UF MTG 6347
Topology 2
Spring 2025

Homework 4

Due Friday, April 4, anytime, on Canvas

Reading. Hatcher pages 230–260.

Problems.

1. §3.3, Exercise 7. For a map $f\colon M \to N$ between connected closed orientable n-manifolds with fundamental classes $[M]$ and $[N]$, the degree of f is defined to be the integer d such that $f_*([M]) = d[N]$, so the sign of the degree depends on the choice of fundamental classes. Show that for any connected closed orientable n-manifold M there is a degree 1 map $M \to S^n$.

2. §3.3, Exercise 6. Given two disjoint connected n-manifolds M_1 and M_2, a connected n-manifold $M_1 \# M_2$, their *connected sum*, can be constructed by deleting the interiors of closed n-balls $B_1 \subset M_1$ and $B_2 \subset M_2$ and identifying the resulting boundary spheres ∂B_1 and ∂B_2 via some homeomorphism between them. (Assume that each B_i embeds nicely in a larger ball in M_i.)

   (a) Show that if M_1 and M_2 are closed then there are isomorphisms $H_i(M_1 \# M_2; \mathbb{Z}) \cong H_i(M_1; \mathbb{Z}) \oplus H_i(M_2; \mathbb{Z})$ for $0 < i < n$, with one exception: If both M_1 and M_2 are nonorientable, then $H_{n-1}(M_1 \# M_2; \mathbb{Z})$ is obtained from $H_{n-1}(M_1; \mathbb{Z}) \oplus H_{n-1}(M_2; \mathbb{Z})$ by replacing one of the two $\mathbb{Z}_2$ summands by a $\mathbb{Z}$ summand. [Euler characteristics may help in the exceptional case.]

   (b) Show that $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - \chi(S^n)$ if M_1 and M_2 are closed.

3. §3.3, Exercise 26. Compute the cup product structure in $H^*(S^2 \times S^8 \# S^4 \times S^6; \mathbb{Z})$, and in particular show that the only nontrivial cup products are those dictated by Poincaré duality. [See Exercise 6. The result has an evident generalization to connected sums of $S^i \times S^{n-i}$'s for fixed n and varying i.]