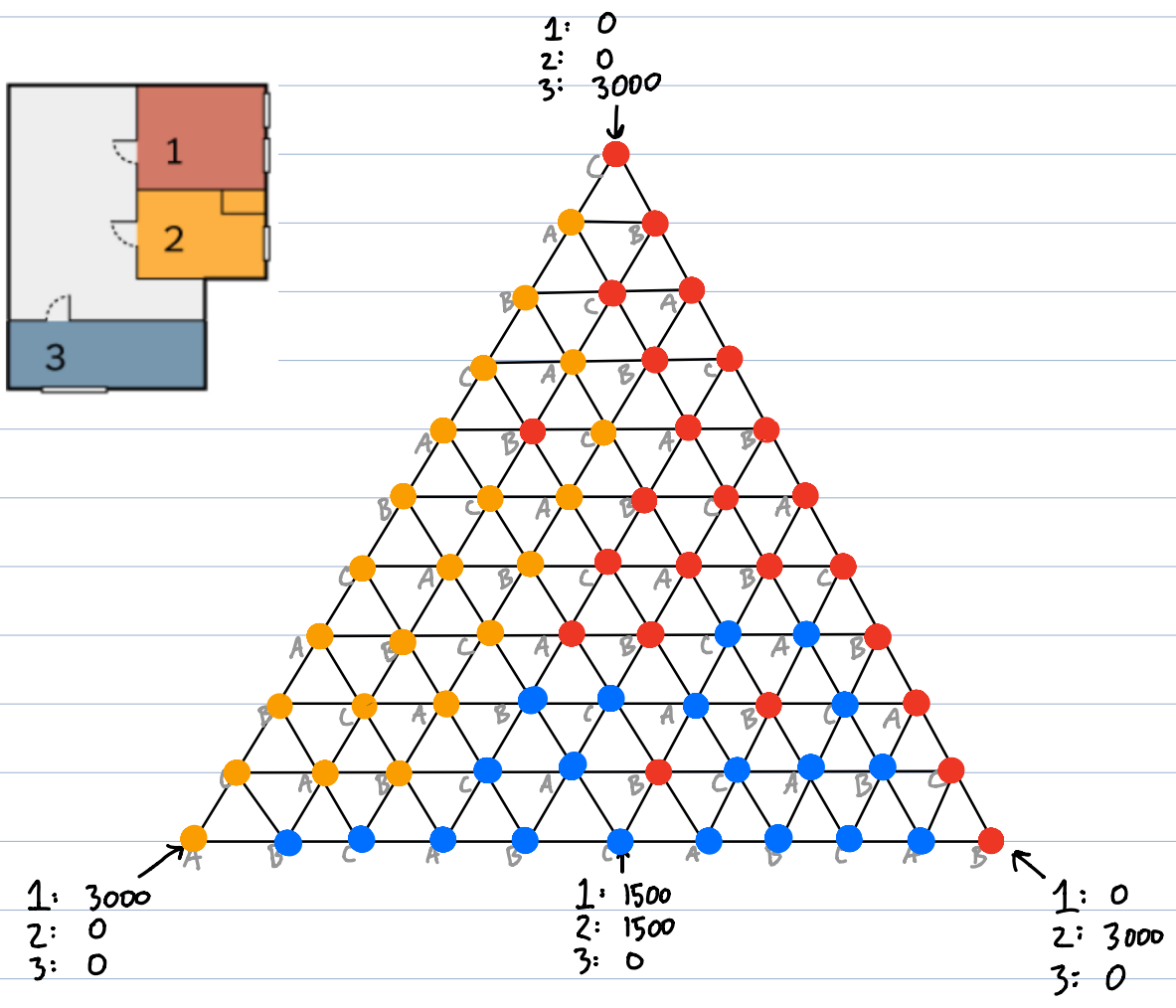
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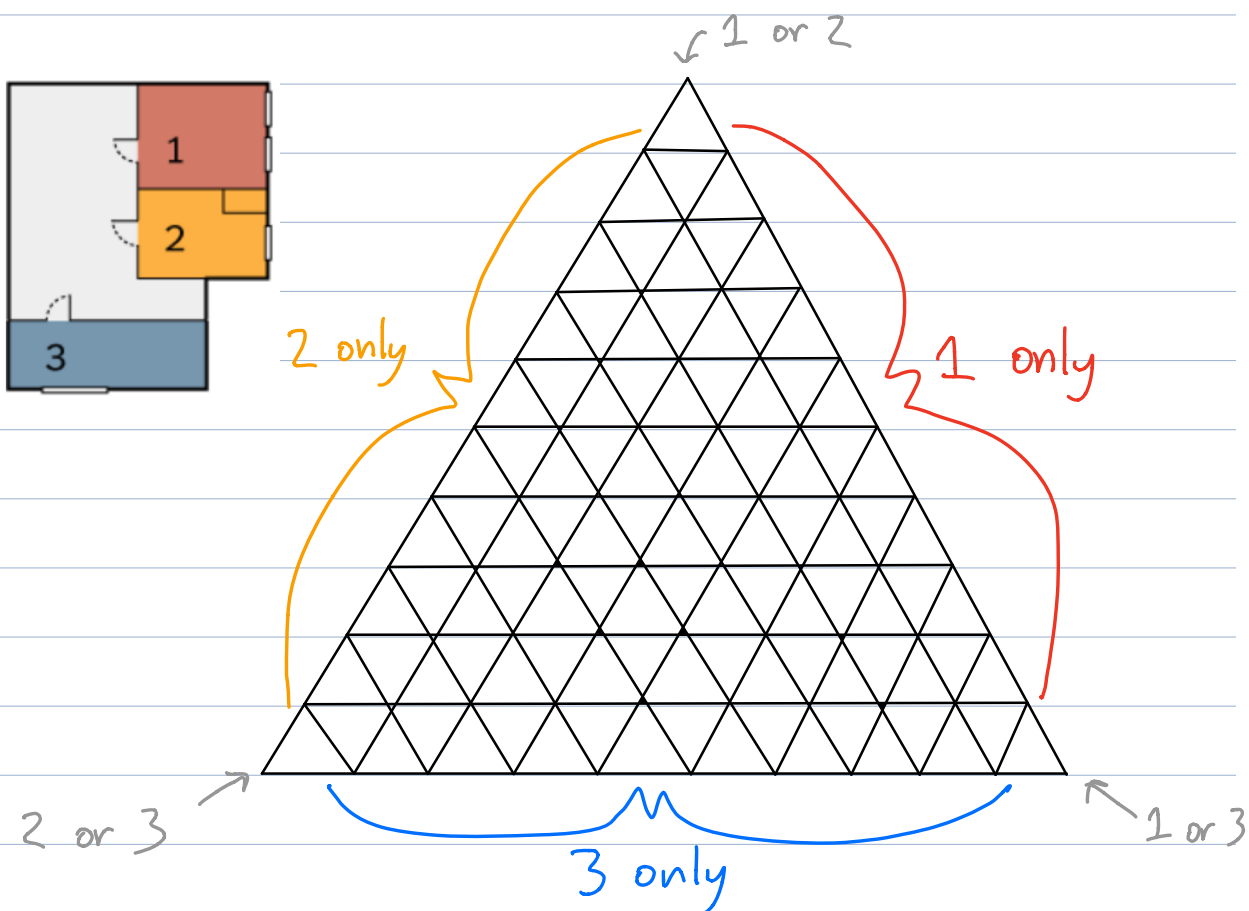
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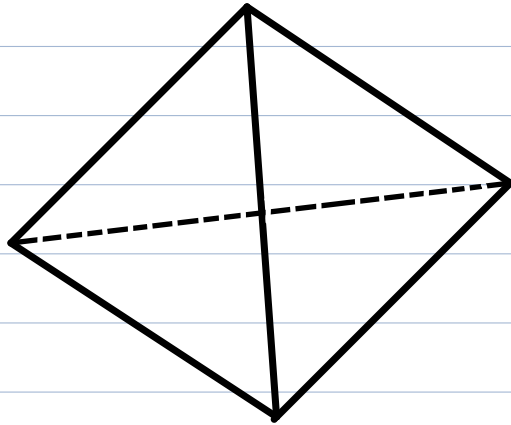
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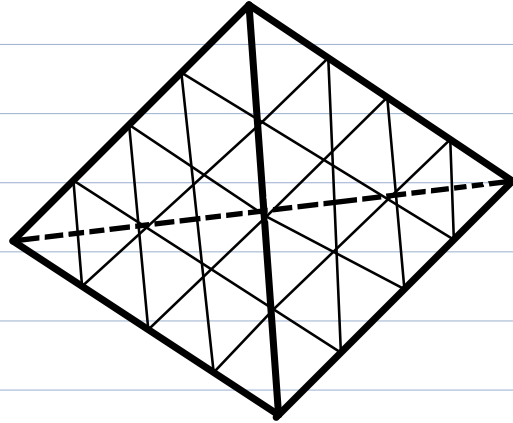
Four Roommates

Tetrahedron instead
of triangle



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Tetrahedron instead
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New York Times Applet

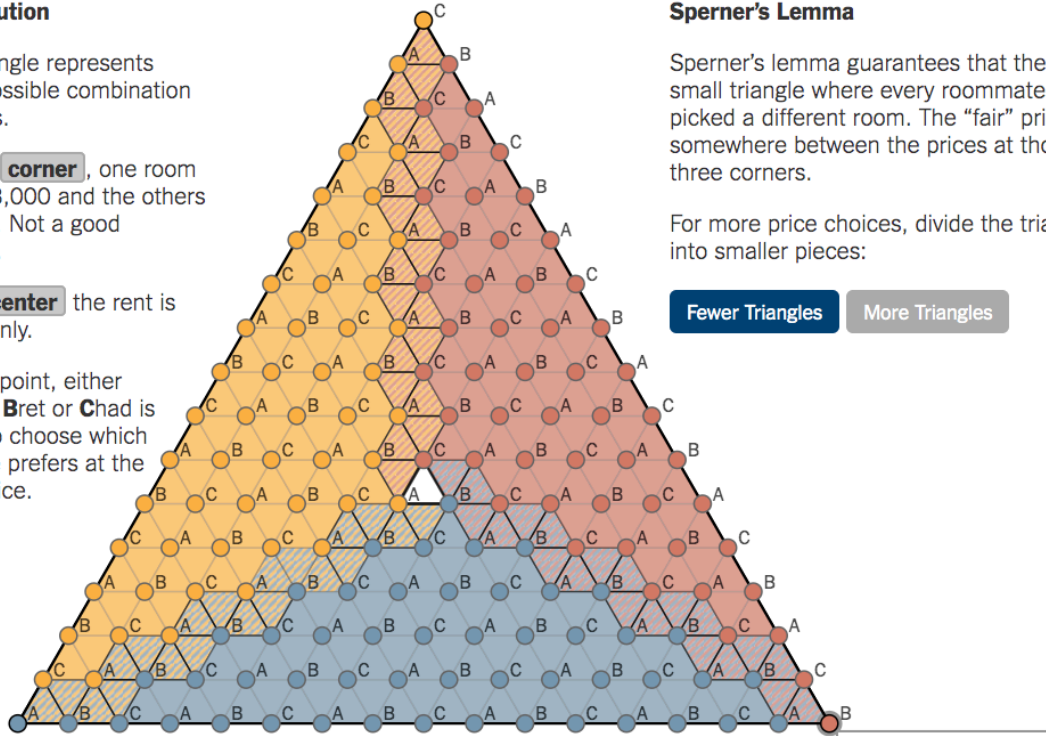
The Solution

This triangle represents every possible combination of prices.

At each **corner**, one room costs \$3,000 and the others are free. Not a good solution.

In the **center** the rent is split evenly.

At each point, either **Ashwin**, **Bret** or **Chad** is asked to choose which room he prefers at the given price.



Sperner's Lemma

Sperner's lemma guarantees that there is a small triangle where every roommate has picked a different room. The "fair" price lies somewhere between the prices at those three corners.

For more price choices, divide the triangle into smaller pieces:

Fewer Triangles

More Triangles

Polling efficiencies, multiple envy-free solutions

New York Times Applet

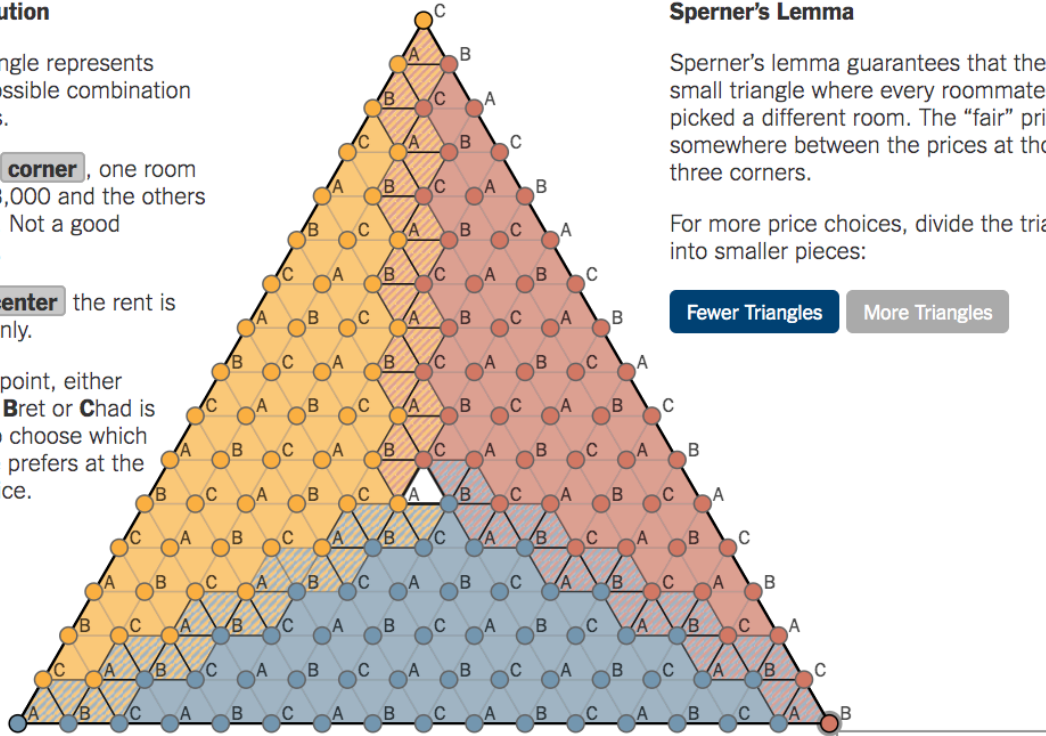
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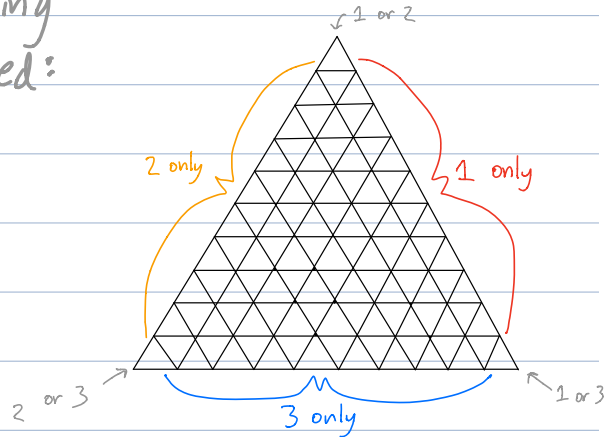
Polling efficiencies, multiple envy-free solutions

Spliddit Split fare, divide goods, assign (academic) credit, distribute chores, ...

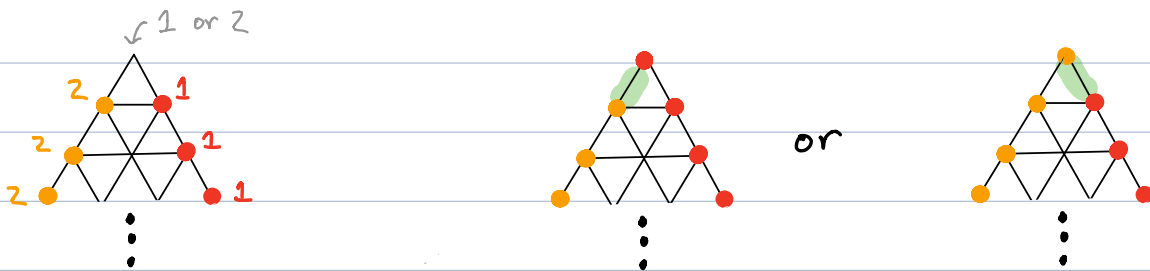
A variant of Sperner's Lemma

Consider a subdivision of a triangle into smaller triangles, such that each vertex is labeled 1, 2, or 3, and such that the following boundary conditions are satisfied:

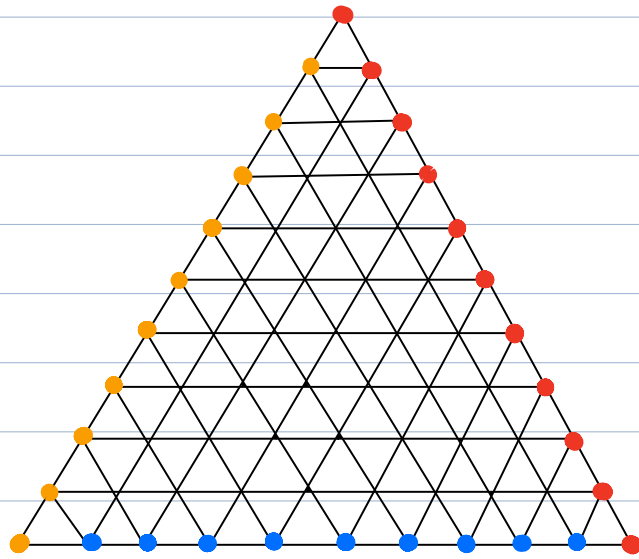
Then there exists a small rainbow triangle with vertex labels 1, 2, 3.



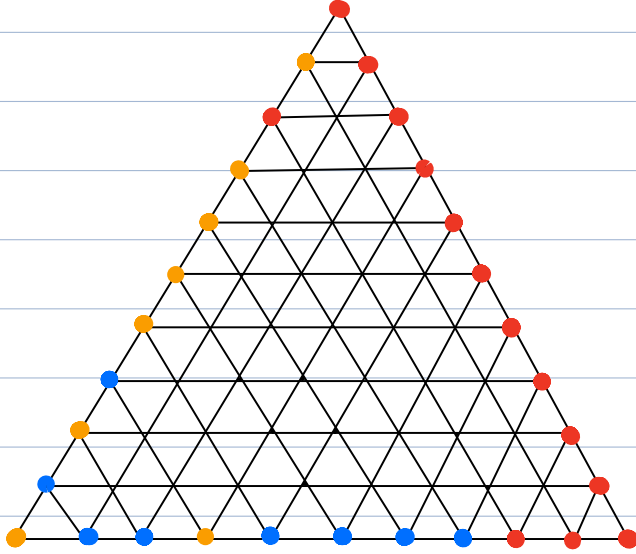
Proof Note there is a unique edge on the boundary with label 1, 2.



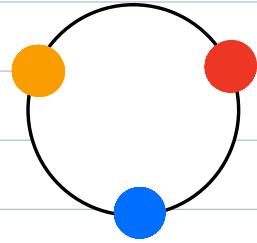
The adjacent triangle has either a second 1, 2 edge, or else a vertex with label 3.



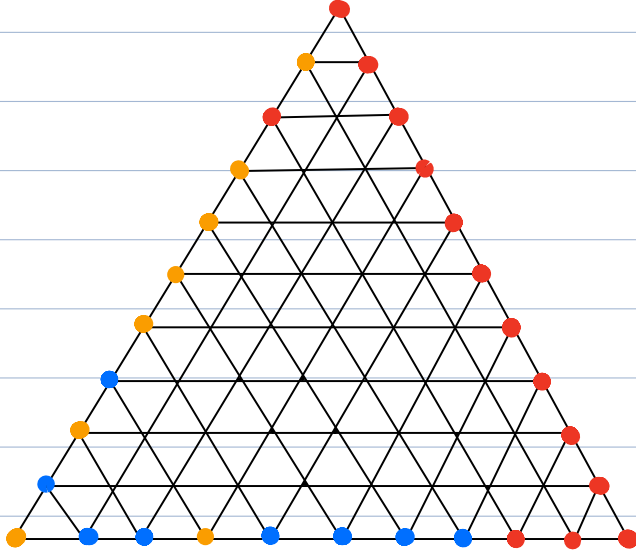
Relaxing the boundary assumptions



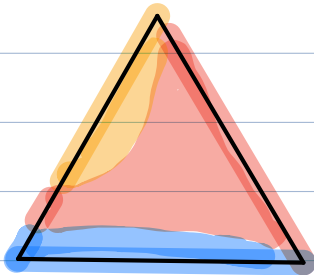
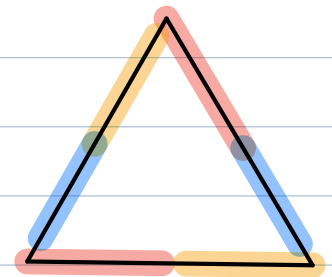
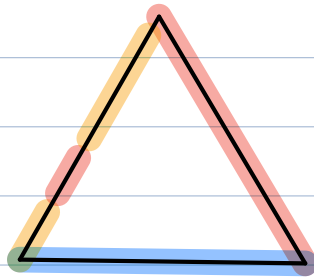
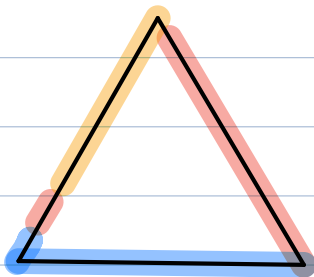
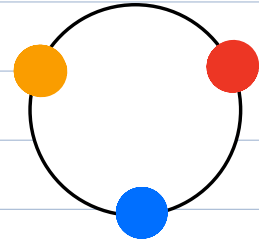
Only need that the "winding number" around the boundary is nonzero.



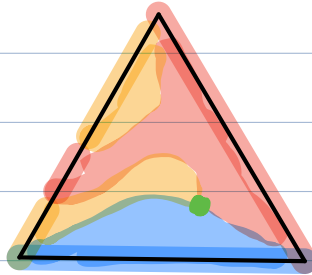
Relaxing the boundary assumptions



Only need that the "winding number" around the boundary is nonzero.



No rainbow triangle



There is a rainbow triangle and an odd # of 1, 2 boundary edges.

Thanks!