Hausdorff vs Gromov-Hausdorff distances

Joint with Florian Frick, Sush Majhi, Nicholas McBride, arXiv 2309.16648

Applied Algebraic Topology
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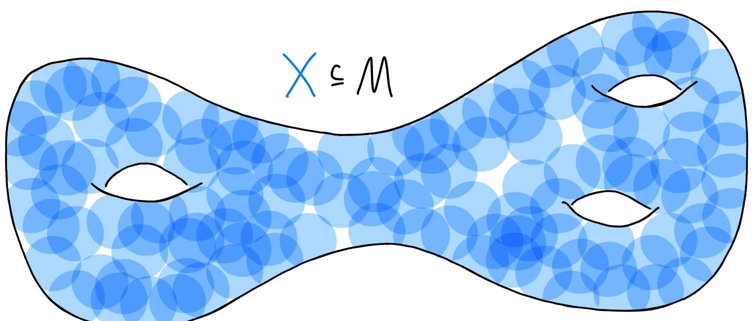
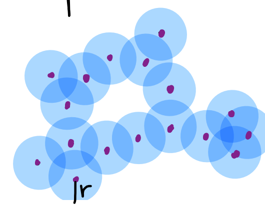
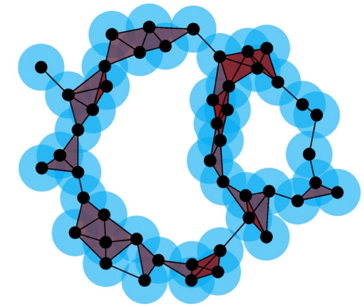


Čech Complexes



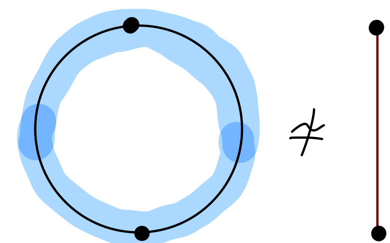


M a metric space, $X \subseteq M$.

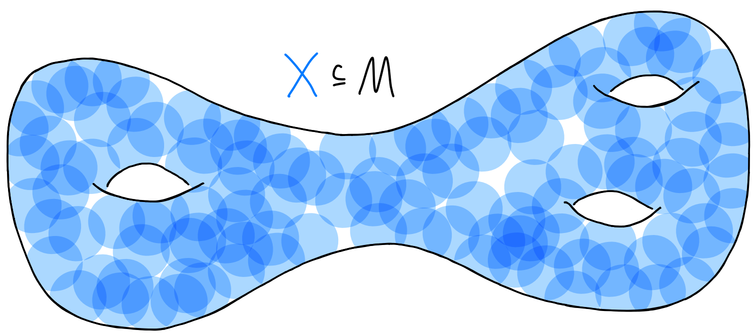
<u>Def</u> The <u>Čech complex</u> $\check{C}(X;r)$ has

- vertex set X

  simplex $[x_0, \ldots, x_k]$ when $\bigcap_{i=0}^{k} B_M(x_i;r) \neq \emptyset$.

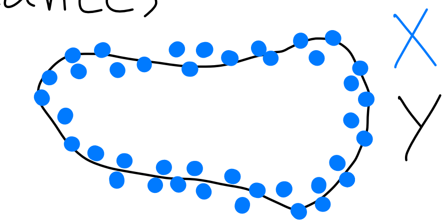


For M a manifold and
$r < \mathrm{ConvRad}(M)$,
$\check{C}(X;r) \simeq$ union of balls.



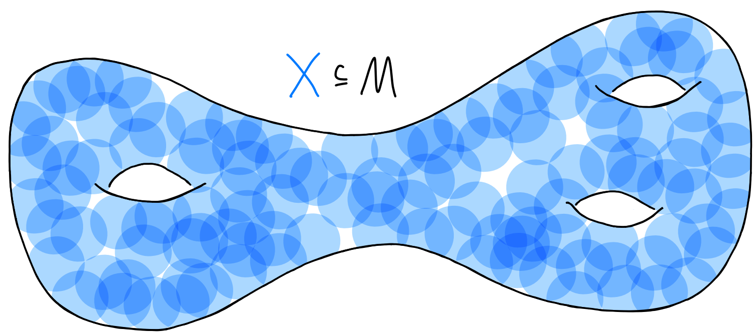
Gromov-Hausdorff distances

X, Y compact metric spaces.

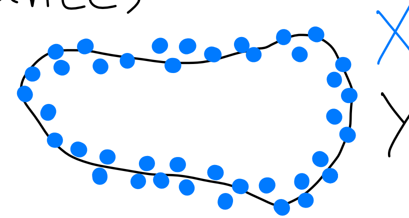


For $X, Y \subseteq M$, their <u>Hausdorff distance</u> is

$$d_H(X,Y) = \inf\{ r>0 \mid X \subseteq Y^r \text{ and } Y \subseteq X^r \}.$$



Gromov-Hausdorff distances



X, Y compact metric spaces.

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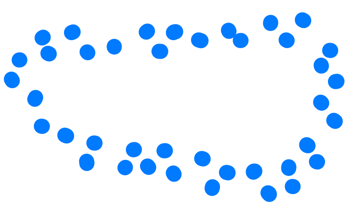
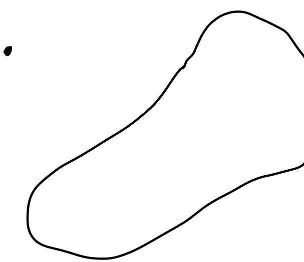
$$d_H(X,Y) = \inf\{r>0 \mid X \subseteq Y^r \text{ and } Y \subseteq X^r\}.$$

For X, Y abstract, their <u>Gromov-Hausdorff distance</u> is

$$d_{GH}(X,Y) = \inf_{\substack{\text{isometric embeddings} \\ X \hookrightarrow Z,\ Y \hookrightarrow Z}} \{d_H^Z(X,Y)\}.$$



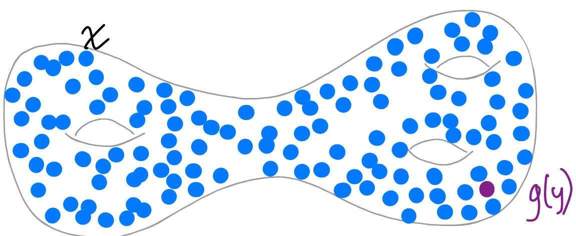


Gromov-Hausdorff distances

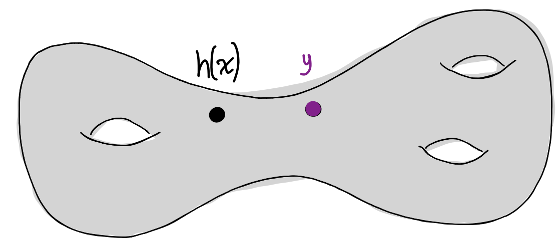
Equivalently:

<u>Def</u>

$$2 \cdot d_{GH}(X,Y) = \inf_{\substack{h: X \to Y \\ g: Y \to X}} \max \{ \mathrm{dis}(h), \mathrm{dis}(g), \mathrm{codis}(h,g) \}.$$



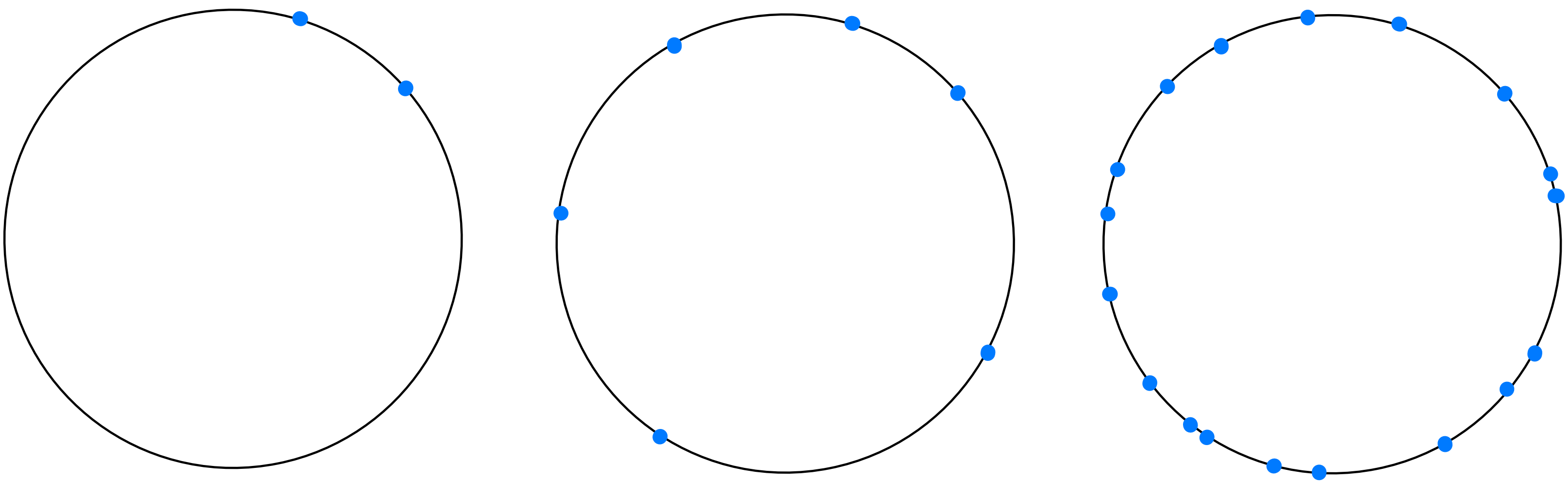
$$X \underset{g}{\overset{h}{\rightleftarrows}} Y$$



$$\mathrm{dis}(h) = \sup_{x,x' \in X} \left| d_X(x,x') - d_Y(h(x), h(x') \right|$$

$$\mathrm{codis}(h,g) = \sup_{x \in X,\, y \in Y} \left| d_X(x, g(y)) - d_Y(h(x), y) \right|$$

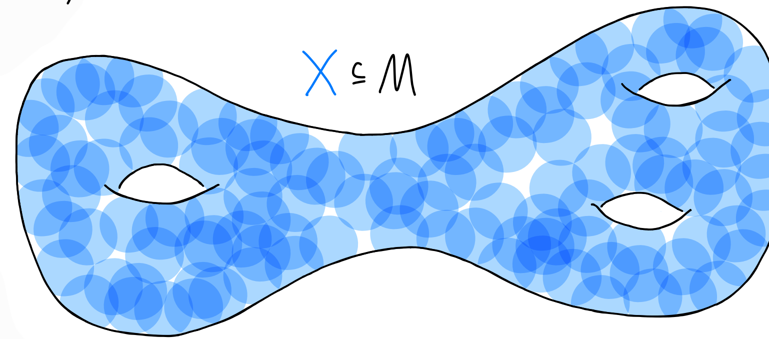
Example of Main Theorem



$d_{GH}(X,M) < d_H(X,M)$

$d_{GH}(X,M) = d_H(X,M)$

Theorem For M a compact manifold and $X \subseteq M$,
$$d_{GH}(X,M) \geq \min\left\{\frac{1}{2} d_H(X,M),\ \frac{1}{6}\,\mathrm{ConvRad}(M)\right\}.$$



Proof We'll show if $d_{GH}(X,M) < \frac{1}{6}\mathrm{ConvRad}(M)$, then $d_{GH}(X,M) \geq \frac{1}{2} d_H(X,M)$.

Fix r with $2 \cdot d_{GH}(X,M) < r < \frac{1}{3}\mathrm{ConvRad}(M)$.
Fix $\varepsilon > 0$ with $3r + 2\varepsilon < \mathrm{ConvRad}(M)$.

$$\begin{tikzcd} \overset{M}{\underset{\mathrm{IS}}{}}\check{C}(M;\varepsilon) \arrow[r, "\bar{g}"] \arrow[rr, bend right, "\bar{h}\circ\bar{g} \simeq \text{inclusion}"'] & \check{C}(X; r+\varepsilon) \arrow[r, "\bar{h}"] & \overset{M}{\underset{\mathrm{IS}}{}}\check{C}(M; 3r+2\varepsilon) \end{tikzcd}$$

If we had $r+\varepsilon < d_H(X,M)$, then we'd have
$\check{C}(X; r+\varepsilon) \simeq$ proper subset of M.

Applying homology (fundamental class) would give

$$\begin{tikzcd} \mathbb{Z}/2 \arrow[r] \arrow[rr, bend right, "\text{isomorphism}"'] & 0 \arrow[r] & \mathbb{Z}/2 \end{tikzcd}$$

Hence $r+\varepsilon \geq d_H(X,M)$ for all such r and ε.

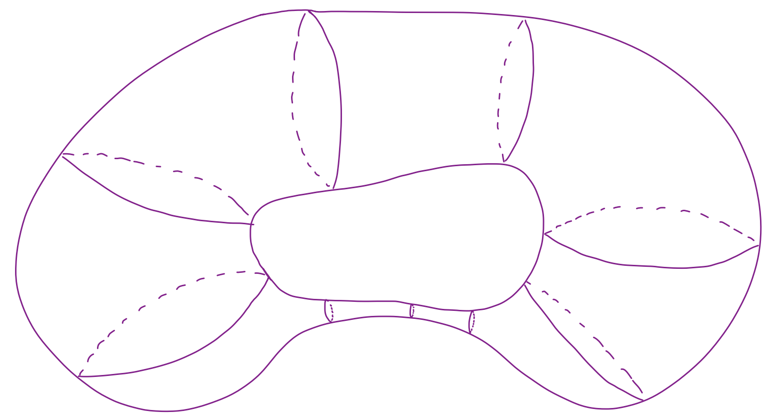
So $2 \cdot d_{GH}(X,M) \geq d_H(X,M)$.

Theorem For M a compact manifold and $X \subseteq M$,

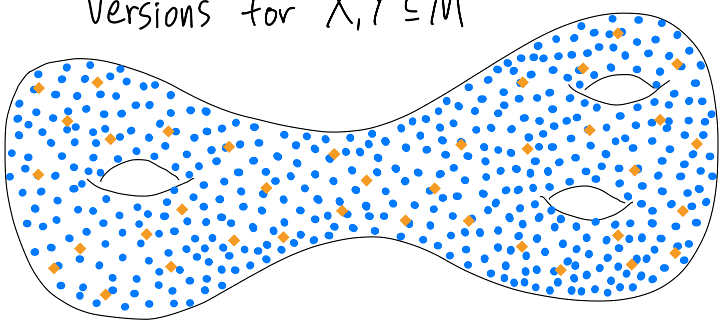
$$d_{GH}(X,M) \geq \min\left\{\frac{1}{2} d_H(X,M),\ \frac{1}{6}\,\mathrm{ConvRad}(M)\right\}.$$

Can be improved to $\sqrt{\frac{n+1}{2n}}$ if M^n has nonpositive sectional curvatures, hence optimal constant 1 when $M=S^1$.

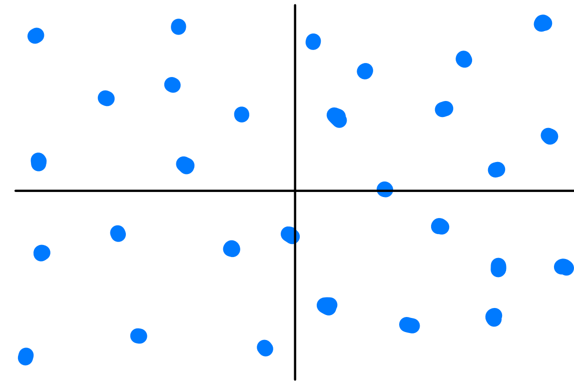
Can be improved to $\frac{1}{3}\,\mathrm{FillRad}(M)$



Versions for $X, Y \subseteq M$

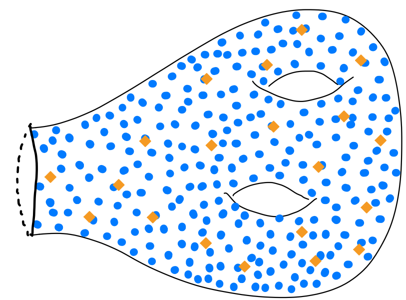


Theorem For M a compact manifold and $X \subseteq M$, $d_{GH}(X,M) \geq \min\left\{\frac{1}{2} d_H(X,M), \ \frac{1}{6}\,\mathrm{ConvRad}(M)\right\}$.



Questions

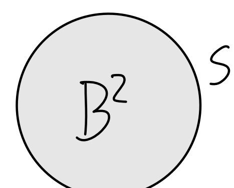
What if M is not bounded, for example $M = \mathbb{R}^n$?



What if M has boundary?

What if M is not a manifold?

Homotopy types of $\check{C}(S^n; r)$ for larger r?



$d_{GH}($ball B^{n+1}, sphere $S^n)$?