UF MTG 6347
Topology 2
Spring 2025

# Homework 3

Due Friday, Marcy 14, anytime, on Canvas

**Reading.** Hatcher pages 206–230.

**Problems.**

1. §3.2, Exercise 3. (a) Using the cup product structure, show there is no map $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^m$ inducing a nontrivial map $H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) \to H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$ if $n > m$. What is the corresponding result for maps $\mathbb{C}P^n \to \mathbb{C}P^m$?

   (b) Prove the Borsuk–Ulam theorem by the following argument. Suppose on the contrary that $f\colon S^n \to \mathbb{R}^n$ satisfies $f(x) \neq f(-x)$ for all $x$. The define $g\colon S^n \to S^{n-1}$ by $g(x) = (f(x) - f(-x))/|f(x) - f(-x)|$, so $g(-x) = -g(x)$ and $g$ induces a map $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^{n-1}$. Show that part (a) applies to this map.

2. §3.2, Exercise 4. Apply the Lefschetz fixed point theorem to show that every map $f\colon \mathbb{C}P^n \to \mathbb{C}P^n$ has a fixed point if $n$ is even, using the fact that $f^*\colon H^*(\mathbb{C}P^n; \mathbb{Z}) \to H^*(\mathbb{C}P^n; \mathbb{Z})$ is a ring homomorphism. When $n$ is odd show there is a fixed point unless $f^*(\alpha) = -\alpha$, for $\alpha$ a generator of $H^2(\mathbb{C}P^n; \mathbb{Z})$. [See Exercise 3 in S2.C for an example of a map without fixed points in this exceptional case.]

3. Let $X$ be two unlinked circles in $\mathbb{R}^3$ and let $Y$ be two simply linked circles. Show that the complements $\mathbb{R}^3 \setminus X$ and $\mathbb{R}^3 \setminus Y$ have isomorphic cohomology groups but different ring structures, and hence are not homotopy equivalent. [See Example 1.23 in Hatcher for how this problem was solved using the fundamental group.]