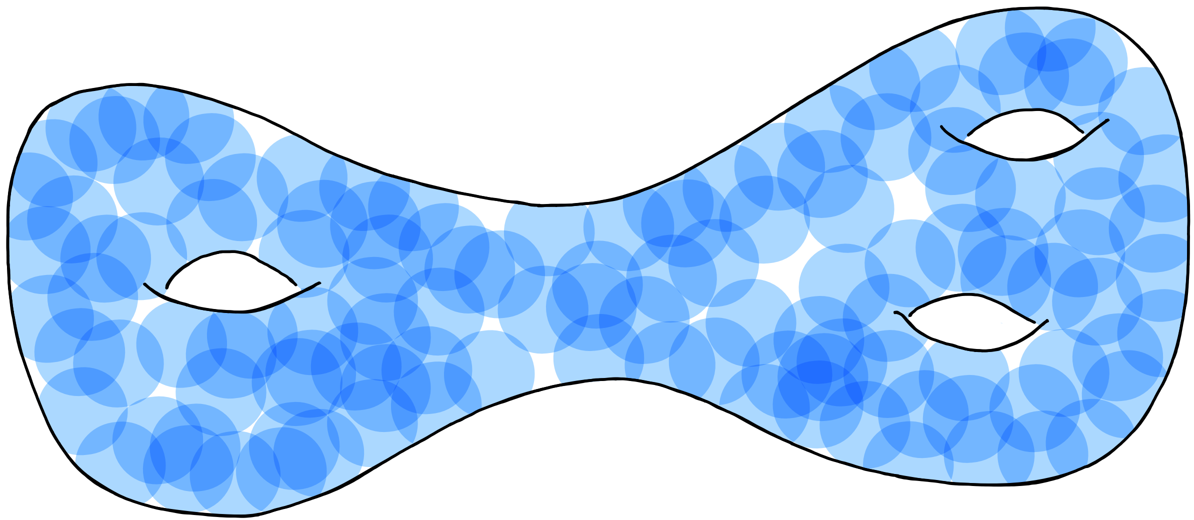
Hausdorff vs Gromov-Hausdorff distances



Henry Adams, University of Florida

Joint with Florian Frick, Sush Majhi, Nicholas McBride, arXiv 2309.16648

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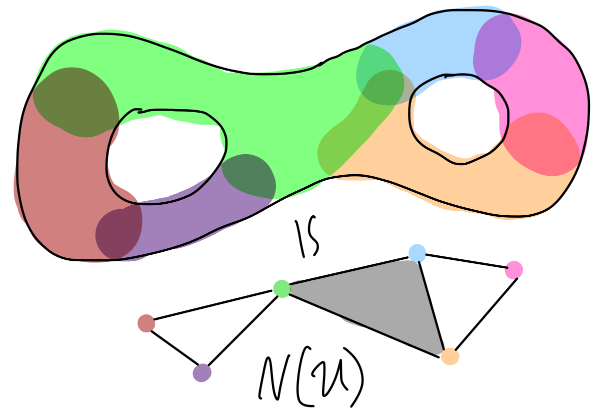
Nerve Lemma

Collection $\mathcal{U}$ of open sets in a metric space M.

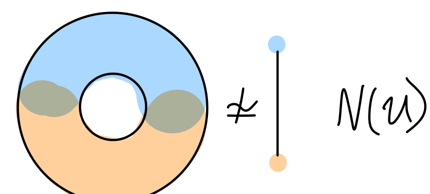
<ins>Def</ins> The nerve complex $N(\mathcal{U})$ has

- a vertex for each set in $\mathcal{U}$
- a k-simplex when $k+1$ sets intersect.

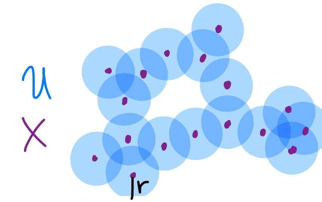


<ins>Nerve lemma</ins> If $\mathcal{U}$ is a good collection, then $N(\mathcal{U}) \simeq$ union of sets in $\mathcal{U}$.

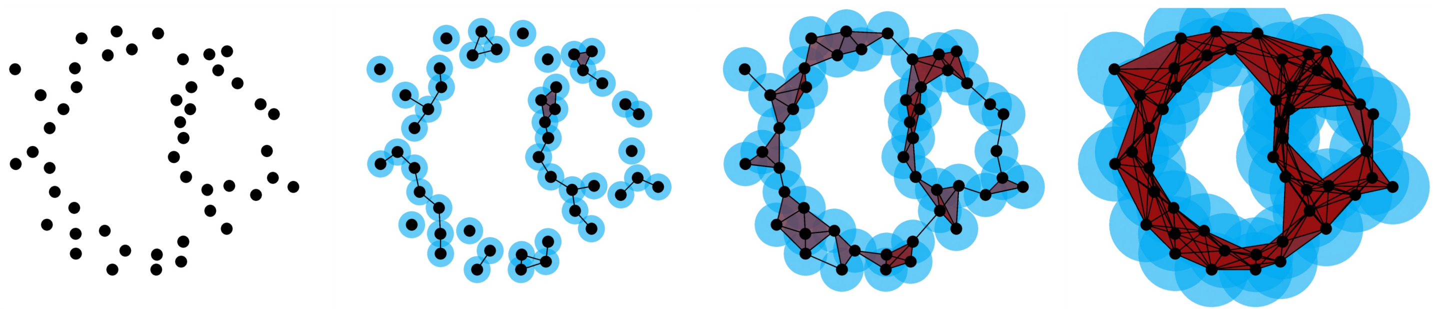
Sets in $\mathcal{U}$ are contractible; intersections empty or contractible.



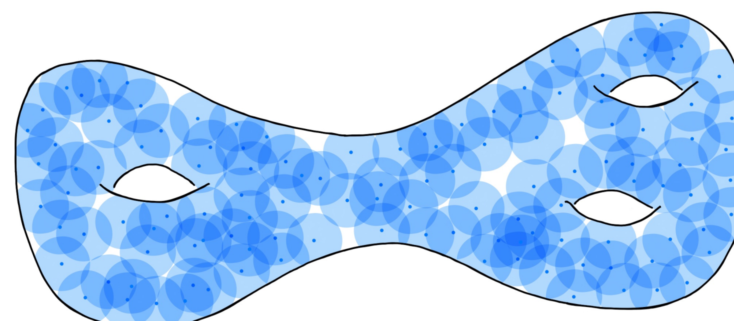
Čech Complexes



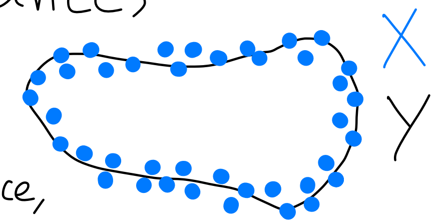
<u>Def</u> When $X \subseteq M$ and $\mathcal{U} = \{B_M(x;r)\}_{x \in X}$,
$N(\mathcal{U}) =: \check{C}(X;r)$ is called a <u>Čech complex</u>.



If M is a manifold
and $r < \mathrm{ConvRad}(M)$,
then $\mathcal{U}$ is a good collection, so
$\check{C}(X;r) \simeq$ union of balls.



Gromov-Hausdorff distances

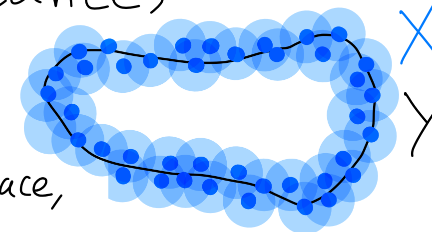


X, Y compact metric spaces.

If X, Y are two subsets of the same metric space, then the <u>Hausdorff distance</u> between them is

$$d_H(X,Y) = \inf\left\{ r > 0 \;\middle|\; X \subseteq \bigcup_{y \in Y} B(y;r),\ Y \subseteq \bigcup_{x \in X} B(x;r) \right\}$$

Gromov-Hausdorff distances



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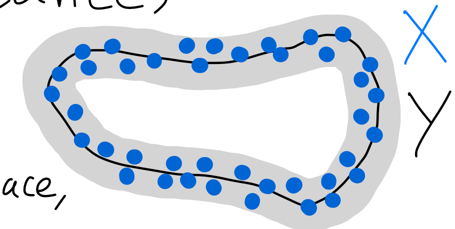

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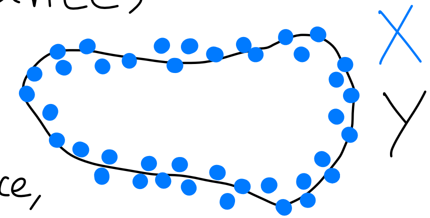

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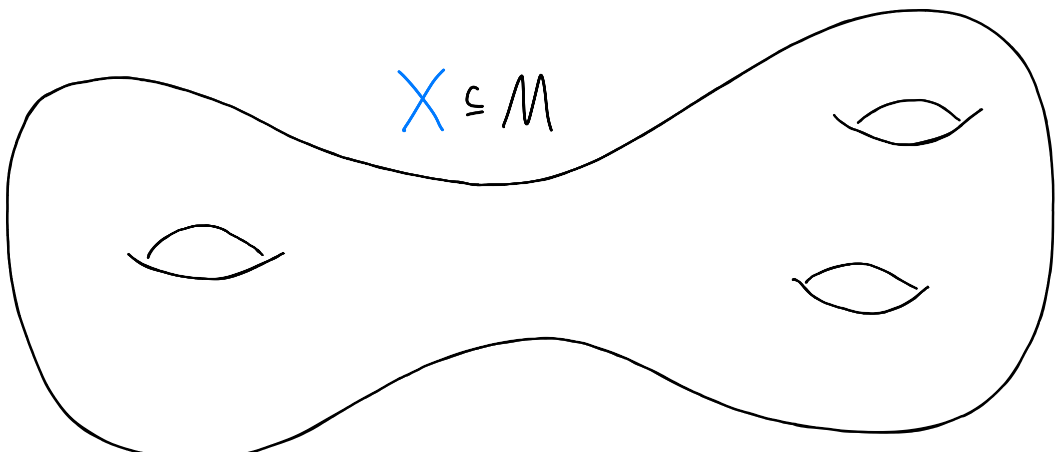

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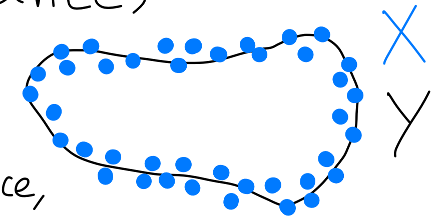

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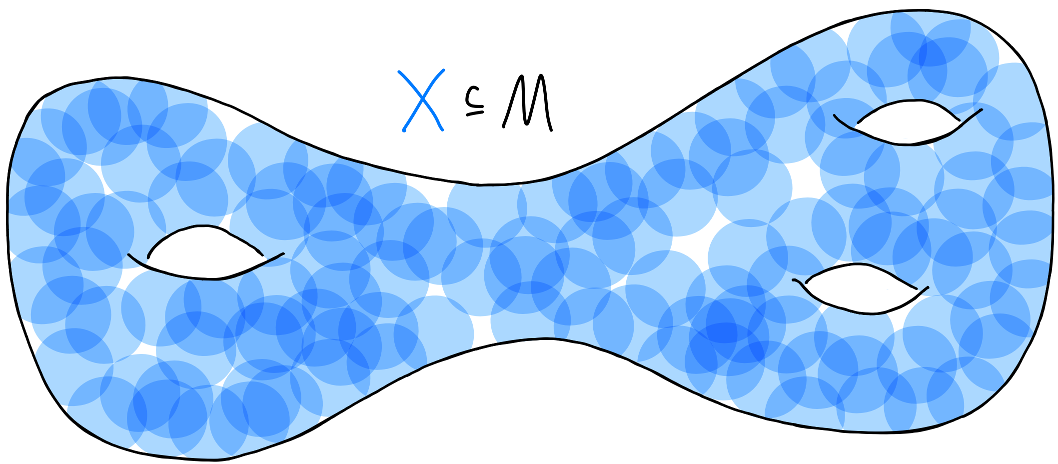

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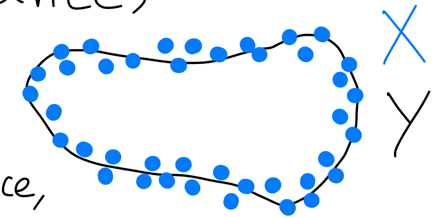

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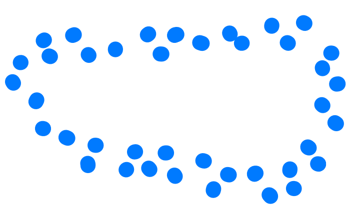
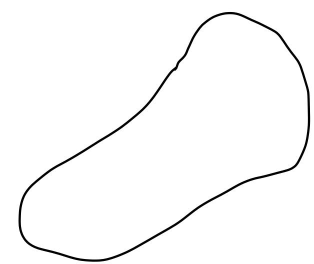
If X, Y are abstract metric spaces, then the <u>Gromov-Hausdorff distance</u> between them is

$$d_{GH}(X,Y) = \inf_{\substack{\text{isometric embeddings} \\ X \hookrightarrow Z,\; Y \hookrightarrow Z}} \left\{ d_H^Z(X,Y) \right\}$$

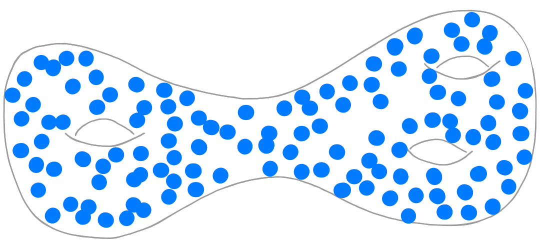




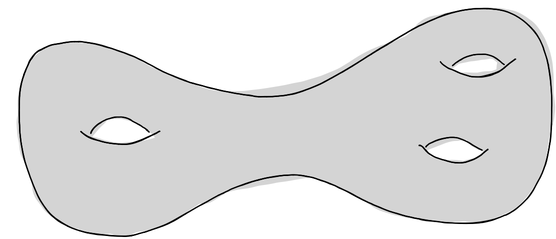
Gromov-Hausdorff distances

X, Y compact metric spaces. Equivalently:

<u>Def</u> $$2 \cdot d_{GH}(X,Y) = \inf_{\substack{h: X \to Y \\ g: Y \to X}} \max\{\mathrm{dis}(h), \mathrm{dis}(g), \mathrm{codis}(h,g)\}.$$



$$\begin{tikzcd} X \arrow[r, bend left, "h"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$



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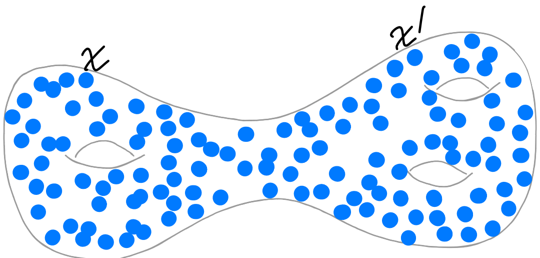
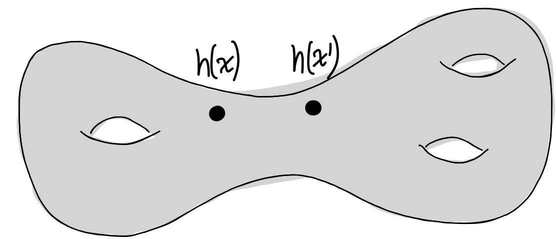


$$X \underset{g}{\overset{h}{\rightleftarrows}} Y$$



$$\mathrm{dis}(h) = \sup_{x, x' \in X} \left| d_X(x,x') - d_Y(h(x), h(x')) \right|$$

Gromov-Hausdorff distances

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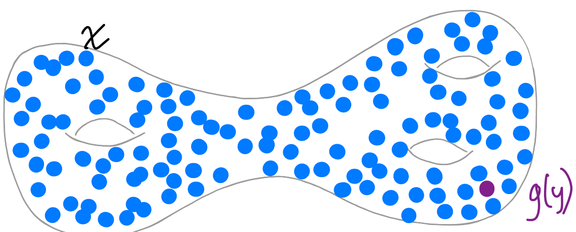

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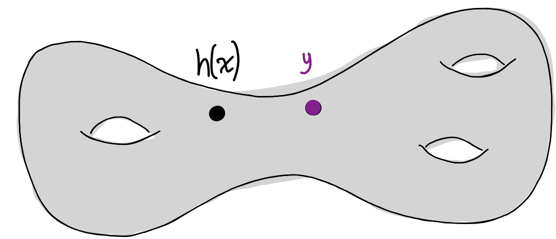
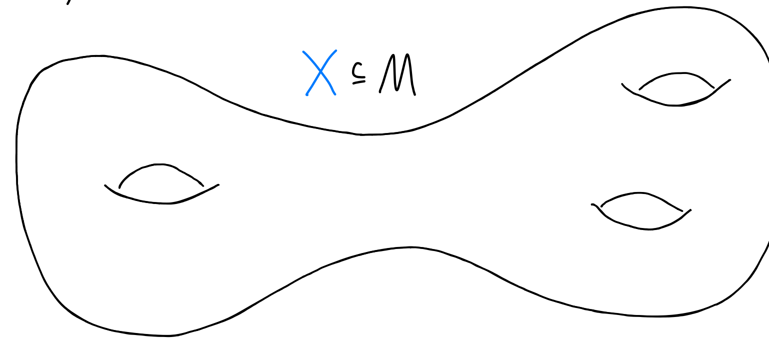


$$\operatorname{dis}(h) = \sup_{x, x' \in X} \left| d_X(x,x') - d_Y(h(x), h(x')) \right|$$

$$\operatorname{codis}(h,g) = \sup_{x \in X,\, y \in Y} \left| d_X(x, g(y)) - d_Y(h(x), y) \right|$$

Theorem For M a compact manifold and $X \subseteq M$,
$$d_{GH}(X,M) \geq \min\left\{\frac{1}{2} d_H(X,M),\ \frac{1}{6}\,\mathrm{ConvRad}(M)\right\}.$$



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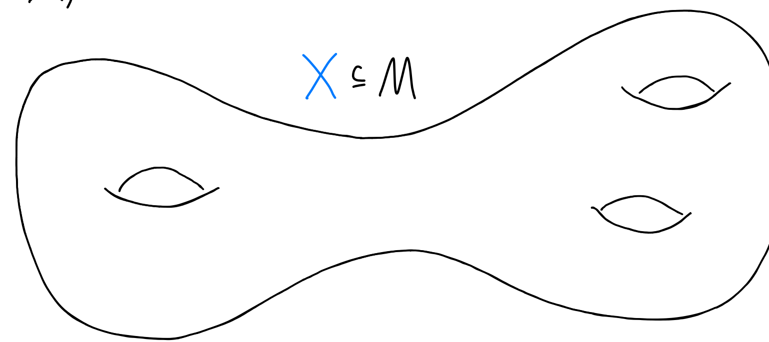

Proof We'll show if $d_{GH}(X,M) < \frac{1}{6}\operatorname{ConvRad}(M)$, then $d_{GH}(X,M) \geq \frac{1}{2} d_H(X,M)$.

Fix r with $2 \cdot d_{GH}(X,M) < r < \frac{1}{3}\operatorname{ConvRad}(M)$.
Fix $\varepsilon > 0$ with $3r + 2\varepsilon < \operatorname{ConvRad}(M)$.

$$\begin{tikzcd}
M \arrow[d, phantom, sloped, "\simeq"] & & M \arrow[d, phantom, sloped, "\simeq"] \\
\check{C}(M;\varepsilon) \arrow[r, "\bar{g}"] \arrow[rr, bend right, "\bar{h}\circ\bar{g} \simeq \text{inclusion}"'] & \check{C}(X; r+\varepsilon) \arrow[r, "\bar{h}"] & \check{C}(M; 3r+2\varepsilon)
\end{tikzcd}$$

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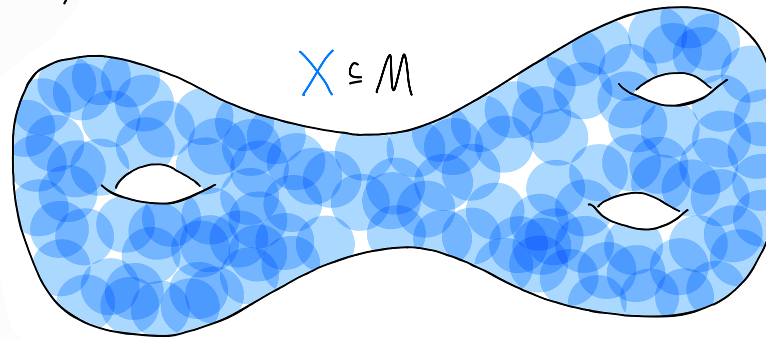

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\end{tikzcd}$$

If we had $r+\varepsilon < d_H(X,M)$, then we'd have
$\check{C}(X; r+\varepsilon) \simeq$ proper subset of M.

Applying homology (fundamental class) would give

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[r] \arrow[rr, bend right, "\text{isomorphism}"'] & 0 \arrow[r] & \mathbb{Z}/2
\end{tikzcd}$$

Hence $r+\varepsilon \geq d_H(X,M)$ for all such r and ε.

So $2 \cdot d_{GH}(X,M) \geq d_H(X,M)$.

Theorem For M a compact manifold and $X \subseteq M$,

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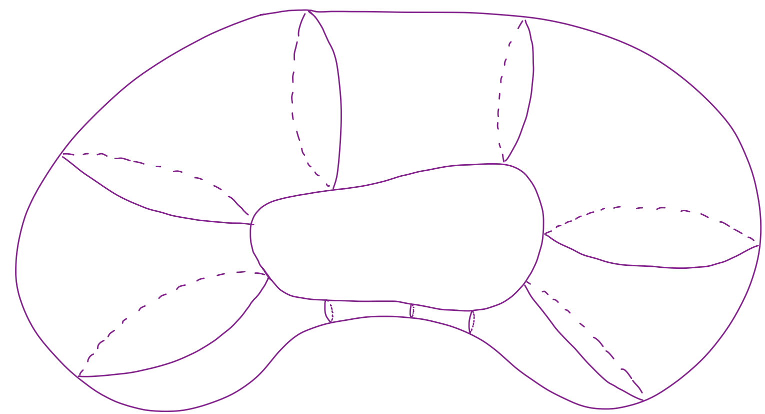
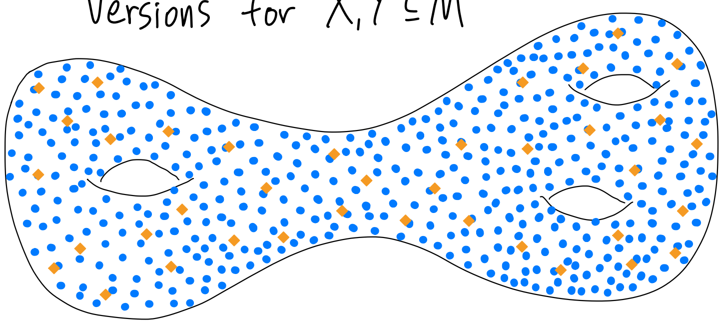
Can be improved to $\sqrt{\frac{n+1}{2n}}$ if M^n has nonnegative sectional curvatures, hence optimal constant 1 when $M=S^1$.

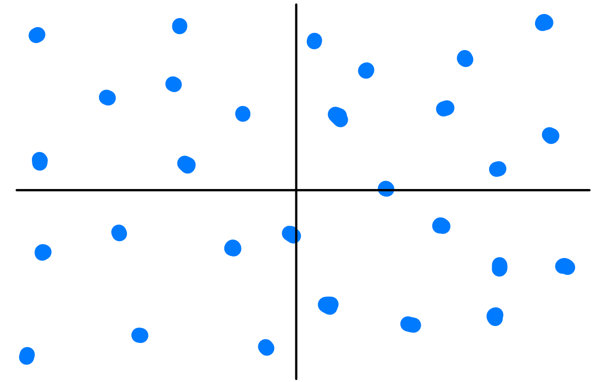
Can be improved to $\frac{1}{3}\,\mathrm{FillRad}(M)$

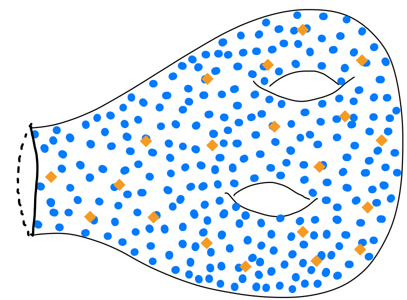


Versions for $X, Y \subseteq M$



<u>Theorem</u> For M a compact manifold and $X \subseteq M$,
$d_{GH}(X, M) \geq \min\left\{\frac{1}{2} d_H(X, M),\ \frac{1}{6}\,\mathrm{ConvRad}(M)\right\}$.





Questions

What if M is not bounded, for example $M = \mathbb{R}^n$?

What if M has boundary?

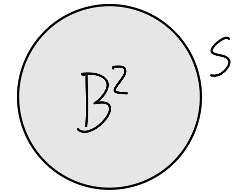
What if M is not a manifold?

Homotopy types of $\check{C}(S^n; r)$ for larger r?

Questions

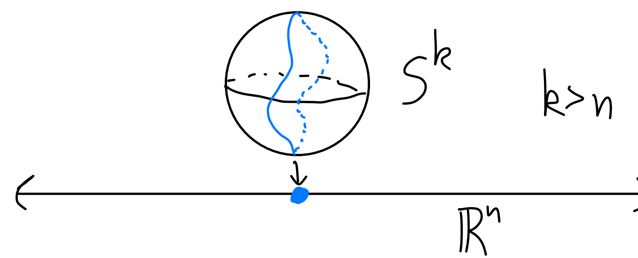
$d_{GH}($ ball B^{n+1}, sphere $S^n)$?

$d_{GH}(S^k, S^n)$?



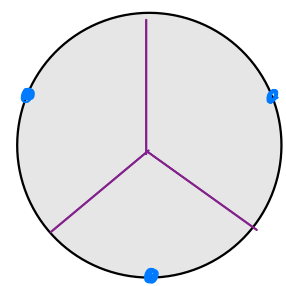
[Lim, Mémoli, Smith], [polymath]

Quantitative versions of Borsuk-Ulam theorem (such as Gromov's waist of sphere theorem) ?



Urysohn widths of balls

$UW_k(B^n)$, $k<n$?



$UW_1(B^2)$