Metric Thickenings, Orbitopes, and Borsuk–Ulam Theorems

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Joint work with Henry Adams[†] and Florian Frick[‡]

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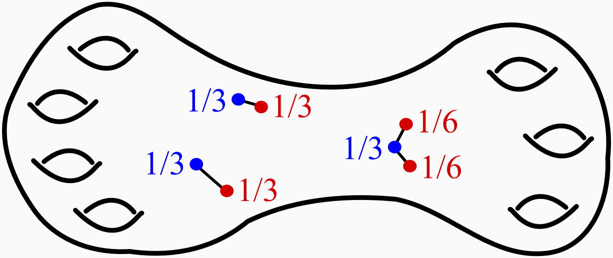
Definition (Adamaszek, Adams, Frick [1])

For a metric space X and $r \geq 0$, the Vietoris–Rips thickening $\mathrm{VR}^m(X;r)$ is the set

$$\mathrm{VR}^m(X;r) = \left\{ \sum_{i=0}^{k} \lambda_i x_i \;\middle|\; k \in \mathbb{N},\ x_i \in X, \text{ and } \mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r \right\}$$

equipped with the 1-Wasserstein metric.

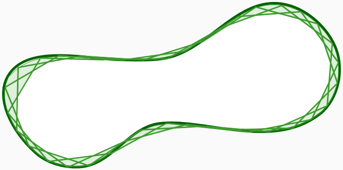


$\mathrm{VR}^m(X;r)$ at small scale parameters

Theorem (Adams, Mirth [2])
Let $X \subseteq \mathbb{R}^n$ with positive reach. Then,

$$\mathrm{VR}^m(X;r) \simeq X$$

for r sufficiently small.

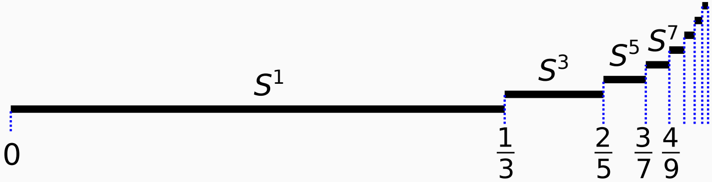


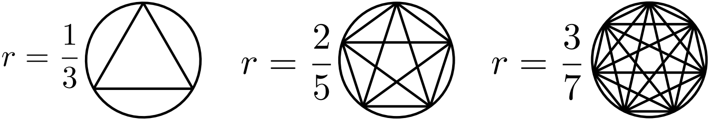
$\mathrm{VR}^m(X;r)$ at large scale parameters

Conjecture

Let S^1 be the circle of unit circumference. Then, for $k \in \mathbb{N}$,

$$\mathrm{VR}^m(S^1;r) \simeq S^{2k-1} \quad \textit{if} \quad \frac{k-1}{2k-1} \leq r < \frac{k}{2k+1}.$$





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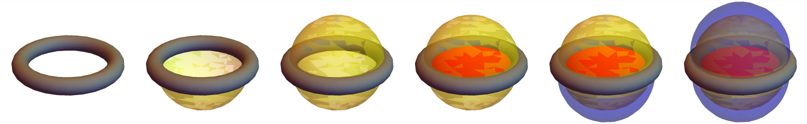

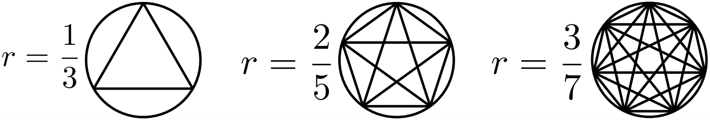

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Desired proof.

For $\frac{k-1}{2k-1} \leq r < \frac{k}{2k+1}$, there exist homotopy equivalences $p \circ \mathrm{SM}_{2k}$ and ι in the following diagram:

$$\mathrm{VR}^m(S^1;r) \xrightarrow{\mathrm{SM}_{2k}} \mathbb{R}^{2k} \setminus \{\vec{0}\} \xrightarrow{p} \partial\mathcal{B}_{2k} \xrightarrow{\iota} \mathrm{VR}^m(S^1;r),$$

where p denotes the radial projection, ι denotes the inclusion, and $\partial\mathcal{B}_{2k} \cong S^{2k-1}$. $\square$

Barvinok–Novik Orbitopes

Fix $k \geq 1$ and define the symmetric moment curve

$$\mathrm{SM}_{2k} \colon \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^{2k}$$

$$\theta \mapsto (\cos(\theta), \sin(\theta), \cos(3\theta), \sin(3\theta), \ldots, \cos((2k-1)\theta), \sin((2k-1)\theta)).$$

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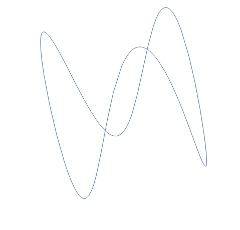
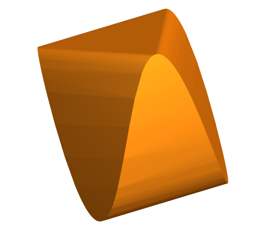
Define the k-th Barvinok–Novik orbitope by

$$\mathcal{B}_{2k} = \mathrm{conv}(\mathrm{SM}_{2k}(S^1)).$$





Theorem (Barvinok, Novik [3])

The proper faces of $\mathcal{B}_4$ *are*

- *the 0-dimensional faces,* $\mathrm{SM}_4(t)$ *for* $t \in S^1$,
- *the 1-dimensional faces,* $\mathrm{conv}(\mathrm{SM}_4(\{t_1, t_2\}))$ *where* $t_1 \neq t_2$ *are the edges of an arc of* S^1 *of length* $\leq \frac{1}{3}$,
- *the 2-dimensional faces,* $\mathrm{conv}(\mathrm{SM}_4(\{t, t + \frac{1}{3}, t + \frac{2}{3}\}))$ *for* $t \in S^1$.

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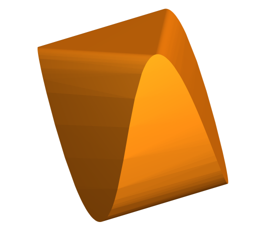
The precise facial structure of $\mathcal{B}_{2k}$ is unknown for $k > 2$.
Known to be *simplicial* and *locally k-neighborly* ([5], [3]).



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$\mathrm{VR}^m(X;r)$ at large scale parameters

So far:

Theorem (Adams, B., Frick)
$\mathrm{VR}^m(S^1;1/3) \simeq S^3$.

Proof.
There exist homotopy equivalences $p \circ \mathrm{SM}_4$ and ι in the following diagram:

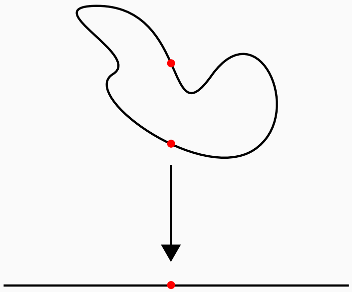
$$\mathrm{VR}^m(S^1;1/3) \xrightarrow{\mathrm{SM}_4} \mathbb{R}^4 \setminus \{\vec{0}\} \xrightarrow{p} \partial\mathcal{B}_4 \xrightarrow{\iota} \mathrm{VR}^m(S^1;1/3)$$

where p denotes the radial projection, ι denotes the inclusion, and $\partial\mathcal{B}_4 \cong S^3$. $\square$

Theorem (Borsuk–Ulam)

Given a continuous function $f\colon S^n \to \mathbb{R}^n$, there exists $x \in S^n$ such that $f(x) = f(-x)$.

Equivalently, given a continuous and odd function $f\colon S^n \to \mathbb{R}^n$, there exists $x \in S^n$ such that $f(x) = \vec{0}$.

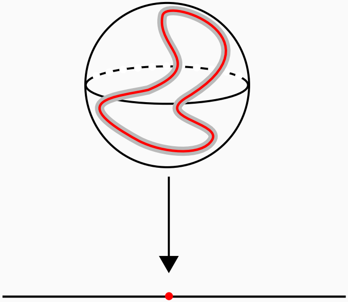


Theorem (Gromov [4])

Given a continuous function $f\colon S^n \to \mathbb{R}^k$ with $k \leq n$, there exists $y \in \mathbb{R}^k$ such that the n-spherical volume of the ε-tubular neighborhood of $f^{-1}(y)$, denoted by $f^{-1}(y) + \varepsilon$, satisfies

$$Vol_n(f^{-1}(y) + \varepsilon) \geq Vol_n(f^{-1}(S^{n-k} + \varepsilon))$$

for every $\varepsilon > 0$.



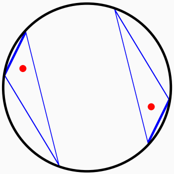
Borsuk–Ulam Theorems

Theorem (Adams, B., Frick)

If $f\colon S^1 \to \mathbb{R}^{2k+1}$ is continuous, there exists a subset $\{x_1, \ldots, x_m\} \subseteq S^1$ of diameter at most $\frac{k}{2k+1}$ and with $m \leq 2k+1$ such that $\sum_{i=1}^m \lambda_i f(x_i) = \sum_{i=1}^m \lambda_i f(-x_i)$, for some choice of convex coefficients λ_i.

Equivalently, if $f\colon S^1 \to \mathbb{R}^{2k+1}$ is continuous and odd, then there exists a subset $X \subseteq S^1$ of diameter at most $\frac{k}{2k+1}$ and size $|X| \leq 2k+1$ such that $\vec{0} \in \mathrm{conv}(f(X))$.

This result is sharp: $f = \mathrm{SM}_{2k}\colon S^1 \to \mathbb{R}^{2k} \subset \mathbb{R}^{2k+1}$ is an odd map such that $\vec{0} \notin \mathrm{conv}(f(X))$ if $\mathrm{diam}(X) < \frac{k}{2k+1}$.



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Proof.

The induced map $f\colon \mathrm{VR}^m(S^1; \frac{k}{2k+1}) \to \mathbb{R}^{2k+1}$ is odd with domain $\mathrm{VR}^m(S^1; \frac{k}{2k+1}) \simeq S^{2k+1}$. By Borsuk–Ulam, this map has a zero, giving a subset X of diameter at most $\frac{k}{2k+1}$ with $\mathrm{conv}(f(X))$ containing the origin. □

Future Work

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- Recent idea: full Borsuk–Ulam type results (i.e., tight bounds) for odd dimensional spheres by taking n-fold joins of S^1.
- A better understanding of metric thickenings of spheres at large scales. (Čech thickenings, different orbitopes, etc.)

References

[1] M. Adamaszek, H. Adams, and F. Frick, *Metric reconstruction via optimal transport*, To appear in the SIAM Journal on Applied Algebra and Geometry, (2018).

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Thank you!