Vietoris–Rips Thickenings of the Circle and Centrally–Symmetric Orbitopes

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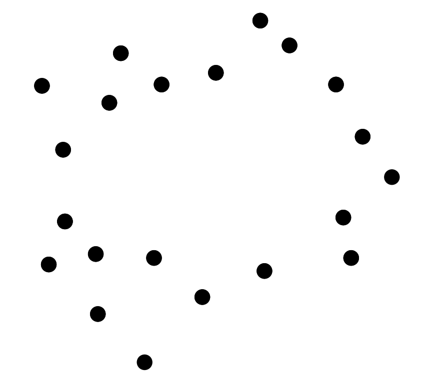
Master's thesis defense – October 19^{th}, 2018

Background

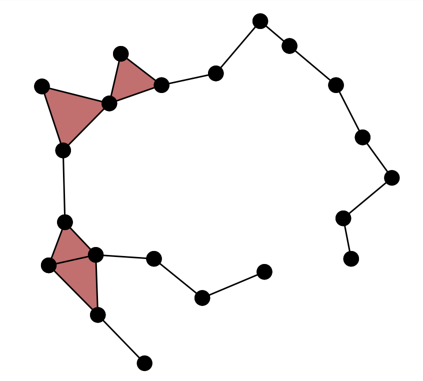
The Vietoris–Rips Complex



Definition

Let X be a metric space and $r > 0$ a scale parameter. The Vietoris–Rips complex of X, denoted $\mathrm{VR}(X; r)$, has vertex set X and a simplex for every finite subset $\sigma \subseteq X$ such that $\mathrm{diam}(\sigma) \le r$.

The Vietoris–Rips Complex



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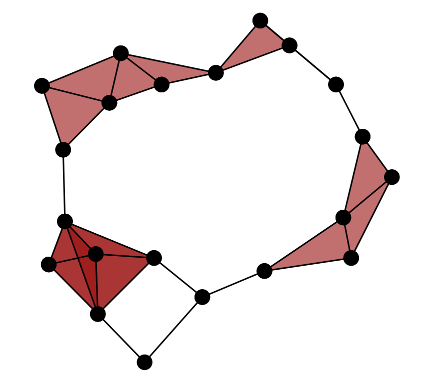

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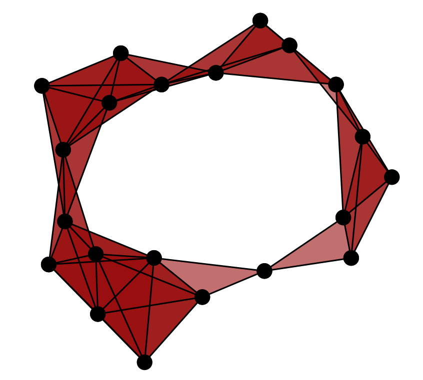

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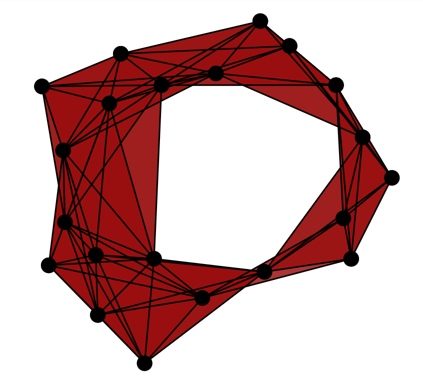

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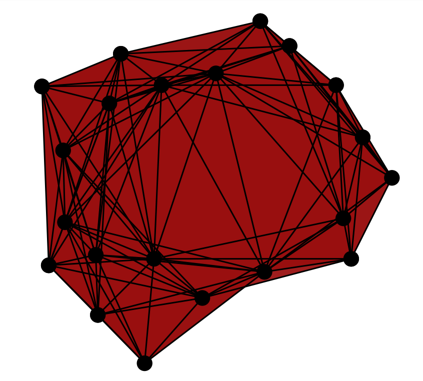

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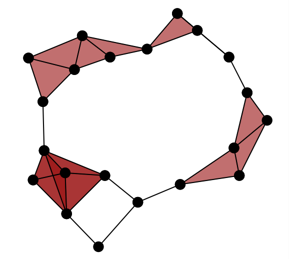
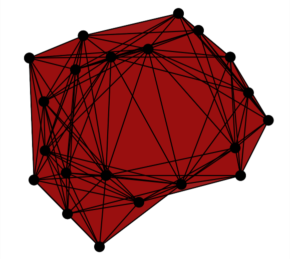
In applications of persistent homology, we consider Vietoris–Rips complexes at *all* scale parameters.

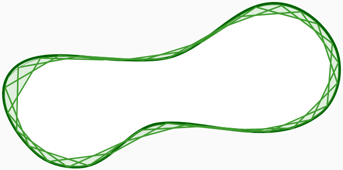




Hausmann's Theorem

Theorem

Let M be a compact Riemannian manifold and $r > 0$ be sufficiently small. Then, $\mathrm{VR}(M; r) \simeq M$. [4]



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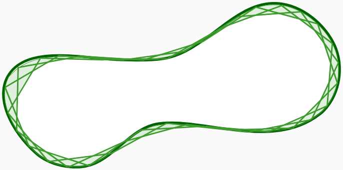

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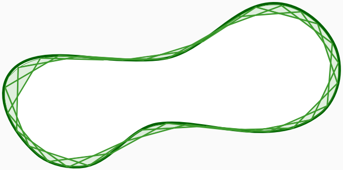

- Downsides of the proof:
  - Hausmann's map $\mathrm{VR}(M; r) \to M$ depends upon a total order of all points in M.
  - $\mathrm{VR}(M; r)$ does not inherit the metric of M. In particular, the inclusion $M \hookrightarrow \mathrm{VR}(M; r)$ is not continuous.

Latschev's Theorem

Theorem

Let M be a closed Riemannian manifold and X a metric space δ–close to M in the Gromov-Hausdorff distance. Then, $\mathrm{VR}(X;r) \simeq M$ *for $r > 0$ sufficiently small.* [5]

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- Again, the value of "sufficiently small" $r > 0$ depends on the curvature of M.
- The value of δ depends on r.

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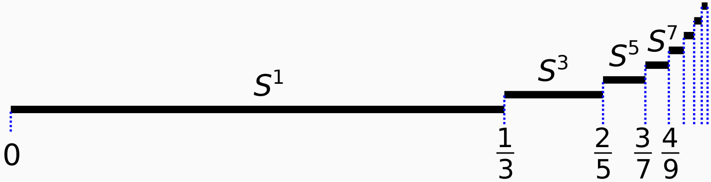
- Generalization of Hausmann's theorem. Applies, in particular, to samplings $X \subseteq M$.
- Again, the value of "sufficiently small" $r > 0$ depends on the curvature of M.
- The value of δ depends on r.
- Same downsides as Hausmann's proof.

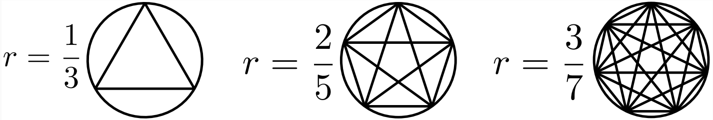
$\mathrm{VR}(X;r)$ for large scale parameters

Theorem (Adamszek, Adams)

Let S^1 be the circle of unit circumference. Then,

$$\mathrm{VR}(S^1;r) \simeq \begin{cases} S^{2\ell+1} & \text{if } \frac{\ell}{2\ell+1} < r < \frac{\ell+1}{2\ell+3} \end{cases} \quad \text{for } \ell \in \mathbb{N}. \qquad [1]$$





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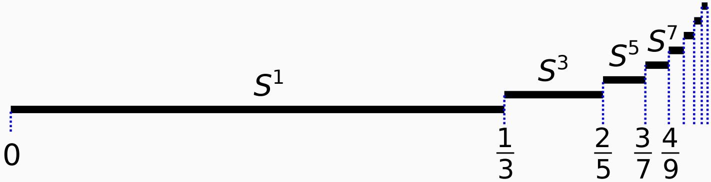

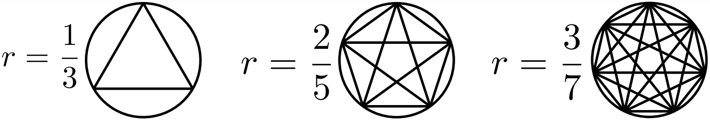

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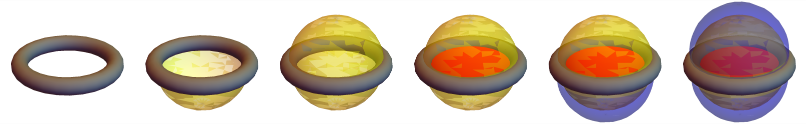
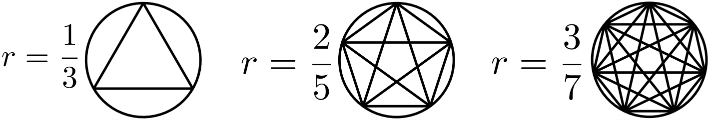




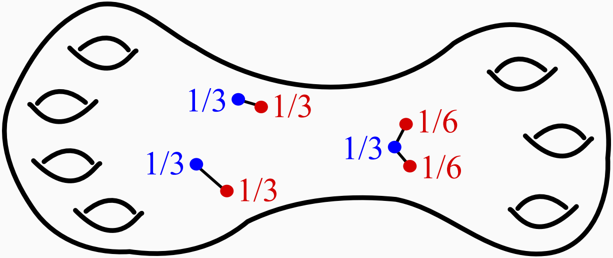
Metric Vietoris–Rips Thickenings

Definition (Adamaszek, Adams, Frick)

For a metric space X and $r \geq 0$, the Vietoris–Rips thickening $\mathrm{VR}^m(X;r)$ is the set

$$\mathrm{VR}^m(X;r) = \left\{ \sum_{i=0}^{k} \lambda_i x_i \mid k \in \mathbb{N},\ x_i \in X, \text{ and } \mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r \right\}$$

equipped with the 1-Wasserstein metric. [2]



Metric Vietoris–Rips Thickenings

- An analogue of Hausmann's theorem holds for the Vietoris–Rips metric thickening $\mathrm{VR}^m(M;r)$.
- Notably, this theorem admits a nicer proof:
  - ◇ The homotopy equivalence $\mathrm{VR}^m(M;r) \to M$ is canonically defined.
  - ◇ The inclusion $M \hookrightarrow \mathrm{VR}^m(M;r)$ is continuous.

Results

Main Theorem

Theorem (Adams, B.)

Let S^1 be the circle of unit circumference, and let $r = \frac{1}{3}$. Then, the Vietoris–Rips metric thickening thickening $\mathrm{VR}^m(S^1; r)$ is homotopy equivalent to S^3.

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Let S^1 be the circle of unit circumference, and let $r = \frac{1}{3}$. Then, the Vietoris–Rips metric thickening thickening $\mathrm{VR}^m(S^1; r)$ *is homotopy equivalent to S^3.*

- 1/3 is the side length of an inscribed equilateral triangle.
- Recall that $\mathrm{VR}(S^1; \frac{1}{3}) \simeq \bigvee^{\infty} S^2$

Main Theorem (proof)

We construct a homotopy equivalence via

$$\mathrm{VR}^m(S^1; 1/3) \xrightarrow{\mathrm{SM}_4} \mathbb{R}^4 \setminus \{\vec{0}\} \xrightarrow{p} \partial\mathcal{B}_4 \xrightarrow{\iota} \mathrm{VR}^m(S^1; 1/3)$$

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Barvinok–Novik Orbitopes

Fix $k \geq 1$ and define the symmetric moment curve

$$\mathrm{SM}_{2k} \colon \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^{2k}$$

$$\theta \mapsto (\cos(\theta), \sin(\theta), \cos(3\theta), \sin(3\theta), \ldots, \cos((2k-1)\theta), \sin((2k-1)\theta)).$$

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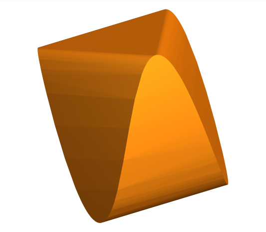

The precise facial structure of $\mathcal{B}_{2k}$ is unknown for $k > 2$.

Our proof technique involves the faces of $\mathcal{B}_4$.

Theorem (4.1 of [3])

The proper faces and subfaces of $\mathcal{B}_4$ *are*

- *0-dimensional faces,* $\mathrm{SM}_4(t)$ *for* $t \in S^1$,
- *1-dimensional faces,* $\mathrm{conv}(\mathrm{SM}_4(\{t_1, t_2\}))$ *where* $t_1 \neq t_2$ *are the edges of an arc of* S^1 *of length less than or equal to* $\frac{1}{3}$,
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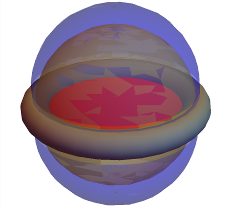

By contrast, the simplicies of $\mathrm{VR}^m(S^1; \frac{1}{3})$ are

- Vertices, $t \in S^1$,
- All simplices of the form $\mathrm{conv}(\{t_1, \ldots, t_l\})$, where $t_1, \ldots, t_m$ belong to an arc of S^1 of length less than or equal to $\frac{1}{3}$,
- 2-simplices, $\mathrm{conv}(\{t, t+\frac{1}{3}, t+\frac{2}{3}\})$ for $t \in S^1$.

Main Theorem (proof)

$$\mathrm{VR}^m(S^1; 1/3) \xrightarrow{\mathrm{SM}_4} \mathbb{R}^4 \setminus \{\vec{0}\} \xrightarrow{p} \partial\mathcal{B}_4 \xrightarrow{\iota} \mathrm{VR}^m(S^1; 1/3)$$

Steps:

(1) Prove ι is well-defined and continuous.

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Main Theorem (proof) – Prove $p \circ \mathrm{SM}_4$ and ι are h. inverses

- $(p \circ \mathrm{SM}_4) \circ \iota = \mathrm{id}_{\partial \mathcal{B}_4}$.
- To show $\iota \circ (p \circ \mathrm{SM}_4) \simeq \mathrm{id}_{\mathrm{VR}^m(S^1;r)}$ we use a linear homotopy H.

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- To show $\iota \circ (p \circ \mathrm{SM}_4) \simeq \mathrm{id}_{\mathrm{VR}^m(S^1;r)}$ we use a linear homotopy H.
- H is well-defined only if $\iota \circ (p \circ \mathrm{SM}_4)$ does not "increase diameter" too much.

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- In general, this "non-diameter increasing" property of p depends on the facial structure of $\mathcal{B}_{2k}$.
- We use Farkas' Lemma to exclude certain cases.
- Continuity of H follows from the fact that $\mathrm{VR}^m(S^1; r)$ is a metric r-thickening.

Main Theorem (proof)

$$\mathrm{VR}^m(S^1;1/3) \xrightarrow{\mathrm{SM}_4} \mathbb{R}^4 \setminus \{\vec{0}\} \xrightarrow{p} \partial\mathcal{B}_4 \xrightarrow{\iota} \mathrm{VR}^m(S^1;1/3)$$

Thus, $\iota \circ (p \circ \mathrm{SM}_4) \simeq \mathrm{id}_{\mathrm{VR}^m(S^1;r)}$ and $\mathrm{VR}^m(S^1;1/3) \simeq \partial\mathcal{B}_4 \cong S^3$.

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- *Geometric* proof of the homotopy type of a Vietoris–Rips metric thickening.
- Conjecture: for $\frac{k-1}{2k-1} \leq r < \frac{k}{2k+1}$,

$$p_{2k} \circ \mathrm{SM}_{2k} \colon \mathrm{VR}^m_{\leq}(S^1;r) \to \partial\mathcal{B}_{2k}$$

  is a homotopy equivalence.

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Thank you!