UF MTG 4303/5317
Introduction to Topology 2
Spring 2024

Homework 4

Due Monday, March 25, anytime, on Canvas

Reading. Munkres §60, 67–68.

Problems.

1. §58 #2(b,e). For each of the following spaces, the fundamental group is either trivial, infinite cyclic, or isomorphic to the fundamental group of the figure eight. Determine for each space which of the three alternatives holds.

   (b) The torus T with a point removed.

   (e) $\mathbb{R}^3$ with the nonnegative x, y, and z axes deleted.

2. §58 #10(b,d). Suppose that to every map $h\colon S^n \to S^n$ we have assigned an integer, denoted by $\deg h$ and called the *degree* of h, such that:

   (i) Homotopic maps have the same degree.

   (ii) $\deg(h \circ k) = (\deg h) \cdot (\deg k)$.

   (iii) The identity map has degree 1, any constant map has degree 0, and the reflection map $\rho(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n, -x_{n+1})$ has degree -1.

   [One can construct such a function, using the tools of algebraic topology. Intuitively, $\deg h$ measures how many times h wraps S^n around itself; the sign tells you whether h preserves orientation or not.] Prove the following:

   (b) If $h\colon S^n \to S^n$ has degree different from $(-1)^{n+1}$, then h has a fixed point.
   [*Hint:* Show that if h has no fixed point, then h is homotopic to the antipodal map $\alpha(x) = -x$.]

   (d) If S^n has a nonvanishing tangent vector field v, then n is odd.
   [*Hint:* If v exists, show the identity map is homotopic to the antipodal map.]

3. §59 #3(a,b).

   (a) Show that $\mathbb{R}^1$ and $\mathbb{R}^n$ are not homeomorphic if $n > 1$.

   (b) Show that $\mathbb{R}^2$ and $\mathbb{R}^n$ are not homeomorphic if $n > 2$.

   It is, in fact, true that $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic if $n \neq m$, but the proof requires more advanced tools of algebraic topology.

4. §60 #5. Consider the covering map indicated in Figure 60.3 (from Munkres). Here, p wraps A_1 around A twice and wraps B_1 around B twice; p maps A_0 and B_0 homeomorphically onto A and B, repectively. Use this covering space to show that the fundamental group of the figure eight is not abelian.

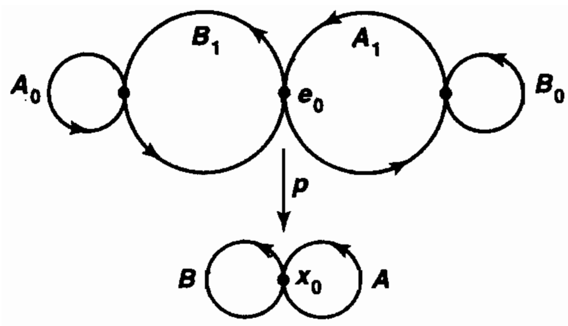

Figure 60.3

Recommend Problems (not to turn in).

- §58 #4
- §58 #9
- §59 #4