UF MTG 4302/5316 **Final**

Name: ________________________________

- Explain your work (efficiently); partial credit is available.
- No notes, books, calculators, or other electronic devices are permitted.
- Please sign below to indicate you accept the following statement:
  "I will not give, receive, or use any unauthorized assistance."

Signature: ____

Problem	Total Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

1 (a) Define a surjective function $f\colon \mathbb{Z}_+ \to \mathbb{Z}$, where $\mathbb{Z}_+ = \{1, 2, 3, \ldots\}$ and $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. (*Remark: This shows that $\mathbb{Z}$ is countable.*)

(b) Prove that $\mathcal{P}(\mathbb{Z}_+)$, the power set of the positive integers, is uncountable.

2 (a) Define what it means for τ to be a *topology* on a set X.

(b) If $\{\tau_\alpha\}_{\alpha \in J}$ is a family of topologies on X, show that $\cap\tau_\alpha$ is a topology on X. (*Remark: The definition of* $\cap\tau_\alpha$ *is* $\cap\tau_\alpha := \{U \mid U \in \tau_\alpha \ \forall \alpha \in J\}$.)

3 (a) For X and Y topological spaces, define what it means for a function $f\colon X \to Y$ to be *continuous*.

(b) Show that if (X, d) is a metric space, then $d\colon X \times X \to \mathbb{R}$ is continuous.

4 Consider the function $f\colon [0,2\pi) \to \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ defined by $f(t) = (\cos(t), \sin(t))$ from the half-open interval to the circle. Show that f is not a homeomorphism.

5 (a) Define what it means for a topological space X to be *path connected.*

(b) Prove that if X is path connected, then X is connected.

6 Recall that a point $x \in \mathbb{R}^\omega$ can be written as $x = (x_1, x_2, \ldots)$ with each $x_i \in \mathbb{R}$. Let $\{x^{(n)}\}_{n \in \mathbb{Z}_+}$ be a sequence of points in $\mathbb{R}^\omega$. Suppose there is a point $x \in \mathbb{R}^\omega$ such that for each coordinate $i \in \mathbb{Z}_+$, the sequence $\{x_i^{(n)}\}_{n \in \mathbb{Z}_+}$ converges to x_i in $\mathbb{R}$.

(a) Show that $\{x^{(n)}\}_{n \in \mathbb{Z}_+}$ converges to x if $\mathbb{R}^\omega$ has the product topology.

(b) Show that $\{x^{(n)}\}_{n \in \mathbb{Z}_+}$ need not converge to x if $\mathbb{R}^\omega$ has the box topology.

7 (a) Define what it means for a topological space X to be *compact*.

(b) Prove that if X is compact and $f\colon X \to Y$ is continuous, then $f(X)$ is compact.

8 (a) Define what it means for a topological space X to be *regular*.

(b) Prove that if X is a compact Hausdorff space, then X is a regular space.

9 Equip Euclidean space $\mathbb{R}^k$ with the metric $\rho(x,y) = \max\{|x_i - y_i| \ : \ i = 1,\ldots,k\}$. Prove that $(\mathbb{R}^k, \rho)$ is a complete metric space.

10 (a) Let X be a topological space. A subset $A \subset X$ has *empty interior* if A contains no open set of X other than the empty set. Prove that if A has empty interior, then $X - A$ is dense in X.

(b) Define what it means for a topological space X to be a *Baire space*.

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