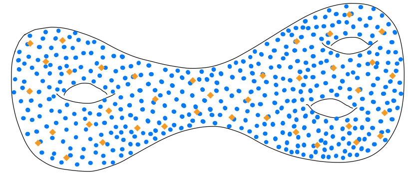
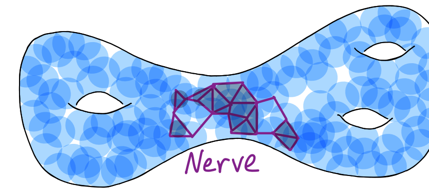




Hausdorff vs. Gromov-Hausdorff distances

Joint with Florian Frick, Sush Majhi, Nicholas McBride

Let M be a connected closed Riemannian n-manifold w/ convexity radius $\rho(M)$.

Thm 1 For $X \subseteq M$,

$$d_{GH}(X,M) \geq \min\left\{\tfrac{1}{2} d_H(X,M),\ \tfrac{1}{6}\rho(M)\right\}.$$

Thm 2 For $X, Y \subseteq M$,

$$d_{GH}(X,Y) \geq \min\left\{\tfrac{1}{2} d_H(X,M) - d_H(Y,M),\ \tfrac{1}{6}\rho(M) - \tfrac{2}{3} d_H(Y,M)\right\}.$$

Rmk Constants can be improved depending on dimension and sectional curvatures of M. For example:

For $X, Y \subseteq S^1$,

$$d_{GH}(X,S^1) \geq \min\left\{d_H(X,S^1),\ \tfrac{1}{3}\rho(S^1)\right\}$$

$$\tfrac{1}{3}\rho(S^1) = \tfrac{\pi}{6}$$

$$d_{GH}(X,Y) \geq \min\left\{d_H(X,S^1) - d_H(Y,S^1),\ \tfrac{1}{3}\rho(S^1) - \tfrac{1}{2} d_H(Y,S^1)\right\}$$

Def For Z a metric space and $X, Y \subseteq Z$, the Hausdorff distance is

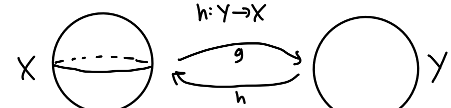
$$d_H^Z(X,Y) = \inf\{\varepsilon > 0 \mid X \subseteq B_Z(Y;\varepsilon),\ Y \subseteq B_Z(X;\varepsilon)\}.$$

Def For X, Y metric spaces, the Gromov-Hausdorff distance is

$$d_{GH}(X,Y) = \inf_{\substack{X \hookrightarrow Z \\ Y \hookrightarrow Z}} d_H^Z(X,Y).$$

Equivalently,

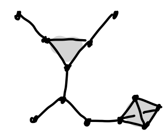
$$2 \cdot d_{GH}(X,Y) = \inf_{\substack{g: X \to Y \\ h: Y \to X}} \max\{\mathrm{dis}(g), \mathrm{dis}(h), \mathrm{codis}(g,h)\}.$$



$$\mathrm{dis}(g) = \sup_{x,x' \in X} |d(x,x') - d(g(x),g(x'))|$$

$$\mathrm{codis}(g,h) = \sup_{x \in X,\, y \in Y} |d(x,h(y)) - d(g(x),y)|$$

Def For X a metric space, the Vietoris-Rips simplicial complex $VR(X;r)$ has
- vertex set X
- simplex $\sigma \subseteq X$ when $\mathrm{diam}(\sigma) \le r$.



Hausmann's Thm For M a manifold and $0 < r < r' < \rho(M)$,

$$M \simeq VR(M;r) \xhookrightarrow{\simeq} VR(M;r') \simeq M$$

Pf Thm 1 If $d_{GH}(X,M) < \frac{1}{6}\rho(M)$, we'll show $d_{GH}(X,M) \ge \frac{1}{2} d_H(X,M)$.

Fix $2 \cdot d_{GH}(X,M) < r < \frac{1}{3}\rho(M)$ and $\varepsilon > 0$ with $3r + 2\varepsilon < \rho(M)$.

$[m_0, \dots, m_k] \longmapsto [h(m_0), \dots, h(m_k)]$

$$\begin{tikzcd}
VR(M;\varepsilon) \arrow[r, "\bar{h}"] \arrow[d, phantom, "\cong"] \arrow[rrrr, bend left, "\simeq"] & VR(X;r+\varepsilon) \arrow[rr, hook] \arrow[dr] & & VR(X;2r+2\varepsilon) \arrow[r, "\bar{g}"] & VR(M;3r+2\varepsilon) \arrow[d, phantom, "\cong"] \\
M & & \mathrm{Nerve}\left(\{B_M(x;r+\varepsilon)\}_{x\in X}\right) \arrow[ur] & & M
\end{tikzcd}$$

Apply $H_n(-;\mathbb{Z}/2)$ with $n = \dim(M)$.

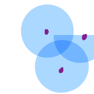
$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[r] \arrow[rrr, bend left, "\text{iso on fundamental class}"] & \cdot \arrow[r] \arrow[dr] & \cdot \arrow[r] & \mathbb{Z}/2 \\
& & 0 \arrow[u] &
\end{tikzcd}$$

If $r + \varepsilon < d_H(X,M)$, then the balls don't cover M, so $H_n(\mathrm{Nerve}) = 0$, contradiction.

Hence $r + \varepsilon \ge d_H(X,M)$ for all such r, ε
$\Rightarrow 2 \cdot d_{GH}(X,M) \ge d_H(X,M)$, as desired.

Rmk How to improve $\frac{1}{2}$ to 1 for $M = S^1$.

Rmk Gromov's filling radius



Pf Thm 2 Let $\delta = 2\cdot d_H(Y,M)$.

Show $d_{GH}(X,Y) < \frac{1}{6}\rho(M) - \frac{1}{3}\delta \implies d_{GH}(X,Y) \geq \frac{1}{2} d_H(X,M) - \frac{\delta}{2}$

Fix $2\cdot d_{GH}(X,Y) < r < \frac{1}{3}\rho(M) - \frac{2}{3}\delta$ and $\varepsilon > 0$ with $3r + 2\delta + 2\varepsilon < \rho(M)$

$$\begin{tikzcd}
\mathrm{VR}(Y;\delta+\varepsilon) \arrow[r, "\bar{h}"] & \mathrm{VR}(X;r+\delta+\varepsilon) \arrow[rr, hook] \arrow[dr, hook] & & \mathrm{VR}(X;2r+2\delta+2\varepsilon) \arrow[r, "\bar{g}"] & \mathrm{VR}(Y;3r+2\delta+2\varepsilon) \arrow[d, hook] \\
\mathrm{VR}(M;\varepsilon) \arrow[u] & & \mathrm{Nerve}(\{B_M(x;r+\delta+\varepsilon)\}_{x\in X}) \arrow[ur] & & \mathrm{VR}(M;3r+2\delta+2\varepsilon)
\end{tikzcd}$$

Apply $H_n(-;\mathbb{Z}/2)$.

Note $r + \delta + \varepsilon < d_H(X,M)$ gives $H_n(\mathrm{Nerve}) = 0$, contradiction.

So $r + \delta + \varepsilon \geq d_H(X,M)$ $\forall$ such r, ε

$\implies 2 d_{GH}(X,Y) \geq d_H(X,M) - \delta$

$\implies d_{GH}(X,Y) \geq \frac{1}{2} d_H(X,M) - d_H(Y,M)$.

Jung's Theorem For M a manifold with sectional curvatures $K \leq 0$,

$\mathrm{VR}(X;r) \hookrightarrow \mathrm{Nerve}\left(\left\{B_M\left(x; r\sqrt{\frac{n}{2n+2}}\right)\right\}_{x\in X}\right)$ $\forall r < \rho(M)$

If $K > 0$,

$\mathrm{VR}(X;r) \hookrightarrow \mathrm{Nerve}\left(\{B_M(x; r\sqrt{\frac{n}{2n+2}}\sqrt{\frac{K}{K+1}} / \sin(\sqrt{\frac{K}{K+1}})\right)$

$\forall r < \min\{\rho(M), \frac{1}{\sqrt{K+1}}\}$

Questions

Optimal constants?

Filling radius

Non-compact manifold, subsets of $\mathbb{R}^n$, Borel-Moore homology

Manifold with boundary

Stratefied space

$d_{GH}(X/G, Y/G)$