Metric Thickenings, Orbitopes, and Borsuk–Ulam Theorems

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Summary

Metric thickenings of a metric space capture local geometric properties of the space. We use the combinatorial and geometric structure of convex bodies in Euclidean space to give geometric proofs of the homotopy type of certain metric thickenings of the circle. Consequently, we discover interconnections between the geometry of circle actions on Euclidean space, the structure of zeros of trigonometric polynomials, and theorems of Borsuk–Ulam type.

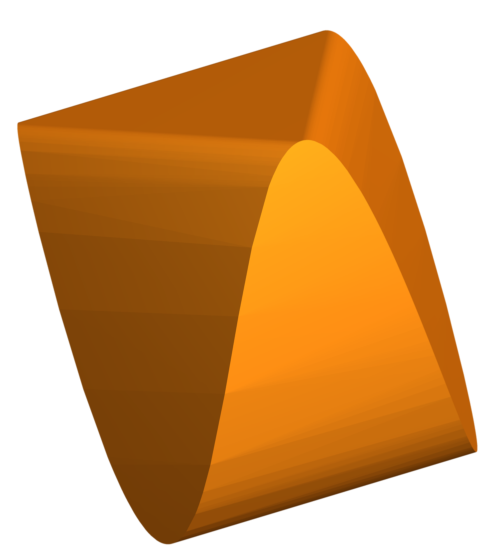
Definitions

- Fix a metric space (X,d). Given a simplicial complex K with vertex set X, the **metric thickening** of K is the metric space $K^m = (|K|, d_W)$, where d_W denotes the 1-Wasserstein metric and

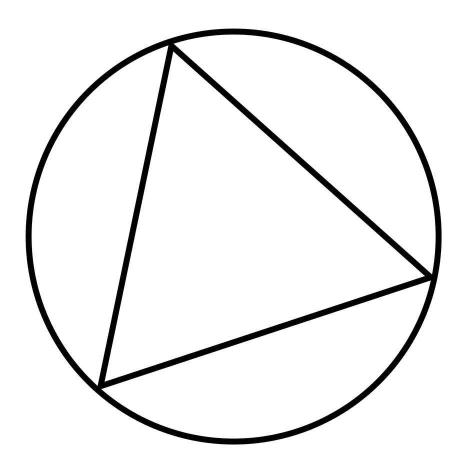
$$|K| = \left\{ \sum_{i=0}^{n} \lambda_i x_i \;\middle|\; n \in \mathbb{N}, \lambda_i \geq 0, \sum_i \lambda_i = 1, \{x_0, \ldots, x_n\} \in K \right\} \quad [1].$$

- Let $\mathrm{VR}(X;r)$ and $\check{\mathrm{C}}(X;r)$ denote the **Vietoris–Rips** and **Čech** simplicial complexes of a metric space X at scale r, respectively. Let $\mathrm{VR}^m(X;r)$ and $\check{\mathrm{C}}^m(X;r)$ denote the metric thickenings of these complexes.
- An **orbitope** is the convex hull of an orbit of a compact group acting linearly on a vector space [2].
- The **Barvinok–Novik orbitope** (see [3]) is $\mathcal{B}_{2k} = \mathrm{conv}(\mathrm{SM}_{2k}(S^1))$, where $\mathrm{SM}_{2k}\colon S^1 \to \mathbb{R}^{2k}$ by

$$\mathrm{SM}_{2k}(t) = (\cos(t), \sin(t), \cos(3t), \sin(3t), \ldots, \cos((2k-1)t), \sin((2k-1)t)).$$



The convex hull of $\{(\cos(t), \sin(t), \cos(3t)) \,|\, t \in S^1\}$. *Not* a Barvinok–Novik orbitope, but some faces of $\mathcal{B}_4$ are visible.



A subset of $\mathrm{VR}(S^1; \frac{1}{3})$. Simplices in $\mathrm{VR}(S^1; \frac{1}{3})$ are all equilateral triangles and all edges of length $\leq \frac{1}{3}$.

Main Theorem ([4])

Equip S^1 with the geodesic metric (of total circumference 1). Then,

$$\mathrm{VR}^m(S^1; \tfrac{1}{3}) \simeq \partial\mathcal{B}_4 \cong S^3.$$

Homotopy equivalences are $p \circ \mathrm{SM}_4$ and ι in the following diagram:

$$\mathrm{VR}^m(S^1; \tfrac{1}{3}) \xrightarrow{\mathrm{SM}_4} \mathbb{R}^4 \setminus \{\vec{0}\} \xrightarrow{p} \partial\mathcal{B}_4 \xrightarrow{\iota} \mathrm{VR}^m(S^1; \tfrac{1}{3}).$$

Here, the domain of SM_4 has been linearly extended to $\mathrm{VR}^m(S^1;r)$, p denotes the radial projection, and ι denotes the inclusion $\sum_i \lambda_i \mathrm{SM}_4(t_i) \mapsto \sum_i \lambda_i t_i$.

Intuition: Simplices contributing to the homotopy type of $\mathrm{VR}^m(S^1; \frac{1}{3})$ are contained in $\partial\mathcal{B}_4$. Consequently, $p \circ \mathrm{SM}_4$ reduces the dimension of $\mathrm{VR}^m(S^1; \frac{1}{3})$ while maintaining the correct topology. In fact, $\mathcal{B}_4$ is *simplicial*, meaning its faces are simplicies, and if $\{\mathrm{SM}_4(t_0), \ldots, \mathrm{SM}_4(t_n)\}$ is a simplex in $\partial\mathcal{B}_4$, then $\{t_0, \ldots, t_n\}$ belongs to the 2-skeleton of $\mathrm{VR}^m(S^1; \frac{1}{3})$ [3].

Conjecture

Equip S^1 with the geodesic metric (of total circumference 1). Then,

$$\mathrm{VR}^m(S^1; r) \simeq S^{2k-1} \quad \text{if} \quad \frac{k-1}{2k-1} \leq r < \frac{k}{2k+1}.$$

Desired proof (outline). For $\frac{k-1}{2k-1} \leq r < \frac{k}{2k+1}$, there exist homotopy equivalences $p \circ \mathrm{SM}_{2k}$ and ι in the following diagram:

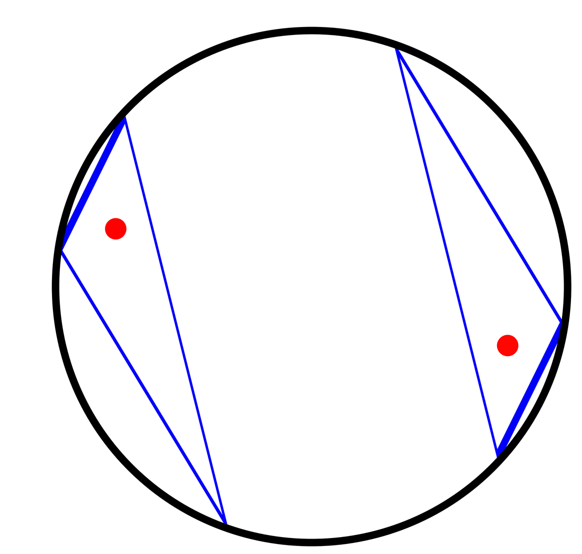
$$\mathrm{VR}^m(S^1;r) \xrightarrow{\mathrm{SM}_{2k}} \mathbb{R}^{2k} \setminus \{\vec{0}\} \xrightarrow{p} \partial\mathcal{B}_{2k} \xrightarrow{\iota} \mathrm{VR}^m(S^1;r),$$

where p denotes the radial projection, ι denotes the inclusion, and $\partial\mathcal{B}_{2k} \cong S^{2k-1}$.

- Continuity depends crucially on the topology of the metric thickening. In fact, $X \hookrightarrow |\mathrm{VR}(X;r)|$ is not continuous if $\mathrm{VR}(X;r)$ is not locally finite, whereas $X \hookrightarrow \mathrm{VR}^m(X;r)$ is *always* continuous.
- Exact structure of $\partial\mathcal{B}_{2k}$ is unknown for $k > 2$, and showing $\iota \circ p \circ \mathrm{SM}_{2k} \simeq \mathrm{id}_{\mathrm{VR}^m(S^1;r)}$ is difficult for $r > \frac{1}{3}$.

Consequences

- **Theorem**. If $f\colon S^1 \to \mathbb{R}^{2k+1}$ is continuous, there exists a subset $\{x_1, \ldots, x_m\} \subseteq S^1$ of diameter at most $\frac{k}{2k+1}$ and with $m \leq 2k+1$ such that $\sum_{i=1}^m \lambda_i f(x_i) = \sum_{i=1}^m \lambda_i f(-x_i)$, for some choice of convex coefficients λ_i. [4]. This result is sharp.
- **Theorem**. Let r_n denote the diameter of an inscribed regular $(n+1)$-simplex in S^n. If $f\colon S^n \to \mathbb{R}^{n+2}$ is continuous, there exists a subset $\{x_1, \ldots, x_m\} \subseteq S^n$ of diameter at most r_n and with $m \leq n+2$ such that $\sum_{i=1}^m \lambda_i f(x_i) = \sum_{i=1}^m \lambda_i f(-x_i)$, for some choice of convex coefficients λ_i [4].



Given $f\colon S^1 \to \mathbb{R}^{2k+1}$, there exist red points "near" the circle that collide under f.

- **Theorem**. Given a subset $X \subseteq S^1$ of diameter less than $\frac{k}{2k+1}$, there exists a raked homogeneous trigonometric polynomial of degree $2k-1$ that is positive on all of the points in X [4]. This result is sharp.

Future Work

- If the orbitopes $\mathcal{B}_{2k}$ are simplicial for all k, it would follow that the $(2k-1)$-dimensional homology, cohomology, and homotopy groups of $\mathrm{VR}^m(S^1;r)$ are nontrivial for $\frac{k-1}{2k-1} \leq r < \frac{k}{2k+1}$.
- The combinatorial structure of the faces of the related Carathéodory orbitopes (see [2]) closely reflects the simplicial structure of $\check{\mathrm{C}}^m(S^1;r)$. A better understanding of the analogous map $\iota \circ p$ may allow for the desired homotopy equivalences, giving $\check{\mathrm{C}}^m(S^1;r) \simeq S^{2l-1}$ for $\frac{l-1}{2l} \leq r < \frac{l}{2(l+1)}$.

References

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