THESIS

VIETORIS–RIPS THICKENINGS OF THE CIRCLE AND CENTRALLY–SYMMETRIC ORBITOPES

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ABSTRACT

VIETORIS–RIPS THICKENINGS OF THE CIRCLE AND CENTRALLY–SYMMETRIC ORBITOPES

Given a metric space X and a scale parameter $r > 0$, the associated Vietoris–Rips simplicial complex, denoted $\mathrm{VR}(X; r)$, has as its simplices all finite subsets of X of diameter at most r. In the case that X is a Riemannian manifold, a result of Jean–Claude Hausmann states that the homotopy type of X is achieved by $\mathrm{VR}(X; r)$ for sufficiently small r. However, this approach does not recover metric information about X, and this deficiency motivates the consideration of a related construction, called the Vietoris–Rips thickening of X, defined via the theory of optimal transport. This construction, which does preserve metric information about X, additionally satisfies an analogue of Hausmann's theorem for sufficiently small r. On the other hand, one often encounters such thickenings given instead by increasingly large values of r in applications of persistent homology, and much less is known about the topological behavior of these constructions. A recently established result due to Adams and Adamaszek provides the homotopy type of the Vietoris–Rips complex of the circle for arbitrarily large values of r. Presently, we determine the homotopy type of the Vietoris–Rips thickening of the circle for a range of values of r. Our primary tools will be an embedding of the metric thickening into Euclidean space via a symmetric moment curve, and the facial structure of the related Barvinok–Novik orbitope.

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Chapter 1

Introduction

Given a metric space M, let X denote a set of points sampled from M, and consider the induced metric on X given by restriction. In general, it is impossible to recover the homotopy type of M from this data alone, even if X is (for example) ε-dense in M. However, Hausmann proved in [14] that for a compact Riemannian manifold M and a sufficiently small scale $r > 0$, the Vietoris–Rips simplicial complex $\mathrm{VR}(M; r)$ achieves the homotopy type of M. A result due to Latschev [16] moreover provides conditions under which it is possible to recover the homotopy type of M from the sampling X, specifically, when M is a closed Riemannian manifold and X is sufficiently close to M in the Gromov-Hausdorff distance. Hence, given a sampling $X \subseteq M$ satisfying this condition, Latschev proved that $\mathrm{VR}(X; r)$ achieves the homotopy type of M for sufficiently small values of r.

Unfortunately, in the case that $\mathrm{VR}(X; r)$ is not locally finite, it is impossible to equip $\mathrm{VR}(X; r)$ with a metric without changing the homeomorphism type, and it is therefore impossible to recover metric information about M. In particular, the topology on $\mathrm{VR}(X; r)$ is such that the natural inclusion $X \hookrightarrow \mathrm{VR}(X; r)$ is not continuous unless X is discrete. An attempt to remedy this issue is given by Adamaszek, Adams, and Frick in [2], in which the authors construct a family of metric spaces $\mathrm{VR}^m(X; r)$, called Vietoris–Rips metric thickenings of X. Critically, metric information about X is described in $\mathrm{VR}^m(X; r)$ via the theory of optimal transport, i.e., the 1-Wasserstein metric, which extends the original metric on X. As the notation suggests, this construction shares similarities with the Vietoris–Rips simplicial complex $\mathrm{VR}(X; r)$. In particular, we may take abstract convex combinations of points in X of diameter at most r, just as in $\mathrm{VR}(X; r)$. Further, we now have a continuous inclusion $X \hookrightarrow \mathrm{VR}^m(X; r)$ for all metric spaces X and $r \geq 0$, and there exists an analogue of Hausmann's theorem regarding the recovery of the homotopy type of the underlying manifold for sufficiently small $r > 0$ [2].

Recently, Vietoris–Rips complexes (and the related Čech simplicial complexes) have been used in topological data analysis and persistent homology; specifically, they allow for a filtration of simplicial complexes associated to a finite collection or sampling of data. From such a filtered simplicial complex, one may apply homology to obtain a persistence module, which encodes some topological properties of the point cloud. Because the point clouds under consideration often naturally arise as samplings of some underlying (and typically unknown) metric space, Latschev's theorem provides an important connection between the topological invariants computed through persistent homology and the underlying space from which the data is sampled. However, given such a sampling of data, the particular scale parameters r at which Latshev's theorem applies are often unknown, and this leads to a difficult and related problem: predicting the behavior of the homotopy type of $\mathrm{VR}(X;r)$ achieved for *arbitrary* (and in particular *large*) values of r. Along these lines, Adamaszek and Adams determined the homotopy type of $\mathrm{VR}(S;r)$ for all values of r [1]. These techniques additionally allow computation of the homotopy types of Vietoris–Rips complex of the n-dimensional torus equipped with the supremum metric (for all values of r) [1, Proposition 10.2], and for ellipses of small eccentricity [3] (for particular values of r depending on the eccentricity of the ellipse).

In light of the aforementioned theorems of Hausmann, Latschev, and Adamsezek et al., we consider the Vietoris–Rips metric thickening of the circle $\mathrm{VR}^m_{\leq}(S^1;r)$, and provide a geometric proof of the homotopy type of this construction for $r = \frac{2\pi}{3}$, the side-length of an inscribed equilateral triangle. To obtain this result, we first define a continuous embedding of $\mathrm{VR}^m(S^1;r)$ into $\mathbb{R}^4$ via a symmetric moment curve. Then, we relate this embedding to the facial structure of the Barvinok–Novik orbitope $\mathcal{B}_4$ [6]. Finally, we obtain the homotopy equivalence $\mathrm{VR}^m_{\leq}(S^1;r) \simeq \partial\mathcal{B}_4 \cong S^3$ via a linear homotopy. To our knowledge, this is the first approach to determine the homotopy type of a Vietoris–Rips thickening by mapping the underlying metric space into a higher-dimensional Euclidean space. This technique is analogous to the "kernel trick" of machine learning, in which data is mapped into a higher dimensional space to illuminate the underlying structure of the data.

Chapter 2

Preliminaries

In this chapter we review background preliminaries on topology, metric spaces, simplicial complexes, metric thickenings, convex geometry, moment curves, and polytopes.

2.1 Basic Topology

We refer the reader to [4, 13] for further background on topology.

Given a set X, let $\mathcal{P}(X)$ denote the power set of X.

Definition 2.1.1. Given a set X, a *topology on* X is a collection of subsets $\mathcal{O} \subseteq \mathcal{P}(X)$ satisfying the following:

1. $\varnothing \in \mathcal{O}$ and $X \in \mathcal{O}$.

2. Given a finite collection $U_1, \ldots, U_n \in \mathcal{O}$, the intersection $\bigcap_{i=1}^n U_i$ belongs to $\mathcal{O}$.

3. Given an arbitrary index set $\mathcal{A}$ and a collection $\{U_\alpha \in \mathcal{O} \mid \alpha \in \mathcal{A}\}$, the union $\bigcup_{\alpha \in \mathcal{A}} U_\alpha$ belongs to $\mathcal{O}$.

Elements of $\mathcal{O}$ are called *open subsets* of X. A *topological space* is an ordered pair $(X, \mathcal{O})$ such that $\mathcal{O}$ is a topology on X. We may simply write X to denote a topological space when the topology is implicitly clear.

Definition 2.1.2. Given any topological space $(X, \mathcal{O})$ and subset $Y \subseteq X$, define the *subspace topology induced by* Y to be $\mathcal{O}' \subseteq \mathcal{P}(Y)$ such that $V \in \mathcal{O}'$ if and only if $V = U \cap Y$ for some $U \in \mathcal{O}$.

Fact 2.1.3. $(Y, \mathcal{O}')$*, as defined above, is a topological space.*

A notion that will be required later (see Section 2.4) is that of compactness.

Definition 2.1.4. Let $(X, \mathcal{O})$ denote a topological space. We say X is *compact* if, given any collection $C \subseteq \mathcal{O}$ with

$$X = \bigcup_{x \in C} x,$$

there exists a finite subset $F \subseteq C$ such that

$$X = \bigcup_{x \in F} x.$$

A subset $Y \subseteq X$ is said to be compact if it is compact as a subspace, i.e., if $(Y, \mathcal{O}')$ is compact.

Sometimes, we may define a topology on a set X given only a particular collection of subsets of X.

Definition 2.1.5. Given a set X and a collection $\mathcal{B} \subseteq \mathcal{P}(X)$, we say $\mathcal{B}$ is a *basis* or *base* for X if the following hold:

1. X is contained in the union of all elements of $\mathcal{B}$.

2. Given $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

From such a basis $\mathcal{B}$ of X, we define *the topology induced by* $\mathcal{B}$, denoted $\mathcal{O}_\mathcal{B}$, to be the collection of all possible unions (including the empty union) of elements of $\mathcal{B}$.

Fact 2.1.6. *Given a set X with a basis $\mathcal{B}$, $(X, \mathcal{O}_\mathcal{B})$ is a topological space.*

Definition 2.1.7. Let $\{(X_i, \mathcal{O}_i) \mid 1 \leq i \leq n\}$ denote a collection of topological spaces. Define the *topological product* of these spaces to be the set $X_1 \times X_2 \times \cdots \times X_n = \prod_{i=1}^{n} X_i$ equipped with the topology generated by the basis

$$\mathcal{B} = \left\{ \prod_{i=1}^{n} U_i \,\middle|\, U_i \in \mathcal{O}_i \right\}.$$

Note that we will require the definition of a topological product to define the notion of homotopy equivalence.

For a set map $f\colon X \to Y$ and subset $U \subseteq Y$, let $f^{-1}(U) = \{x \in X \mid f(x) \in U\}$ denote the *preimage* of U.

Definition 2.1.8. Let $(X_1, \mathcal{O}_1)$ and $(X_2, \mathcal{O}_2)$ be topological spaces. A map $f\colon X_1 \to X_2$ is said to be *continuous* if $f^{-1}(U) \in \mathcal{O}_1$ for all $U \in \mathcal{O}_2$.

Definition 2.1.9. Let $(X, \mathcal{O})$ denote a topological space. The *identity function on* X, denoted id_X, is the unique and continuous map $\mathrm{id}_X\colon X \to X$ such that $\mathrm{id}_X(x) = x$ for all $x \in X$.

Continuous maps between topological spaces allow us to define two important notions of topological equivalence: homeomorphism and homotopy equivalence.

Definition 2.1.10. Let $(X_1, \mathcal{O}_1)$ and $(X_2, \mathcal{O}_2)$ be topological spaces. We say X_1 and X_2 are *homeomorphic*, denoted $X_1 \cong X_2$, if there exist continuous maps $f\colon X_1 \to X_2$ and $g\colon X_2 \to X_1$ such that $g \circ f = \mathrm{id}_{X_1}$ and $f \circ g = \mathrm{id}_{X_2}$.

Fact 2.1.11. *The spaces X_1 and X_2 are homeomorphic if and only if there exists a continuous, bijective map $f\colon X_1 \to X_2$ with a continuous inverse.*

In order to define homotopy equivalence, we will need an intermediate definition:

Definition 2.1.12. We say two continuous maps $f, g\colon X \to Y$ are *homotopic*, written $f \simeq g$, if there exists a continuous map $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. Such a map H is called a *homotopy*.

Definition 2.1.13. Let $(X_1, \mathcal{O}_1)$ and $(X_2, \mathcal{O}_2)$ be topological spaces. We say X_1 and X_2 are *homotopy equivalent*, denoted $X_1 \simeq X_2$, if there exist continuous maps $f\colon X_1 \to X_2$ and $g\colon X_2 \to X_1$ such that $g \circ f \simeq \mathrm{id}_{X_1}$ and $f \circ g \simeq \mathrm{id}_{X_2}$.

2.2 Metric Spaces and the Euclidean Metric

Metric spaces are topological spaces which are furthermore equipped with the notion of a distance. Let $\mathbb{R}^+ = \{t \in \mathbb{R} \mid t \geq 0\}$.

Definition 2.2.1. Given a set M, a *metric on* M is any function $d\colon M \times M \to \mathbb{R}^+$ such that the following hold for all $x, y, z \in M$:

1. $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

A *metric space* is an ordered pair (M, d) such that d is a metric on M.

Notably, any metric space (M, d) may be construed as a topological space in the following way: given $z \in M$ and $0 < r \in \mathbb{R}$, let $B(z; r) = \{z \in M \mid d(z, x) < r\}$, be the *open ball of radius* r *centered at* z. Let $\mathcal{B} = \{B(z; r) \mid z \in M, 0 < r \in \mathbb{R}\} \subseteq \mathcal{P}(M)$ denote the collection of all such balls. One may check that $\mathcal{B}$ forms a basis for M; hence, $(M, \mathcal{O}_{\mathcal{B}})$ is a topological space.

Definition 2.2.2. Given a set of points $S \subseteq M$ of a metric space (M, d), define the *diameter of* S, denoted $\operatorname{diam}(S)$, by

$$\operatorname{diam}(S) = \begin{cases} \sup\limits_{x,y \in S} d(x, y) & \text{if } \sup\limits_{x,y \in S} d(x, y) \in \mathbb{R} \\ \infty & \text{otherwise.} \end{cases}$$

Definition 2.2.3. Let (M, d) be a metric space, and let $S \subseteq M$. Let $d' = d|_{S \times S}$ denote the restriction of d to $S \times S \subseteq X \times X$. Then, the *metric subspace of* M *induced by restriction* is the tuple (S, d'). One may check that (S, d') is, indeed, a metric space.

Definition 2.2.4. Let (X, d_X) and (Z, d) be metric spaces. Following [12] and [2], we say Z is a *metric thickening* of X if:

1. $X \subseteq Z$, and
2. $d_X(x, y) = d(x, y)$ for all $(x, y) \in X \times X$.

If furthermore there exists some $r \geq 0$ such that for every $z \in Z$ we have $d(z, x) \leq r$ for some $x \in X$, then we additionally say that Z is a *metric r-thickening* of X.

An important example of a metric thickening is given in Section 2.5.

Let $\mathbb{R}^n$ denote the n-fold Cartesian product of the set of real numbers. Under coordinate-wise addition and scalar multiplication, recall that $\mathbb{R}^n$ has the structure of a vector space. Define the *standard inner product* $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ by

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = \sum_{i=1}^{n} x_i y_i,$$

and define the *norm* $||\cdot||$ of an element $x \in \mathbb{R}^n$ by $||x|| = \langle x, x \rangle^{1/2}$. Finally, define $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $d(x, y) = ||y - x||$.

Definition 2.2.5. The function d above is a metric on $\mathbb{R}^n$, called the *Euclidean metric*, and we will refer to the metric space $(\mathbb{R}^n, d)$ as (n-dimensional) *Euclidean space*.

2.3 Simplicial Complexes

Simplicial complexes are a combinatorial way to define a topological space. Given a set X, let $\operatorname{fin}\mathcal{P}(X)$ denote the collection of finite subsets of X.

Definition 2.3.1. Let X be a set. We say a collection $\varnothing \neq K \subseteq \operatorname{fin}\mathcal{P}(X)$ forms an *abstract simplicial complex* if, for every element $\sigma \in K$ and nonempty subset $\sigma' \subseteq \sigma$, the subset σ' also belongs to K. Elements $\sigma \in K$ are called *faces* of the simplicial complex K, and we define the *vertex set* of K to be the union of all faces of K. Finally, singleton elements of the vertex set of K are called *vertices* of K.

Definition 2.3.2. A subset $L \subseteq K$ of an abstract simplicial complex K is called a *subcomplex of* K if L is itself an abstract simplicial complex. In the case that a subcomplex $L \subseteq K$ is a face of K, L is called a *simplex* of K.

To an abstract simplicial complex, we may associate a topological space as follows.

Definition 2.3.3. Let K denote an abstract simplicial complex, let V denote its set of vertices, and let

$$|K| = \left\{ f\colon V \to [0,1] \,\middle|\, \operatorname{supp}(f) \in K, \sum_{v \in V} f(v) = 1 \right\}.$$

Here, $\operatorname{supp}(f) = \{v \in V \mid f(v) \neq 0\}$ is the support of f. Next, give $[0,1]^V$ the topology induced as the direct limit of $[0,1]^S$ as S varies over all finite subsets of V, and equip $|K|$ with the induced subset topology [20]. The space $|K|$ with this induced topology is called *the geometric realization of K*.

Given any metric space X and scale parameter $r \geq 0$, we may define a particular simplicial complex with vertex set X.

Definition 2.3.4. Let X be a metric space and fix $r \geq 0$. The *Vietoris–Rips simplicial complex of X with scale parameter r*, denoted $\mathrm{VR}_{\leq}(X; r)$ (resp. $\mathrm{VR}_{<}(X; r)$), has X as its vertex set and a face $\sigma \subseteq X$ whenever $\operatorname{diam}(\sigma) \leq r$ (resp. $\operatorname{diam}(\sigma) < r$).

A related construction, called the *Čech simplicial complex of X with scale parameter r*, denoted $\check{\mathrm{C}}(X; r)$, has X as a vertex set and a face $\sigma \subseteq X$ whenever

$$\bigcap_{x \in \sigma} B(x; r) \neq \varnothing.$$

2.4 Wasserstein Metric and Optimal Transport

The Wasserstein or optimal transport metric gives a notion of distance between probability measures defined on a metric space.

Definition 2.4.1. Let X be a set. A subset $\Sigma \subseteq \mathcal{P}(X)$ is called a *σ-algebra on X* if it satisfies the following:

1. $X \in \Sigma$.
2. Given $A \in \Sigma$, the absolute complement $X \setminus A$ belongs to Σ.
3. Given a countable collection $\{A_i\}_{i=1}^{\infty}$ of elements of Σ, the union $\cup_i A_i$ belongs to Σ.

Note that the last two properties additionally imply that Σ is closed under finite intersections.

Definition 2.4.2. Let X be a set and let Σ be a σ-algebra on X. A function $\mu\colon \Sigma \to \mathbb{R}^+ \cup \{\infty\}$ is called a *measure* if it satisfies the following:

1. $\mu(\varnothing) = 0$.

2. If $\{A_i\}_{i=1}^{\infty}$ is a countable collection of pairwise-disjoint elements of Σ, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Definition 2.4.3. Given a nonempty set X, a σ-algebra Σ on X, and a measure μ on Σ, the triple (X, σ, μ) is called a *measure space*. If, in addition, $\mu(X) = 1$, we call (X, σ, μ) a *probability space*.

Definition 2.4.4. Let X and Y be sets equipped with σ-algebras Σ_X and Σ_Y, respectively. A function $f\colon X \to Y$ is called *measurable* if, for all $A \in \Sigma_Y$, we have $f^{-1}(A) \in \Sigma_X$.

In particular, in the case that $Y = \mathbb{R}$, the function f is measurable if Σ_X contains the preimage of all intervals of the form (t, ∞), i.e., if $\{x \mid f(x) > t\} \in \Sigma_X$ for all $t \in \mathbb{R}$.

Given a measure space (X, σ, μ) and a real-valued measurable function f on X, one may define *the Lebesgue integral* $\int_X f(x)\mathrm{d}\mu(x) = \int_X f(x)\mathrm{d}\mu$, which is an element of the extended real numbers $\mathbb{R} \cup \{\pm\infty\}$. The definition of such an integral is lengthy and beyond the scope of this exposition. For a proper treatment of Lebesgue integration, see [21].

Definition 2.4.5. Let (M, d) be a metric space, let $\mathcal{B}$ denote the collection of open balls of M, and let $(M, \mathcal{O}_{\mathcal{B}})$ denote the topological space induced by $\mathcal{B}$ (see Section 2.2). The *Borel σ-algebra over* $(M, \mathcal{O}_{\mathcal{B}})$, written $\mathscr{B}(M)$, is defined to be the intersection of all σ-algebras on M containing $\mathcal{O}_{\mathcal{B}}$. Equivalently, a subset of M belongs to $\mathscr{B}(M)$ if and only if it can be obtained through taking countable unions, countable intersections, and absolute complements of open sets of M. Elements of $\mathscr{B}(M)$ are called *Borel sets* of M.

Definition 2.4.6. A measure μ defined on a Borel σ-algebra $\mathscr{B}(M)$ is called *inner regular* if $\mu(B) = \sup\{\mu(K) \mid B \supseteq K \text{ compact}\}$ for all $B \in \mathscr{B}(M)$. μ is called *locally-finite* if, for all $x \in X$, there exists a neighborhood $x \in U \in \mathscr{B}(M)$ such that $\mu(U) < \infty$. Finally, μ is called a *Radon measure* if it is both inner regular and locally-finite.

To define the 1-Wasserstein metric, we first introduce some notation. The following is taken from [2]. Let (M, d) be a metric space. Let $\mathscr{P}(M)$ denote the set of Radon probability measures μ on M such that for some (hence, all) $x_0 \in M$, we have $\int_M f(x, x_0)\mathrm{d}\mu(x) < \infty$. Next, define a metric on $M \times M$ by setting the distance between $(x_1, x_2), (x_1', x_2') \in M \times M$ to be $d(x_1, x_1') + d(x_2, x_2')$. Given $\mu, \nu \in \mathscr{P}(M)$, let $\Pi(\mu, \nu) \subseteq \mathscr{P}(M \times M)$ denote the set of Radon probability measures on $M \times M$ such that $\mu(A) = \pi(A \times M)$ and $\nu(A) = \pi(M \times A)$ for all Borel sets $A \subseteq M$. Note that $\pi \in \Pi(\mu, \nu)$ is a joint probability measure with marginals μ and ν.

Definition 2.4.7. Let (M, d) be a metric space. The 1-*Wasserstein metric on* $\mathscr{P}(M)$ is defined by

$$d_{\mathscr{P}(M)}(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \int_{M \times M} d(x, y)\, \mathrm{d}\pi$$

Fact 2.4.8. $\big(\mathscr{P}(M), d_{\mathscr{P}(M)}\big)$*, as defined above, is a metric space.*

Given $x \in M$, let $\delta_x \in \mathscr{P}(M)$ denote the Dirac probability measure with mass one at x. One may check that the map $x \mapsto \delta_x$ is an isometry onto its image; hence, $d_{\mathscr{P}(M)}$ extends the metric $d\colon M \times M \to \mathbb{R}$ to $\mathscr{P}(M) \times \mathscr{P}(M) \to \mathbb{R}$, and we therefore write $d_{\mathscr{P}(M)} = d$.

2.5 Vietoris–Rips Metric Thickenings

The material in this section is taken from [2]. Given a metric space M, let $x_0, \ldots, x_k \in M$. Write $\mu = \sum_{i=0}^k \lambda_i \delta_{x_i}$, where $\lambda_i > 0$ and $\sum_{i=0}^k \lambda_i = 1$. Then, for all $y \in M$, we have $\int_M d(x, y)\mathrm{d}\mu = \sum_{i=0}^k \lambda_i d(x_i, y) < \infty$. This fact, together with elementary properties of probability spaces, proves:

Fact 2.5.1. *The measure* μ*, as defined above, belongs to* $\mathscr{P}(M)$.

Therefore, we may define the following:

Definition 2.5.2. (Definition 3.1 of [2]) Let (M, d) be a metric space and $r \geq 0$. The *Vietoris–Rips thickening* is the following set,

$$\mathrm{VR}^m_{\leq}(M; r) = \left\{ \sum_{i=0}^{k} \lambda_i \delta_{x_i} \,\middle|\, \mathrm{diam}(\{x_0, \ldots, x_k\} \leq r \right\} \text{ or}$$
$$\mathrm{VR}^m_{<}(M; r) = \left\{ \sum_{i=0}^{k} \lambda_i \delta_{x_i} \,\middle|\, \mathrm{diam}(\{x_0, \ldots, x_k\} < r \right\},$$

equipped with the restriction of the 1-Wasserstein metric. By convention, $\mathrm{VR}^m_{<}(X; r) = \varnothing$ and $\mathrm{VR}^m_{\leq}(X; r) = (X, d)$.

Contrary to the situation for the usual Vietoris–Rips complex, the embedding $X \to \mathrm{VR}^m_{\leq}(M; r)$ into the Vietoris–Rips metric thickening given by $x \mapsto \delta_x$ is continuous (an isometry, in fact). Therefore, we naturally identify points $\sum_{i=0}^{k} \lambda_i \delta_{x_i}$ with $\sum_{i=0}^{k} \lambda_i x_i$ and consider X to be a subset of $\mathrm{VR}^m_{\leq}(M; r)$. In fact, more is true:

Lemma 2.5.3. $\mathrm{VR}^m_{\leq}(M; r)$ *is an* r*-thickening of* X*.*

Proof. We follow the proof of Lemma 3.5 of [2]. It is clear that the 1-Wasserstein metric extends the metric on X. Additionally, given $\sum_i \lambda_i x_i \in \mathrm{VR}^m_{\leq}(M; r)$, note

$$d\left(\sum_i \lambda_i x_i, X\right) \leq d\left(\sum_i \lambda_i x_i, x_0\right) = \sum_i \lambda_i d(x_i, x_0) \leq r \sum_i \lambda_i = r.$$

□

A useful reinterpretation of the 1-Wasserstein metric on $\mathrm{VR}^m(X; r)$ is as follows: given $x, x' \in \mathrm{VR}^m(X; r)$ with $x = \sum_{i=0}^{k} \lambda_i x_i$ and $x' = \sum_{j=0}^{k'} \lambda'_j x'_j$, define a *matching* p between x and x' to be any collection of non-negative real numbers $\{p_{i,j}\}_{i,j}$ such that $\sum_{j=0}^{k'} p_{i,j} = \lambda_i$ and $\sum_{i=0}^{k} p_{i,j} = \lambda'_j$. Then, define the *cost of the matching* p to be $\mathrm{cost}(p) = \sum_{i,j} p_{i,j} \tilde{d}(x_i, x'_j)$, where $\tilde{d}$ denotes the original metric on X.

Fact 2.5.4. *The* 1*-Wasserstein metric* d *on* $\mathrm{VR}^m(X;r)$ *satisfies*

$$d(x,x') = \inf\{\text{cost}(p) \mid p \text{ is a matching between } x \text{ and } x'\}$$

for all $x, x' \in \mathrm{VR}^m(X;r)$*. In general, this is an equivalent definition to Definition 2.4.7 whenever* x *and* x' *have finite support.*

2.6 Convex Geometry

Convex geometry is the study of convex sets, especially polytopes and their facial structures. We say a subset $X \subseteq \mathbb{R}^n$ is *convex* if $\{\lambda x + (1-\lambda)y \mid 0 \leq \lambda \leq 1\} \subseteq X$ for all $x, y \in X$. In other words, X is convex if and only if the line segment joining any two points of X is contained in X. Given an arbitrary subset $Y \subseteq \mathbb{R}^n$, we may construct the set

$$\mathrm{Conv}(Y) = \left\{ \sum_{i=1}^{k} \lambda_i v_i \,\middle|\, k \in \mathbb{N}, v_i \in Y, \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\},$$

called the *convex hull* of Y. This set is the unique minimal convex set containing Y, i.e., given any convex set Z with $Y \subseteq Z$, we have $\mathrm{Conv}(Y) \subseteq Z$.

Given two vectors $v, p \in \mathbb{R}^n$ with n nonzero, we define

$$H(v,p) = \{x \in \mathbb{R}^n \mid \langle v, x-p \rangle = 0\}$$

to be the *hyperplane* through p with normal vector v. One may show that the set $\mathbb{R}^n \setminus H(v,p)$ is given by two disconnected regions, and we therefore define

$$H(v,p)^+ = \{x \in \mathbb{R}^n \mid \langle v, x-p \rangle \geq 0\},$$

called the *upper halfspace* of $\mathbb{R}^n$ determined by $H(v,p)$. We say a hyperplane $H(v,p)$ *supports* a set $X \subseteq \mathbb{R}^n$, and is a *supporting hyperplane* of X, if $X \subseteq H(v,p)^+$.

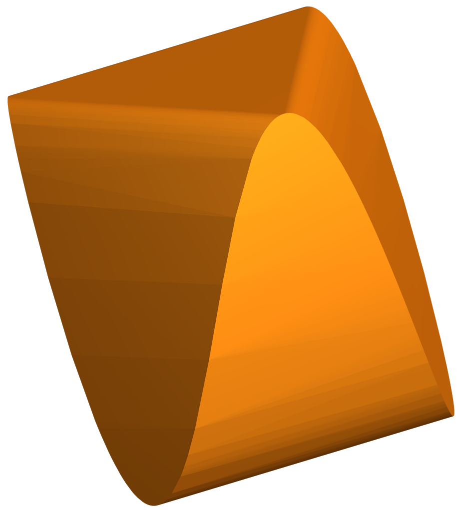

Figure 2.1: The convex hull of the set of points $\{(\cos(t), \sin(t), \cos(3t))\}$ in $\mathbb{R}^3$.

Definition 2.6.1. Given a convex set $Y \subseteq \mathbb{R}^n$, define a *face* F of Y to be any nonempty intersection of Y with a supporting hyperplane of Y. In other words, $F \subseteq Y$ is a face of Y if and only if there exist $v, p \in \mathbb{R}^n$ with v nonzero such that

$$\varnothing \neq F = Y \cap H(v,p) \qquad \text{and} \qquad Y \subseteq H(v,p)^+.$$

Under this definition, F is often called a *proper face* of Y.

Example 2.6.2. Figure 2.1 shows the convex hull of the image of the map $f\colon \mathbb{R} \to \mathbb{R}^3$ defined by $t \mapsto (\cos(t), \sin(t), \cos(3t))$. The proper faces of this convex set are

- the 0-dimensional faces (vertices) $f(t)$ for $t \in [0, 2\pi)$,
- the 1-dimensional faces (edges) $\mathrm{Conv}(f(\{t_0 + \varepsilon, t_0 - \varepsilon\}))$, for $t_0 \in \{0, \frac{\pi}{3}, \frac{2\pi}{3}, \ldots, \frac{5\pi}{3}\}$, and $0 < \varepsilon < \frac{2\pi}{6}$, and
- the 2-dimensional faces (triangles) $\mathrm{Conv}(f(\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}))$ and $\mathrm{Conv}(f(\{\frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}\}))$.

2.7 Conventions Regarding S^1

We define the *circle of radius one* S^1 to be the image of the map $S^1\colon \mathbb{R} \to \mathbb{R}^2$ defined by $t \mapsto (\cos(t), \sin(t))$. Where the meaning is clear, we may write S^1 for both the map and its image. We do this for convenience, as we will often lift a point $S^1(t)$ on the circle to a point $\mathrm{SM}_{2k}(t)$ on the centrally symmetric moment curve, defined below. Because $S^1(t) = S^1(t+2\pi)$, we identify S^1 with $\mathbb{R}/2\pi\mathbb{Z}$, where the positive orientation on $\mathbb{R}$ corresponds to the counterclockwise orientation on S^1. Finally, we equip S^1 with the geodesic metric (of total circumference 2π), though our results also hold when S^1 is instead equipped with the restriction of the Euclidean metric on $\mathbb{R}^2$.

Unless otherwise stated, we will always take a representative $t \in S^1$ belonging to $[0, 2\pi)$.

2.8 Moment Curves

In $\mathbb{R}^d$, the moment curve is the algebraic curve $\mathbf{x}\colon \mathbb{R} \to \mathbb{R}^d$ defined by

$$\mathbf{x}(t) = (t, t^2, \ldots, t^d).$$

Motivating the consideration of this function is the combinatorial structure associated to the convex hull of $n > d \geq 2$ distinct points on the curve. In this context, the d-dimensional cyclic polytope with n vertices is defined by

$$C_d(n) = \mathrm{Conv}\left\{\mathbf{x}(t_1), \ldots, \mathbf{x}(t_n)\right\},$$

for $t_1 < t_2 < \cdots < t_n$. It has been shown that the combinatorial structure of this polytope is independent of the particular values of t_i chosen [24]. Another notable combinatorial property of this polytope is its $d/2$-*neighborliness*: every set of $d/2$ vertices or less defines a face of C_d. Furthermore, the *upper bound theorem* states that among all polytopes in $\mathbb{R}^d$ with n vertices, the cyclic polytopes achieve the maximal possible number of $(d-1)$-dimensional faces [24].

A related construction is given by the so-called trigonometric moment curve $M_{2k}\colon \mathbb{R} \to \mathbb{R}^{2k}$ defined by

$$M_{2k}(t) = (\cos(t), \sin(t), \cos(2t), \sin(2t), \ldots, \cos(kt), \sin(kt)),$$

assuming $d = 2k$. The associated polytope given by the convex hull of $n > d$ points on M_{2k} has been proven by Gale [11] to be combinatorially equivalent to $C_d(n)$.

Chapter 3

Related Work

3.1 Results of Hausmann and Latschev

Vietoris–Rips complexes are often used in applications of persistent homology due to their ease of computation and accompanying theoretical guarantees. Given a finite dataset sampled from an underlying unknown manifold M, what properties of M can be recovered from this finite sampling? The use of Vietoris–Rips complexes in computational topology is justified largely by the theorems of Hausmann and Latschev, which provide theoretical guarantees that one may recover the homotopy type of M from the Vietoris–Rips complex on M or on a sampling of M.

Given a manifold M, one may expect $\mathrm{VR}(M; r)$ to achieve the homotopy type of M for some scale parameter $r \geq 0$. The following theorem, due to Hausmann [14, Theorem 3.5], provides an affirmative answer for sufficiently nice manifolds and r sufficiently small.

Theorem 3.1.1. *Let M be a Riemannian manifold with positive injectivity radius and bounded sectional curvature. Then $\mathrm{VR}(M; \varepsilon) \simeq M$ for sufficiently small $\varepsilon > 0$ depending on the curvature of M.*

Hausmann's proof of Theorem 3.1.1 depends on a noncanonical choice of a total ordering of all the points of M. In addition, because $\mathrm{VR}(M; r)$ is not metrizable if it is not locally finite, the natural inclusion $M \hookrightarrow \mathrm{VR}(M; r)$ is not in general continuous.

Latschev provides the following generalization of Hausmann's theorem in [16]:

Theorem 3.1.2. *Given a closed Riemannian manifold M, for $\varepsilon > 0$ sufficiently small there exists a $\delta > 0$ such that $\mathrm{VR}(X; \varepsilon) \simeq M$ for all metric spaces X δ-close to M in the Gromov–Hausdorff distance.*

The precise value of ε for which the theorem applies again depends upon the curvature of the manifold. Note that, in particular, Theorem 3.1.2 applies to *finite samplings* $X \subseteq M$ sufficiently close to M in Gromov–Hausdorff distance—in practice, a finite set X is chosen to be the vertex set of a Vietoris–Rips complex.

3.2 Metric Reconstruction via Optimal Transport

Our work determines the homotopy type of the Vietoris–Rips metric thickening of the circle for particular scale parameters. The construction of this metric thickening was first introduced by Adamaszek, Adams, and Frick in [2]. Here, we give a brief summary of their work.

3.2.1 Vietoris–Rips Metric Thickenings

Let X denote a metric space and let $r > 0$. The Vietoris–Rips simplicial complex, $\mathrm{VR}(X; r)$, is a common construction used to associate a topological space to X, traditionally considered in the case that X is a finite dataset or sampling of a manifold. However, in the case that X is not finite, it is impossible to equip $\mathrm{VR}(X; r)$ with a metric without changing the homeomorphism type. This means that $\mathrm{VR}(X; r)$ necessarily destroys the metric information about the underlying space X. This motivates the consideration of the Vietoris–Rips metric thickening, $\mathrm{VR}^m(X; r)$, as defined above in Definition 2.5.2, which does preserve metric information about X.

The most essential properties of the Vietoris–Rips metric thickening are as follows:

Theorem 3.2.1 (Main Theorem of [2])**.** *Let X be a metric space and $r > 0$.*

1. *Metric space $\mathrm{VR}^m(X; r)$ is an r-thickening of X; in particular the Gromov–Hausdorff distance between X and $\mathrm{VR}^m(X; r)$ is at most r.*

2. *If $\mathrm{VR}(X; r)$ is locally finite, then $\mathrm{VR}^m(X; r)$ is homeomorphic to $\mathrm{VR}(X; r)$.*

3. *If X is discrete, then $\mathrm{VR}^m(X; r)$ is homotopy equivalent to $\mathrm{VR}(X; r)$.*

4. *If M is a complete Riemannian manifold with curvature bounded from above and below, then $\mathrm{VR}^m(M; r)$ is homotopy equivalent to M for r sufficiently small.*

Item (4) is an analogue of Hausmann's theorem (cf. [14]). Remarkably, the homotopy equivalence $\mathrm{VR}^m(M; r) \to M$ is canonically defined in this setting, in contrast to Hausmann's noncanonical homotopy equivalence $\mathrm{VR}(M;r) \to M$ depending on a total ordering of all the points of M. Additionally, the homotopy inverse is given by the now-continuous inclusion $M \hookrightarrow \mathrm{VR}^m(M;r)$.

3.3 Vietoris–Rips complexes at large scale parameters

While the theorems of Hausmann, Latschev, and Adamsezek et al. describe conditions under which the homotopy type of a manifold is recoverable from a Vietoris–Rips simplicial or metric construction for sufficiently small $r > 0$, much less is known about the topological behavior of these constructions for large values of r. Further, the precise value of "sufficiently small" r is often unknown in practice, and large values of r commonly arise in applications of persistent homology.

An important result toward understanding the topology of such constructions for large values of r is given by Adamaszek and Adams in [1], in which the authors determine the homotopy type of $\mathrm{VR}(S;r)$ for all values of r. Below, we provide a summary of their results.

3.3.1 The Vietoris–Rips Complexes of a Circle

Let $\mathfrak{c}$ denote the cardinality of the continuum. We have the following:

Theorem 3.3.1 (Theorems 7.4 and 7.6 of [1]). *Let $0 < r < \pi$. There are homotopy equivalences*

$$\mathrm{VR}_<(S^1;r) \simeq S^{2l+1} \quad \text{if } \tfrac{2\pi l}{2l+1} < r \leq \tfrac{2\pi(l+1)}{2l+3},$$

$$\mathrm{VR}_\leq(S^1;r) \simeq \begin{cases} S^{2l+1} & \text{if } \frac{2\pi l}{2l+1} < r < \frac{2\pi(l+1)}{2l+3}, \\ \bigvee^{\mathfrak{c}} S^{2l} & \text{if } r = \frac{2\pi l}{2l+1}, \end{cases}$$

where $l = 0, 1, 2, \ldots$.

In particular, as r increases, $\mathrm{VR}(S^1;r)$ obtains the homotopy type of odd-dimensional spheres S^1, S^3, S^5, …, until it becomes contractible. Additionally, at critical values of r, $\mathrm{VR}_\leq(S^1;r)$ obtains the homotopy type of an infinite wedge sum of an even-dimensional sphere.

To establish this result, the authors of [1] use cyclic graphs and an associated numerical invariant called the *winding fraction* to compute the homotopy type of a clique complex of a cyclic graph. These results are first applied to Vietoris–Rips complexes of samplings of points of S^1, and then extended to Vietoris–Rips complexes of arbitrary (potentially infinite) subsets of S^1.

Notably, an analogous description of the homotopy types achieved by the the Čech complexes $\check{\mathrm{C}}_{\leq}(S^1; r)$ and $\check{\mathrm{C}}_{<}(S^1; r)$ is also determined in [1].

Theorem 3.3.1 is, to our knowledge, the first computation of the homotopy types of $\mathrm{VR}(M; r)$ for a non-contractible connected manifold M at arbitrary values of r.

Adamaszek, Adams, and Reddy [3] have also determined the homotopy type of the Vietoris–Rips simplicial complex of an ellipse of sufficiently small eccentricity for a range of scale parameters r (bounded above by a constant depending on the eccentricity of the ellipse).

3.4 A Centrally Symmetric Version of the Cyclic Polytope

The combinatorial structure associated to polytopes of moment curves, and in particular, their neighborliness, arises in the context of linear optimization. In specific, certain underdetermined systems $Ax = y$ given by combinatorial optimization problems often have solutions associated to centrally symmetric k-neighborly polytopes [10]. Additionally, Donoho [8, 9] has shown a connection between sparse solutions of l^1 optimization problems and the construction of certain k-neighborly polytopes for sufficiently large k.

The centrally symmetric moment curve is analogous to the trigonometric moment curve (Section 2.8), with the additional property that is symmetric under reflecting through the origin.

Definition 3.4.1. For $k \in \mathbb{N}$, the *centrally symmetric moment curve* $\mathrm{SM}_{2k} \colon \mathbb{R} \to \mathbb{R}^{2k}$ is defined by

$$\mathrm{SM}_{2k}(t) = (\cos t, \sin t, \cos 3t, \sin 3t, \ldots, \cos(2k-1)t, \sin(2k-1)t).$$

Because $\mathrm{SM}_{2k}(t) = \mathrm{SM}_{2k}(t + 2\pi)$, SM_{2k} defines a map $\mathrm{SM}_{2k} \colon S^1 \to \mathbb{R}^{2k}$, where we identify the domain S^1 with $\mathbb{R}/2\pi\mathbb{Z}$.

Note that $\mathrm{SM}_{2k}(t + \pi) = -\mathrm{SM}_{2k}(t)$. Therefore, we say SM_{2k} is *centrally symmetric* about the origin.

The connection to linear optimization has motivated the study of the combinatorial properties of *bicyclic polytopes,* defined for $X = \{t_1, \ldots, t_n\} \subseteq S^1$ by

$$\mathcal{B}_{2k}(X) = \text{Conv}\{\text{SM}_{2k}(t_1), \ldots, \text{SM}_{2k}(t_n)\}$$

(see Section 3.5) by Barvinok and Novik in [6]. The authors of [6] note that such a bicyclic polytope is centrally symmetric, i.e., x belongs to the polytope if and only if $-x$ belongs to the polytope, in the case that X is a centrally symmetric subset of the circle S^1. Additionally, they conjecture that, among all d-dimensional centrally symmetric polytopes with n vertices, the number of faces of centrally symmetric $\mathcal{B}_{2k}(X)$ asymptotically approaches the largest number of faces in every dimension as n grows and d is fixed.

3.5 Barvinok–Novik Orbitopes

The centrally symmetric orbitope, also called the Barvinok–Novik orbitope [6], is defined by

$$\mathcal{B}_{2k} = \text{Conv}(\text{SM}_{2k}(S^1)) \subseteq \mathbb{R}^{2k}.$$

This convex body is not the convex hull of a finite set of points; it is an *orbitope* instead of a polytope [19]. While the facial structure of $\mathcal{B}_{2k}$ is in general an open question, the faces of $\mathcal{B}_4$ are known.

Let $B \subset \mathbb{R}^{2k}$ denote a convex body. Faces of B are defined by the intersection of B with a supporting hyperplane, i.e., the intersection of B with the zero-set of an affine function A on $\mathbb{R}^{2k}$ satisfying $A(x) \geq 0$ for all $x \in B$. The value of an affine function $A(x)$ on SM_{2k} is represented by a trigonometric polynomial

$$A(t) = c + \sum_{j=1}^{k} a_j \cos[(2j-1)t] + \sum_{j=1}^{k} b_j \sin[(2j-1)t].$$

Following section 3.3 of [6], substitute $z = e^{it}$ into $A(t)$ to write

$$A(t) = z^{2k-1} D(z),$$

where

$$D(z) = cz^{2k-1} + \sum_{j=1}^{k} \frac{a_j - ib_j}{2} z^{2j+2k-2} + \sum_{j=1}^{k} \frac{a_j + ib_j}{2} z^{2k-2j} = cz^{2k-1} + \sum_{j=0}^{2k-1} d_{2j} z^{2j}.$$

One may check that $D(z)$ satisfies

$$D(z) = z^m \overline{D(1/\overline{z})}, \qquad \text{where } m = 4k - 2.$$

Polynomials satisfying this relation are referred to as *self-inversive polynomials*, and have been studied for their connections to complex numbers and number theory, as well as, more recently, algebraic curves, coding theory, and reduction theory of binary forms (see [17] and [15], respectively). Because all odd terms, excluding possibly the middle term, of $D(z)$ vanish, such a polynomial is called a *raked self-inversive polynomial* by the authors of [6].

With this in mind, we see that the faces of $\mathcal{B}_{2k}$ are defined by raked self-inversive polynomials of degree at most $4k-2$ whose roots of modulus one have even multiplicity. In particular, given such a polynomial $D(z)$ with roots $\{e^{it_0}, \ldots, e^{it_s}\}$ of modulus one, $\mathrm{Conv}\{\mathrm{SM}_{2k}(t_0), \ldots, \mathrm{SM}_{2k}(t_s)\}$ defines a face of $\mathcal{B}_{2k}$, and each face arises in this way.

In the case $k = 2$, it is possible to use properties of raked self-inversive polynomials to determine all possible configurations of roots corresponding to faces of $\mathcal{B}_4$. This establishes the following theorem.

Theorem 3.5.1 (Theorem 4.1 of [6])**.** *The proper faces of* $\mathcal{B}_4$ *are*

- *the 0-dimensional faces (vertices)* $\mathrm{SM}_4(t)$ *for* $t \in S^1$*,*
- *the 1-dimensional faces (edges)* $\mathrm{Conv}(\mathrm{SM}_4(\{t_1, t_2\}))$ *where* $t_1 \neq t_2$ *are the edges of an arc of* S^1 *of length less than* $\frac{2\pi}{3}$*, and*
- *the 2-dimensional faces (triangles)* $\mathrm{Conv}(\mathrm{SM}_4(\{t, t + \frac{2\pi}{3}, t + \frac{4\pi}{3}\}))$ *for* $t \in S^1$.

Note that *some* of the faces of $\mathcal{B}_4$ are visible in the convex hull in Figure 2.1, whose underlying curve is one projection of the image of SM_4 from $\mathbb{R}^4$ to $\mathbb{R}^3$.

While the precise facial structure of $\mathcal{B}_{2k} = \mathcal{B}_{2k}(S^1)$ is not known for $k > 2$, certain neighborliness-results have been established by Barvinok, Lee, and Novik:

Theorem 3.5.2 (Theorem 1.1 of [5]). *For every positive integer k there exists a number*

$$\frac{\pi}{2} < \phi_k < \pi$$

such that for an arbitrary open arc $\Gamma \subseteq S^1$ of length ϕ_k and arbitrary distinct $n < k$ points $t_0, \ldots, t_n \in \Gamma$, the set

$$\mathrm{Conv}\left(\{\mathrm{SM}_{2k}(t_0), \ldots, \mathrm{SM}_{2k}(t_n)\}\right)$$

is a face of $\mathcal{B}_{2k}$.

Additionally, Vinzant has proven the following regarding edges of $\partial\mathcal{B}_{2k}$:

Theorem 3.5.3 (Theorem 1 of [23]). *For $\alpha \neq \beta \in [0, 2\pi]$, the line segment*

$$\mathrm{Conv}\left(\{\mathrm{SM}_{2k}(\alpha), \mathrm{SM}_{2k}(\beta)\}\right)$$

is an exposed edge of $\mathcal{B}_{2k}$ if $|\alpha - \beta| < \frac{2\pi(k-1)}{2k-1}$, and not an edge of $\mathcal{B}_{2k}$ if $|\alpha - \beta| > \frac{2\pi(k-1)}{2k-1}$.

Presently, we explore the close relationship between the facial structure of $\mathcal{B}_{2k}$ and the structure of $\mathrm{VR}^m(S^1; r)$ for particular values of r and k. This relationship will be made explicit in Chapter 4.

Chapter 4

Results

4.1 Outline of the proof of the Main Theorem.

We are now prepared to state and prove our main result, Theorem 4.1.1.

Theorem 4.1.1. *Let $r = \frac{2\pi}{3}$ denote the side-length of an equilateral triangle inscribed in S^1. Then,* $\mathrm{VR}^m_{\le}(S^1; r) \simeq S^3$.

To prove this result, we construct homotopy equivalences between $\mathrm{VR}^m_{\le}(S^1; 2\pi/3)$ and $\partial\mathcal{B}_4 \cong S^3$ given by the maps $p \circ \mathrm{SM}_4$ and ι in the following diagram:

$$\mathrm{VR}^m_{\le}(S^1; 2\pi/3) \xrightarrow{\mathrm{SM}_4} \mathbb{R}^4 \setminus \{\vec{0}\} \xrightarrow{p} \partial\mathcal{B}_4 \xrightarrow{\iota} \mathrm{VR}^m_{\le}(S^1; 2\pi/3).$$

This construction will proceed as follows.

1. Define the radial projection $p\colon \mathbb{R}^4 \setminus \{\vec{0}\} \to \partial\mathcal{B}_4$. Extend the domain of SM_4 to $\mathrm{VR}^m_{\le}(S^1; r)$, and prove that the image of SM_4 misses the origin in $\mathbb{R}^4$, so that the composition $p \circ \mathrm{SM}_4$ is well-defined.

2. Define the inclusion $\iota\colon \partial\mathcal{B}_4 \to \mathrm{VR}^m_{\le}(S^1; r)$

3. Prove that $p \circ \mathrm{SM}_4$ and ι are homotopy inverses.

4.2 Extend the domain of SM_4 to $\mathrm{VR}^m_{\le}(S^1; r)$, and show the image of SM_4 misses the origin

Define the radial projection map $p_{2k}\colon \mathbb{R}^{2k} \setminus \{\vec{0}\} \to \partial\mathcal{B}_{2k} \simeq S^{2k-1}$. As $\mathcal{B}_{2k}$ is a convex body containing the origin in its interior, each ray emanating from the origin intersects $\partial\mathcal{B}_{2k}$ exactly once. Hence, p_{2k} is well-defined. Throughout, we write $p = p_4$.

In general, we may extend $\mathrm{SM}_{2k}\colon S^1 \to \mathbb{R}^{2k}$ to $\mathrm{SM}_{2k}\colon \mathrm{VR}^m_{\leq}(S^1;r) \to \mathbb{R}^{2k}$ via

$$\sum_i \lambda_i S^1(\theta_i) \mapsto \sum_i \lambda_i \mathrm{SM}_{2k}(\theta_i).$$

Because SM_{2k} restricted to S^1 is continuous and bounded, Lemma 5.2 of [2] proves that the extension to all of $\mathrm{VR}^m_{\leq}(S^1;r)$ is continuous.

Next, observe that the composition $p \circ \mathrm{SM}_4\colon \mathrm{VR}^m_{\leq}(S^1;r) \to \partial\mathcal{B}_4$ is well-defined if and only if $\vec{0} \notin \mathrm{Im}(\mathrm{SM}_4\colon \mathrm{VR}^m_{\leq}(S^1;r) \to \mathbb{R}^4)$.

Throughout this section, we fix $k \in \mathbb{N}$ and prove instead the general case:

Theorem 4.2.1. *Given* $0 \leq t_0 < t_1 < \cdots < t_{2k} < 2\pi$ *such that* $\mathrm{diam}(\{t_0,\ldots,t_{2k}\}) < \mathcal{C} < \pi$, $\mathrm{Conv}\{\mathrm{SM}_{2k}(t_0),\ldots,\mathrm{SM}_{2k}(t_{2k})\}$ *does not contain* $\vec{0}$ *if* $\mathcal{C} \leq \frac{2\pi k}{2k+1}$, *and this bound is sharp.*

Therefore, $p \circ \mathrm{SM}_4$ is well-defined for $r < \mathcal{C} = \frac{4\pi}{5}$.

To prove Theorem 4.2.1, we may restrict attention to simplices of $\mathrm{VR}^m_{\leq}(S^1;r)$ of dimension $2k$ or less by Carathéodory's theorem. Therefore, suppose $\{t_0,\ldots,t_{2k}\} \subset S^1$ is such that the origin is contained in the convex hull of $\{\mathrm{SM}_{2k}(t_0),\ldots,\mathrm{SM}_{2k}(t_{2k})\}$. Then, there exist scalars $\lambda_i \geq 0$ such that $\vec{0} = \sum_{i=0}^{2k} \lambda_i \mathrm{SM}_{2k}(t_i)$ and $\sum_{i=0}^{2k} \lambda_i = 1$. In this way, we obtain a system of $2k$ equations:

$$\sum_{i=0}^{2k} \lambda_i \cos(nt_i) = 0 \quad \text{for} \quad n = 1,3,\ldots,2k-1, \text{ and}$$
$$\sum_{i=0}^{2k} \lambda_i \sin(nt_i) = 0 \quad \text{for} \quad n = 1,3,\ldots,2k-1.$$

Therefore, let

$$M_{2k} = \begin{pmatrix} \cos(t_0) & \cos(t_1) & \dots & \cos(t_{2k}) \\ \sin(t_0) & \sin(t_1) & \dots & \sin(t_{2k}) \\ \cos(3t_0) & \cos(3t_1) & \dots & \cos(3t_{2k}) \\ \sin(3t_0) & \sin(3t_1) & \dots & \sin(3t_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ \cos((2k-1)t_0) & \cos((2k-1)t_1) & \dots & \cos((2k-1)t_{2k}) \\ \sin((2k-1)t_0) & \sin((2k-1)t_1) & \dots & \sin((2k-1)t_{2k}) \end{pmatrix}$$

and consider the vector equation $M_{2k}\vec{\lambda} = \vec{0}$. As a step toward proving Theorem 4.2.1, we first describe the nullspace of M_{2k}. To obtain this description, it will be useful to state the following proposition, which for example can be found in [18, Section 2.8.1].

Proposition 4.2.2. *An $n \times n$ matrix of the form*

$$V = \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix}$$

satisfies

$$\det(V) = \prod_{1 \le i < j \le n} (a_j - a_i).$$

A matrix of this form is called a *Vandermonde matrix.*

Lemma 4.2.3. *Let A denote the $2k \times 2k$ matrix whose columns are* $\mathrm{SM}_{2k}(t_1), \mathrm{SM}_{2k}(t_2), \dots, \mathrm{SM}_{2k}(t_{2k})$. *Then*

$$\det(A) = \kappa \left(\prod_{1 \le j < l \le 2k} \sin(t_l - t_j) \right)$$

for some nonzero constant κ depending only on k.

Proof. We would like to thank Harrison Chapman for key insights behind this proof. Given $f\colon \mathbb{R} \to \mathbb{C}$, define the function $f\colon \mathbb{R}^{2k} \to \mathbb{C}^{2k}$ via

$$f(\underline{t}) = \begin{pmatrix} f(t_1) & f(t_2) & \cdots & f(t_{2k}) \end{pmatrix}^{\mathrm{T}} \in \mathbb{C}^{2k}.$$

To prove this lemma, we will perform elementary row and column operations to A to obtain a Vandermonde matrix. In order to clarify the proof of the general case, we consider first the case $k = 2$, with

$$A = \begin{pmatrix} \cos(t_1) & \cos(t_2) & \cos(t_3) & \cos(t_4) \\ \sin(t_1) & \sin(t_2) & \cos(t_3) & \sin(t_4) \\ \cos(3t_1) & \cos(3t_2) & \cos(t_3) & \cos(3t_4) \\ \sin(3t_1) & \sin(3t_2) & \cos(t_3) & \sin(3t_4) \end{pmatrix}.$$

Write

$$A^{\mathrm{T}} = \begin{pmatrix} \cos(\underline{t}) & \sin(\underline{t}) & \cos(3\underline{t}) & \sin(3\underline{t}) \end{pmatrix}$$

for $\underline{t} = (t_1, t_2, t_3, t_4)^T$, and observe that

$$\begin{aligned} \det(A) &= \det\left(A^{\mathrm{T}}\right) \\ &= \det\begin{pmatrix} \frac{e^{i\underline{t}}+e^{-i\underline{t}}}{2} & \frac{e^{i\underline{t}}-e^{-i\underline{t}}}{2i} & \frac{e^{3i\underline{t}}+e^{-3i\underline{t}}}{2} & \frac{e^{3i\underline{t}}-e^{-3i\underline{t}}}{2i} \end{pmatrix} \\ &= \frac{1}{2^4}(-i)^2 \det\begin{pmatrix} e^{i\underline{t}}+e^{-i\underline{t}} & e^{i\underline{t}}-e^{-i\underline{t}} & e^{3i\underline{t}}+e^{-3i\underline{t}} & e^{3i\underline{t}}-e^{-3i\underline{t}} \end{pmatrix}. \end{aligned}$$

Next, let C_i denote the i-th column of the above matrix and perform the following column operations:

$$C_1 \mapsto C_1 + C_2 \qquad C_2 \mapsto C_2 - \frac{1}{2}C_1 \qquad C_3 \mapsto C_3 + C_4 \qquad C_4 \mapsto C_4 - \frac{1}{2}C_3.$$

It follows that

$$\det(A) = -\frac{1}{2^4}\det\begin{pmatrix}2e^{i\underline{t}} & -e^{-i\underline{t}} & 2e^{3i\underline{t}} & -e^{-3i\underline{t}}\end{pmatrix}$$
$$= -\frac{1}{2^2}\det\begin{pmatrix}e^{i\underline{t}} & e^{-i\underline{t}} & e^{3i\underline{t}} & e^{-3i\underline{t}}\end{pmatrix}$$

by factoring out column multiples. Letting $\omega = e^{-3i(t_1+t_2+t_3+t_4)}$, we may factor e^{-3it_j} from row j to obtain

$$\det(A) = -\frac{1}{4}\omega\det\begin{pmatrix}e^{4i\underline{t}} & e^{2i\underline{t}} & e^{6i\underline{t}} & \underline{1}\end{pmatrix}.$$

Re-ordering rows by the positive permutation $(4\ 1\ 3)$ gives

$$\det(A) = -\frac{1}{4}\omega\det\begin{pmatrix}\underline{1} & e^{2i\underline{t}} & e^{4i\underline{t}} & e^{6i\underline{t}}\end{pmatrix}$$
$$= -\frac{1}{4}\omega\prod_{1\le i<j\le 4}\left(e^{2it_j} - e^{2it_i}\right)$$

by Proposition 4.2.2. Finally, into each factor $(e^{2it_j} - e^{2it_i})$, extract a factor of $e^{i(t_j+t_i)}$ from ω to obtain

$$\det(A) = -\frac{1}{4}\prod_{1\le i<j\le 4}\left(e^{i(t_j-t_i)} - e^{i(t_j-t_i)}\right)$$
$$= -\frac{1}{4}\prod_{1\le i<j\le 4}2i\sin(t_j-t_i)$$
$$= -\frac{(2i)^6}{4}\prod_{1\le i<j\le 4}\sin(t_j-t_i)$$
$$= 16\prod_{1\le i<j\le 4}\sin(t_j-t_i).$$

Next, we consider the general case, with

$$A = \begin{pmatrix} \cos(t_1) & \cos(t_2) & \dots & \cos(t_{2k}) \\ \sin(t_1) & \sin(t_2) & \dots & \sin(t_{2k}) \\ \cos(3t_1) & \cos(3t_2) & \dots & \cos(3t_{2k}) \\ \sin(3t_1) & \sin(3t_2) & \dots & \sin(3t_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ \cos((2k-1)t_1) & \cos((2k-1)t_2) & \dots & \cos((2k-1)t_{2k}) \\ \sin((2k-1)t_1) & \sin((2k-1)t_2) & \dots & \sin((2k-1)t_{2k}) \end{pmatrix}.$$

Write

$$A^{\mathrm{T}} = \begin{pmatrix} \cos(\underline{t}) & \sin(\underline{t}) & \cos(3\underline{t}) & \sin(3\underline{t}) & \cdots & \cos((2k-1)\underline{t}) & \sin((2k-1)\underline{t}) \end{pmatrix}$$

for $\underline{t} = (t_1, t_2, \dots, t_{2k})^T$, and observe that

$$\begin{aligned} \det(A) &= \det\left(A^{\mathrm{T}}\right) \\ &= \det\begin{pmatrix} \frac{e^{i\underline{t}}+e^{-i\underline{t}}}{2} & \frac{e^{i\underline{t}}-e^{-i\underline{t}}}{2i} & \frac{e^{3i\underline{t}}+e^{-3i\underline{t}}}{2} & \frac{e^{3i\underline{t}}-e^{-3i\underline{t}}}{2i} & \dots & \frac{e^{(2k-1)i\underline{t}}+e^{-(2k-1)i\underline{t}}}{2} & \frac{e^{(2k-1)i\underline{t}}-e^{-(2k-1)i\underline{t}}}{2i} \end{pmatrix} \\ &= \frac{1}{2^{2k}}(-i)^k \det\begin{pmatrix} e^{i\underline{t}}+e^{-i\underline{t}} & e^{i\underline{t}}-e^{-i\underline{t}} & \cdots & e^{(2k-1)i\underline{t}}+e^{-(2k-1)i\underline{t}} & e^{(2k-1)i\underline{t}}-e^{-(2k-1)i\underline{t}} \end{pmatrix}. \end{aligned}$$

Next, let C_i denote the i-th column of the above matrix. For $i = 1, 3, \dots, 2k-1$, perform the following column operations: $C_i \mapsto C_i + C_{i+1}$, $C_{i+1} \mapsto C_{i+1} - \frac{1}{2}C_i$. It follows that

$$\begin{aligned} \det(A) &= \frac{1}{2^{2k}}(-i)^k \det\begin{pmatrix} 2e^{i\underline{t}} & -e^{-i\underline{t}} & 2e^{3i\underline{t}} & -e^{-3i\underline{t}} & \cdots & 2e^{(2k-1)i\underline{t}} & -e^{-(2k-1)i\underline{t}} \end{pmatrix} \\ &= \frac{i^k}{2^k}\det\begin{pmatrix} e^{i\underline{t}} & e^{-i\underline{t}} & e^{3i\underline{t}} & e^{-3i\underline{t}} & \cdots & e^{(2k-1)i\underline{t}} & e^{-(2k-1)i\underline{t}} \end{pmatrix} \end{aligned}$$

by factoring out column multiples. Letting $\omega = e^{-(2k-1)i(t_1+t_2+\cdots+t_{2k})}$, we may factor $e^{-(2k-1)it_j}$ from row j to obtain

$$\det(A) = \frac{i^k}{2^k}\omega \det \begin{pmatrix} e^{((2k-1)+1)i\underline{t}} & e^{((2k-1)-1)i\underline{t}} & \dots & e^{((2k-1)+(2k-1))i\underline{t}} & e^{((2k-1)-(2k-1))i\underline{t}} \end{pmatrix}$$
$$= \frac{i^k}{2^k}\omega \det \begin{pmatrix} e^{2ki\underline{t}} & e^{(2k-2)i\underline{t}} & e^{(2k+2)i\underline{t}} & e^{(2k-4)i\underline{t}} & \dots & e^{2(2k-1)i\underline{t}} & \underline{1} \end{pmatrix}.$$

Re-ordering rows by a permutation σ gives

$$\det(A) = \operatorname{sign}(\sigma)\frac{i^k}{2^k}\omega \det \begin{pmatrix} \underline{1} & e^{2i\underline{t}} & e^{4i\underline{t}} & \dots & e^{(2(2k-1))i\underline{t}} \end{pmatrix}$$
$$= \operatorname{sign}(\sigma)\frac{i^k}{2^k}\omega \prod_{1\le i<j\le 2k} \left(e^{2it_j} - e^{2it_i}\right)$$

by Proposition 4.2.2. Finally, into each factor $(e^{2it_j} - e^{2it_i})$, extract a factor of $e^{i(t_j+t_i)}$ from ω to obtain

$$\det(A) = \operatorname{sign}(\sigma)\frac{i^k}{2^k} \prod_{1\le i<j\le 2k} \left(e^{i(t_j-t_i)} - e^{i(t_j-t_i)}\right)$$
$$= \operatorname{sign}(\sigma)\frac{i^k}{2^k} \prod_{1\le i<j\le 2k} 2i\sin(t_j - t_i)$$
$$= \operatorname{sign}(\sigma)\frac{i^k}{2^k}(2i)^{(2k-1)k} \prod_{1\le i<j\le 2k} \sin(t_j - t_i)$$
$$= \operatorname{sign}(\sigma) i^{2k^2} 2^{2k(k-1)} \prod_{1\le i<j\le 2k} \sin(t_j - t_i)$$
$$= \operatorname{sign}(\sigma) 2^{2k(k-1)} \prod_{1\le i<j\le 2k} \sin(t_j - t_i).$$

This completes the proof of Lemma 4.2.3. □

The following corollary is immediate.

Corollary 4.2.4. *For $0 \le i \le 2k$, let $M_{2k,i}$ denote the $2k \times 2k$ matrix obtained by removing the i-th column of M_{2k}. Then,*

$$\det(M_{2k,i}) = \kappa \left(\prod_{\substack{0\le j<l\le 2k \\ j,l\ne i}} \sin(t_l - t_j) \right),$$

for some nonzero constant κ depending only on k.

Theorem 4.2.5. *The nullspace of M_{2k} is one-dimensional and is spanned by*

$$\vec{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{2k})^T,$$

where

$$\lambda_i = (-1)^i \prod_{\substack{0 \le j < l \le 2k \\ j,l \ne i}} \sin(t_l - t_j).$$

Proof. Because M_{2k} has $2k$ rows and $2k+1$ columns, it has nullity at least one. Further, by Corollary 4.2.4, observe that $M_{2k,0}$ is invertible if and only if no two points t_l, t_j are antipodal for $1 \le l, j \le 2k$. Hence, M_{2k} contains $2k$ linearly independent columns and has nullity exactly one.

Next, we prove $\vec{\lambda}$ is contained in the nullspace of M_{2k}. To ease notation, write

$$M_{2k}\vec{\lambda} = \begin{pmatrix} C_1 & S_1 & C_3 & S_3 & \cdots & C_{2k-1} & S_{2k-1} \end{pmatrix}^{\mathrm{T}}.$$

For $n = 1, 3, 5, \ldots, 2k-1$, note

$$\begin{aligned} C_n = \sum_{i=0}^{2k} \cos(nt_i)\lambda_i &= \sum_{i=0}^{2k} (-1)^i \cos(nt_i) \prod_{\substack{0 \le j < l \le 2k \\ j,l \ne i}} \sin(t_l - t_j) \\ &= \frac{1}{\kappa} \sum_{i=0}^{2k} (-1)^i \cos(nt_i) \det(M_{2k,i}). \end{aligned}$$

Hence, C_n is equal to $\frac{1}{\kappa}$ times the determinant of the matrix

$$\begin{pmatrix} \cos(nt_0) & \cos(nt_1) & \dots & \cos(nt_{2k}) \\ \cos(t_0) & \cos(t_1) & \dots & \cos(t_{2k}) \\ \sin(t_0) & \sin(t_1) & \dots & \sin(t_{2k}) \\ \cos(3t_0) & \cos(3t_1) & \dots & \cos(3t_{2k}) \\ \sin(3t_0) & \sin(3t_1) & \dots & \sin(3t_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ \cos((2k-1)t_0) & \cos((2k-1)t_1) & \dots & \cos((2k-1)t_{2k}) \\ \sin((2k-1)t_0) & \sin((2k-1)t_1) & \dots & \sin((2k-1)t_{2k}) \end{pmatrix}.$$

Since $n = 2j - 1$ for some $1 \leq j \leq k$, the first row of this matrix is equal to one of the other rows. Therefore the matrix is singular, giving that $C_n = 0$.

Similarly, it follows that S_n is equal to $\frac{1}{\kappa}$ times the determinant of

$$\begin{pmatrix} \sin(nt_0) & \sin(nt_1) & \dots & \sin(nt_{2k}) \\ \cos(t_0) & \cos(t_1) & \dots & \cos(t_{2k}) \\ \sin(t_0) & \sin(t_1) & \dots & \sin(t_{2k}) \\ \cos(3t_0) & \cos(3t_1) & \dots & \cos(3t_{2k}) \\ \sin(3t_0) & \sin(3t_1) & \dots & \sin(3t_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ \cos((2k-1)t_0) & \cos((2k-1)t_1) & \dots & \cos((2k-1)t_{2k}) \\ \sin((2k-1)t_0) & \sin((2k-1)t_1) & \dots & \sin((2k-1)t_{2k}) \end{pmatrix}.$$

For the same reasons as before, it follows that $S_n = 0$. $\square$

For convenience, we rescale $\vec{\lambda}$ by $0 \neq \gamma = \prod_{0 \leq j < l \leq 2k} \frac{1}{\sin(t_l - t_j)}$ to obtain

$$\gamma\vec{\lambda} = \left(\frac{1}{\alpha_0(t_0, \dots, t_{2k})}, \dots, \frac{1}{\alpha_{2k}(t_0, \dots, t_{2k})} \right),$$

where

$$\alpha_i(t_0, \dots, t_{2k}) = \prod_{\substack{0 \le j \le 2k \\ j \ne i}} \sin(t_j - t_i).$$

Recall that entries of $\vec{\lambda}$ correspond to coefficients in the linear combination $\vec{0} = \sum_{i=0}^{2k} \lambda_i \mathrm{SM}_{2k}(t_i)$. In particular, we are concerned only with *convex* linear combinations. Hence, after normalizing $\vec{\lambda}$ (and potentially rescaling by -1), it is necessary that each entry λ_i is positive. In other words, the origin may be contained in the convex hull of $\{\mathrm{SM}_{2k}(t_0), \dots, \mathrm{SM}_{2k}(t_{2k})\}$ only in the case that the terms $\alpha_i(t_0, \dots, t_{2k})$ share the same sign.

To relate the sign of each term $\alpha_i(t_0, \dots, t_{2k})$ to the configuration of points $t_0, \dots, t_{2k} \in S^1$, we first prove some intermediate lemmas. Throughout the remainder of this section, assume that the points $t_0, \dots, t_{2k} \in S^1$ are ordered by index with a counterclockwise orientation.

Definition 4.2.6. Let $a, b \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$, where $a \ne b$, and where a and b are each identified with a point in $[0, 2\pi)$. Define the *open arc* $(a, b)_{S^1}$ as follows:

$$(a, b)_{S^1} = \begin{cases} \{t \in S^1 \mid a < t < b\} & \text{if } a < b \\ \{t \in S^1 \mid a < t < b + 2\pi\} & \text{if } a > b. \end{cases}$$

Define the *closed arc* $[a, b]_{S^1}$ similarly.

Definition 4.2.7. Let $t_0, \dots, t_{2k} \in S^1$ be distinct, with no two points antipodal. For $i = 0, \dots, 2k$ define

$$\chi(t_i) = \#\{t_j \mid t_j \in (t_i + \pi, t_i)\}.$$

Lemma 4.2.8. *Let $t_0, \dots, t_{2k} \in S^1$ be distinct, with no two points antipodal. Then, $\sum_{i=0}^{2k} \chi(t_i) = k(2k+1)$.*

Proof. Note that since no two points are antipodal, we have that $t_j \in (t_i + \pi, t_i)$ if and only if $t_i \notin (t_j + \pi, t_j)$. Define

$$\mathbf{1}_{i,j} = \begin{cases} 1 & \text{if } t_j \in (t_i + \pi, t_i) \\ 0 & \text{otherwise.} \end{cases}$$

Note $\mathbf{1}_{i,j} + \mathbf{1}_{j,i} = 1$ for all $i \neq j$, and $\mathbf{1}_{i,i} = 0$ for all i. Therefore we have

$$\sum_{i=0}^{2k} \chi(t_i) = \sum_{i=0}^{2k} \#\{t_j \mid t_j \in (t_i + \pi, t_i)\} = \sum_{i,j=0}^{2k} \mathbf{1}_{i,j} = \frac{(2k+1)(2k)}{2} = k(2k+1).$$

□

Lemma 4.2.9. *Let $t_0, \dots, t_{2k} \in S^1$ be distinct, with no two points antipodal. Then, $\chi(t_i) = \chi(t_j) \pmod 2$ for some $i, j \in \{1, \dots, 2k\}$ if and only if $\operatorname{sign}(\alpha_i(t_0, \dots, t_{2k})) = \operatorname{sign}(\alpha_j(t_0, \dots, t_{2k}))$.*

Proof. Observe that $\operatorname{sign}(\alpha_i(t_0, \dots, t_{2k})) = (-1)^{\chi(t_i)}$. □

Lemma 4.2.10. *Let $t_0, \dots, t_{2k} \in S^1$ be distinct, not all contained in a semicircle, with no two points antipodal. Then, $1 \geq \chi(t_{i+1}) - \chi(t_i)$ for $0 \leq i \leq 2k$, where we set $t_{2k+1} = t_0$.*

Proof. Observe that the open arc $(t_{i+1} + \pi, t_i)_{S^1}$ contains exactly $\chi(t_{i+1}) - 1$ points. Hence, $(t_i + \pi, t_{i+1} + \pi)_{S^1}$ must contain exactly $\chi(t_i) - (\chi(t_{i+1}) - 1)$ points. Because this number is nonnegative, it follows that $1 \geq \chi(t_{i+1}) - \chi(t_i)$. □

Lemma 4.2.11. *Let $t_0, \dots, t_{2k} \in S^1$, with no two points antipodal. Then, the numbers $\alpha_i(t_0, \dots, t_{2k})$ have the same sign for all $0 \leq i \leq 2k$ if and only if $\chi(t_i) = k$ for all i.*

Proof. In the case that $\chi(t_i) = k$ for all i, it is straightforward to verify that the numbers $\alpha_i(t_0, \dots, t_{2k})$ are all positive or are all negative by considering the sign of each constituent sine function.

Conversely, suppose the numbers $\alpha_i(t_0, \dots, t_{2k})$ have the same sign. Then, by Lemma 4.2.9, the numbers $\chi(t_i)$ have the same parity. Further, in the case k is odd (resp. even), Lemma 4.2.8 implies each $\chi(t_i)$ is odd (resp. even). Therefore, in either case, we may write $\chi(t_i) = k + 2n_i$ for some integer $n_i \in \mathbb{Z}$. Note that Lemma 4.2.8 implies

$$k(2k+1) = \sum_{i=0}^{2k} \chi(t_i) = \sum_{i=0}^{2k} k + 2n_i = k(2k+1) + 2\sum_{i=0}^{2k} n_i,$$

and it follows that $\sum_{i=0}^{2k} n_i = 0$. Therefore, it is sufficient to prove that $n_i = n_j$ for all i, j. Toward that end, define $t_{2k+1} = t_0$ and $n_{2k+1} = n_0$ and observe

$$0 = \sum_{i=0}^{2k} n_{i+1} = \sum_{i=0}^{2k}(n_{i+1} + (-n_i + n_i)) = \sum_{i=0}^{2k}((n_{i+1} - n_i) + n_i)$$
$$= \sum_{i=0}^{2k}(n_{i+1} - n_i) + \sum_{i=0}^{2k} n_i$$
$$= \sum_{i=0}^{2k}(n_{i+1} - n_i).$$

It cannot be the case that all of the points t_i are contained in a semicircle, since then $\chi(t_i)$ would obtain all of the values $0, 1, \ldots, 2k$, contradicting the fact that these values have the same parity. Therefore, we may apply Lemma 4.2.10 to obtain

$$1 \geq (k + 2n_{i+1}) - (k + 2n_i) = 2(n_{i+1} - n_i),$$

which implies $0 \geq n_{i+1} - n_i$. Thus, since the points $t_0, \ldots, t_{2k}$ are ordered counterclockwise, it follows that $n_{i+1} = n_i$ for all $0 \leq i \leq 2k$, proving the claim.

□

We are now prepared to prove Theorem 4.2.1.

Proof of Theorem 4.2.1. If $\operatorname{diam}(\{t_0, \ldots, t_{2k}\}) < \mathcal{C} \leq \frac{2\pi k}{2k+1}$, observe that $\chi(t_i) \neq k$ for some $0 \leq i \leq 2k$. Then, Theorem 4.2.5 and Lemma 4.2.11 together imply that there do not exist positive scalars λ_i such that $\vec{0} = \sum_{i=0}^{2k} \lambda_i \mathrm{SM}_{2k}(t_i)$.

To see that this bound is sharp, let $t_i \in S^1$ such that $t_i = t_0 + i\frac{2\pi}{2k+1}$ for $i = 1, \ldots, 2k$. Then, Theorem 4.2.5 and Lemma 4.2.11 imply that there exists a vector of norm one with positive entries contained in the nullspace of M_{2k} (in fact, $\vec{0} = \sum_{i=0}^{2k} \frac{1}{2k+1}\mathrm{SM}_{2k}(t_i)$ in this case). □

Remark 4.2.12. If the orbitopes $\mathcal{B}_{2k}$ were known to be simplicial (as we conjecture to be the case), then together with Remark 4.3.2, Theorem 4.2.1 would imply that $\partial\mathcal{B}_{2l+2} \cong S^{2l+1}$ is a retract of

$\mathrm{VR}^m_{\leq}(S^1;r)$ for $\frac{2\pi l}{2l+1} \leq r < \frac{2\pi(l+1)}{2l+3}$ and $0 \leq l$. It would then follow that the $(2l+1)$-dimensional homology, cohomology, and homotopy groups of $\mathrm{VR}^m_{\leq}(S^1;r)$ are nontrivial for all $0 \leq l$.

4.3 Define the inclusion ι

For $r \geq \frac{2\pi}{3}$, define $\iota\colon \partial\mathcal{B}_4 \to \mathrm{VR}^m_{\leq}(S^1;r)$ as follows: given an n-dimensional face of $\mathcal{B}_4$ of the form $\mathrm{Conv}(\mathrm{SM}_4(\{\theta_0,\ldots,\theta_n\}))$, define

$$\iota\left(\sum_{i=0}^{n} \lambda_i \mathrm{SM}_4(\theta_i)\right) = \sum_{i=0}^{n} \lambda_i S^1(\theta_i).$$

In light of Theorem 3.5.1 ι is well-defined; indeed each face of $\partial\mathcal{B}_4$ has diameter at most $\frac{2\pi}{3}$. We also prove in the following lemma that ι is continuous.

Lemma 4.3.1. *Let $r \geq \frac{2\pi}{3}$. The map $\iota\colon \partial\mathcal{B}_4 \to \mathrm{VR}^m_{\leq}(S^1;r)$ is continuous.*

Proof. We will show that $(p \circ \mathrm{SM}_4)|_{\iota(\partial\mathcal{B}_4)}\colon \iota(\partial\mathcal{B}_4) \to \partial\mathcal{B}_4$ is an bijective continuous function from a compact space to a Hausdorff space. It follows from [4, Theorem 3.7] that $(p \circ \mathrm{SM}_4)|_{\iota(\partial\mathcal{B}_4)}$ is a homeomorphism, with a continuous inverse $\iota\colon \partial\mathcal{B}_4 \to \iota(\partial\mathcal{B}_4)$. Therefore $\iota\colon \partial\mathcal{B}_4 \to \mathrm{VR}^m_{\leq}(S^1;r)$ is continuous.

The fact that $(p \circ \mathrm{SM}_4)|_{\iota(\partial\mathcal{B}_4)}$ is an bijective function follows from Theorem 3.5.1. The space $\partial\mathcal{B}_4$ is Hausdorff since it inherits the subspace topology from Euclidean space. Finally, to see that $\iota(\partial\mathcal{B}_4)$ is compact, we note that $\iota(\partial\mathcal{B}_4)$ is a closed subset $\mathscr{P}(S^1)$, the space of all Radon probability measures on S^1 equipped with the Wasserstein metric. Since S^1 is compact, it follows that $\mathscr{P}(S^1)$ is compact by [22, Remark 6.19], and therefore $\iota(\partial\mathcal{B}_4)$ is compact as a closed subset of a compact space. □

Remark 4.3.2. It follows from Theorem 3.5.1 that $\mathcal{B}_4$ is a *simplicial* orbitope, meaning that all of its faces are simplices. To our knowledge, it is not known whether $\mathcal{B}_{2k}$ is simplicial for $k \geq 3$, although we conjecture this to be the case. If $\mathcal{B}_{2k}$ is simplicial, then an analogous map $\iota\colon \partial\mathcal{B}_{2k} \to \mathrm{VR}^m_{\leq}(S^1;r)$ can be defined for $r \geq \frac{2\pi(k-1)}{2k-1}$; this map will be well-defined by [23, Theorem 1], and continuous by an argument analogous to Lemma 4.3.1.

4.4 Show $p \circ \mathrm{SM}_4$ and ι are homotopy inverses.

Observe that $(p \circ \mathrm{SM}_4) \circ \iota = \mathrm{id}_{\partial \mathcal{B}_4}$. Hence, it remains to show that $\iota \circ (p \circ \mathrm{SM}_4) \simeq \mathrm{id}_{\mathrm{VR}^m_{\leq}(S^1;r)}$.

Theorem 4.4.1. *Let* $x = \sum_{i=0}^n \lambda_i t_i \in \mathrm{VR}^m_{\leq}(S^1; \frac{2\pi}{3})$ *and write* $\iota \circ (p \circ \mathrm{SM}_4)(x) = \sum_{j=0}^m \mu s_j$ *for some* $s_j \in S^1$*. Then,* $\mathrm{diam}(\{t_0, \ldots, t_n\} \cup \{s_0, \ldots, s_m\}) = \mathrm{diam}(\{t_0, \ldots, t_n\})$.

In order to prove Theorem 4.4.1, we first introduce a definition and prove a number of intermediate lemmas.

Definition 4.4.2. Let $\vec{v_0}, \ldots, \vec{v_k} \in \mathbb{R}^m$. Let

$$\mathrm{Cone}(\{\vec{v_0}, \ldots, \vec{v_k}\}) = \left\{ \sum_{i=0}^{k} \lambda_i \vec{v_i} \,\middle|\, \lambda_i \geq 0 \right\}$$

denote the *convex cone generated by* $\{\vec{v_0}, \ldots, \vec{v_k}\}$.

The next lemma, Farkas' Lemma, gives a charagerization of when a vector lies in a convex cone.

Lemma 4.4.3 (Farkas' Lemma [7])**.** *Let* $M \in \mathbb{R}^{m \times n}$ *and* $\vec{v} \in \mathbb{R}^m$*, and let* $\vec{a_i}$ *denote the columns of* M *for* $1 \leq i \leq n$*. Then, exactly one of the following is true:*

1. *There exists* $\vec{x} \in (\mathbb{R}^+)^n$ *such that* $M\vec{x} = \vec{v}$*.*
2. *There exists* $\vec{y} \in \mathbb{R}^m$ *such that* $\vec{a_i}^{\mathsf{T}} \vec{y} \geq 0$ *for all* i *and* $\vec{v}^{\mathsf{T}} \vec{y} < 0$*.*

We can use Farkas' Lemma to study how cones intersect.

Lemma 4.4.4. *Let* $\vec{u_0}, \ldots, \vec{u_n}, \vec{v_0}, \ldots, \vec{v_k} \in \mathbb{R}^m$*. If there exists* $\vec{y} \in \mathbb{R}^m$ *such that*

$$\vec{u_i}^{\mathsf{T}} \vec{y} \geq 0 \quad \textit{for} \quad 0 \leq i \leq n \quad \textit{and} \quad \vec{v_i}^{\mathsf{T}} \vec{y} < 0 \quad \textit{for} \quad 0 \leq i \leq k,$$

then

$$\mathrm{Cone}\left(\{\vec{u_0}, \ldots, \vec{u_n}\}\right) \cap \mathrm{Cone}\left(\{\vec{v_0}, \ldots, \vec{v_k}\}\right) = \vec{0}.$$

Proof. Suppose such a vector $\vec{y} \in \mathbb{R}^m$ exists, and let $\vec{0} \neq \vec{v} = \sum_{i=0}^{k} \lambda_i \vec{v_i} \in \mathrm{Cone}(\{\vec{v_0}, \ldots, \vec{v_k}\})$. Then, because there exists some $0 \leq j \leq k$ with $\lambda_j > 0$,

$$\vec{v}^{\mathrm{T}}\vec{y} = \sum_{i=0}^{k} \lambda_i \vec{v_i}^{\mathrm{T}} \vec{y} \leq \lambda_j \vec{v_j}^{\mathrm{T}} \vec{y} < 0.$$

Hence, by Lemma 4.4.3, $\vec{v}$ is not contained in the convex cone generated by $\{\vec{u}_0, \ldots, \vec{u}_n\}$. □

Lemma 4.4.5. *Fix a positive integer $k \geq 2$, and let $n \geq 0$. Let distinct $t_0, \ldots, t_n \in S^1$ be given with a counterclockwise order. Let distinct $s_0, \ldots, s_m \in S^1$ for some $0 \leq m \leq 2k-2$ be given with a counterclockwise order such that*

1. $\{s_0, \ldots, s_m\} \cap \{t_0, t_1, \ldots, t_n\} = \varnothing$,
2. *no two elements of $\{s_0, \ldots, s_m\}$ are antipodal, and*
3. *there exists an arc $\Gamma = (\gamma_1, \gamma_2)_{S^1}$ of length π such that*
   (a) $[t_0, t_n]_{S^1} \subseteq \Gamma$,
   (b) $\{s_0, \ldots, s_m\} \cap \{\gamma_1, \gamma_2\} = \varnothing$, *and*
   (c) *at most $k-1$ elements of $\{s_0, \ldots, s_m\}$ are contained within Γ.*

Then,

$$\mathrm{Cone}\left(\{\mathrm{SM}_{2k}(t_0), \ldots, \mathrm{SM}_{2k}(t_n)\}\right) \cap \mathrm{Cone}\left(\{\mathrm{SM}_{2k}(s_0), \ldots, \mathrm{SM}_{2k}(s_m)\}\right) = \vec{0}.$$

Proof. Throughout, for convenience, consider points $\mathrm{SM}_{2k}(t) \in \mathbb{R}^{2k}$ to be written as column vectors. Let $0 \leq N \leq \min(k-1, m+1)$ denote the number of elements of $\{s_0, \ldots, s_m\}$ contained in Γ. In the case $N > 0$, we may assume without loss of generality that $s_0, \ldots, s_{N-1} \in \Gamma$.

Observe, by Lemma 4.4.4, it is sufficient to find $\vec{y} \in \mathbb{R}^{2k}$ such that $(\mathrm{SM}_{2k}(t_i))^{\mathrm{T}}\,\vec{y} \geq 0$ for $0 \leq i \leq n$ and $(\mathrm{SM}_{2k}(s_i))^{\mathrm{T}}\,\vec{y} < 0$ for $1 \leq i \leq m$. Toward defining such a vector $\vec{y}$, fix points $v_1, \ldots, v_{2k-1} \in S^1$ as follows:

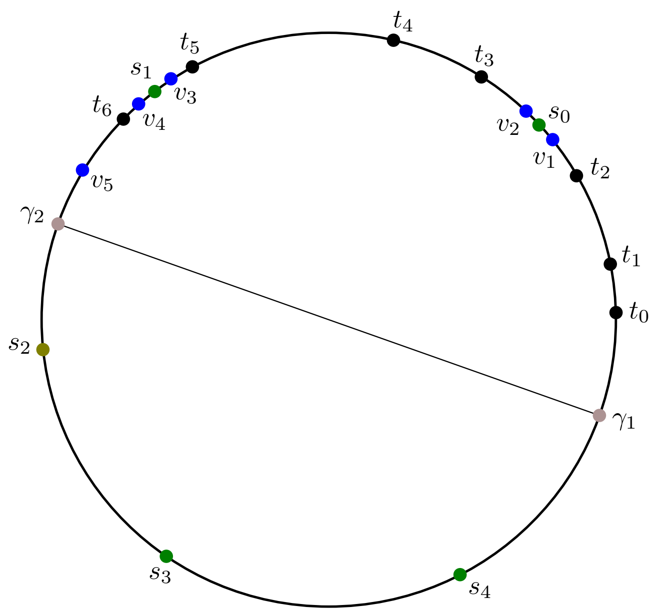

Figure 4.1: An example of points $\{t_0, \ldots, t_6\}$ (black), $\{s_0, \ldots, s_4\}$ (green), and $\{\gamma_1, \gamma_2\}$ (gray) in S^1 satisfying the hypotheses of Lemma 4.4.5 for $k = 3$ and $n = 6$. Points $\{v_1, \ldots, v_5\}$ (blue) are defined in the proof of Lemma 4.4.5, and are used to construct a vector satisfying the hypotheses of Lemma 4.4.4.

1. In the case $N > 0$, define $v_{2i+1} = s_i - \varepsilon$ and $v_{2i+2} = s_i + \varepsilon$ with $\varepsilon > 0$ small enough such that the N intervals of the form $(v_{2i+1}, v_{2i+2})_{S^1}$ are disjoint, and furthermore

$$(v_{2i+1}, v_{2i+2})_{S^1} \cap \{t_0, \ldots, t_n\} = \varnothing$$

and

$$(v_{2i+1} + \pi, v_{2i+2} + \pi)_{S^1} \cap \{s_0, \ldots, s_m\} = \varnothing$$

for $0 \le i \le N-1$. Note that such points v_{2i+1} and v_{2i+2} must exist, because no two elements of $\{s_0, \ldots, s_m\}$ are antipodal. This defines $v_1, v_2, \ldots, v_{2N}$.

2. For $2N < i \le 2k-1$, choose v_i such that

$$\max\{t_n, v_{2N}, s_m + \pi\} < v_{2N+1} < v_{2N+2} < \cdots < v_{2k-1} < \gamma_2.$$

Next, for $t \in S^1$, define the $2k \times 2k$ matrix

$$N_t = \begin{pmatrix} \mathrm{SM}_{2k}(t) & \mathrm{SM}_{2k}(v_1) & \mathrm{SM}_{2k}(v_2) & \cdots & \mathrm{SM}_{2k}(v_{2k-2}) & \mathrm{SM}_{2k}(v_{2k-1}) \end{pmatrix}.$$

By Corollary 4.2.4,

$$\det(N_t) = \kappa \left(\prod_{1 \le l \le 2k-1} \sin(v_l - t) \right) \left(\prod_{1 \le j < l \le 2k-1} \sin(v_l - v_j) \right),$$

where κ is a nonzero constant that depends only on k. By construction, since $v_1, \ldots, v_{2k-1}$ live in an arc of length less than π, note that

$$\prod_{1 \le j < l \le 2k-1} \sin(v_l - v_j) > 0.$$

Hence,

$$\operatorname{sign}\left(\frac{1}{\kappa} \det(N_t) \right) = \operatorname{sign}\left(\prod_{1 \le l \le 2k-1} \sin(v_l - t) \right).$$

Therefore, define

$$\rho(t) = \#\left\{ l \in \{1, \ldots, 2k-1\} \mid (v_l - t \pmod{2\pi}) > \pi \right\}$$

for $t \in S^1$ and note that

$$\operatorname{sign}\left(\frac{1}{\kappa} \det(N_t) \right) = (-1)^{\rho(t)}.$$

Now, for $j = 1, \ldots, 2k$, let N^j denote the submatrix of N_t obtained by deleting the first column and jth row of N_t, and define $\vec{y} \in \mathbb{R}^{2k}$ by

$$y_j = \frac{(-1)^j}{\kappa} \det(N^j).$$

By considering the computation of $\det(N_t)$ by performing cofactor expansion along the first column of N_t, we note that

$$\operatorname{sign}\left(\left(\mathrm{SM}_{2k}(t_i)\right)^{\mathrm{T}}\vec{y}\right) = \operatorname{sign}\left(\frac{1}{\kappa}\det(N_t)\right) = (-1)^{\rho(t)}.$$

When we consider the case $t = t_i$ for $0 \leq i \leq n$, we note by construction that $\rho(t_i)$ is even for each t_i, and so $\left(\mathrm{SM}_{2k}(t_i)\right)^{\mathrm{T}}\vec{y} \geq 0$ for $0 \leq i \leq n$.

On the other hand, observe that $\rho(s_i) = 2i + 1$ for $0 \leq i \leq N - 1$, and it follows that

$$\operatorname{sign}\left((\mathrm{SM}_{2k}(s_i))^{\mathrm{T}}\vec{y}\right) = (-1)^{\rho(s_i)} = -1$$

for $0 \leq i \leq N - 1$.

Finally, for $N \leq i \leq m$, note that each pair $\{v_j, v_{j+1}\}$ for $1 \leq j \leq 2N - 1$ has zero net effect on the parity of $\rho(s_i)$ by the fact that $(v_j + \pi, v_{j+1} + \pi)_{S^1} \cap \{s_0, \ldots, s_m\} = \varnothing$. Therefore,

$$\operatorname{sign}\left((\mathrm{SM}_{2k}(s_i))^{\mathrm{T}}\vec{y}\right) = \operatorname{sign}\left(\prod_{1 \leq l \leq 2k-1} \sin(v_l - s_i)\right) = (-1)^{(2k-1)-2N} = -1$$

for $N \leq i \leq m$. □

The following is both a corollary and a generalization of Lemma 4.4.5, where now some of the t_i and s_j points may coincide.

Corollary 4.4.6. *Let $k \geq 2$, $n \geq 0$, and $t_0, \ldots, t_n \in S^1$ be given as in Lemma 4.4.5. Let distinct $s_0, \ldots, s_m \in S^1$ for some $0 \leq m \leq \min(\{2k - 2, n - 1\})$ be given with a counterclockwise order such that*

1. *$\{s_0, \ldots, s_m\} \cap \{t_0, t_1, \ldots, t_n\} = \{t_p\}_{p \in I}$ for some $I \subseteq \{0, \ldots, n\}$ with $|I| \leq m$,*
2. *no two elements of $\{s_0, \ldots, s_m\}$ are antipodal, and*
3. *there exists an arc $\Gamma = (\gamma_1, \gamma_2)_{S^1}$ of length π such that*
   - *(a) $[t_0, t_n]_{S^1} \subseteq \Gamma$,*
   - *(b) $\{s_0, \ldots, s_m\} \cap \{\gamma_1, \gamma_2\} = \varnothing$, and*

(c) at most $k-1$ elements of $\{s_0, \ldots, s_m\}$ are contained within Γ.

Then,

$$\text{Cone}\left(\{\text{SM}_{2k}(t_0), \ldots, \text{SM}_{2k}(t_n)\}\right) \cap \text{Cone}\left(\{\text{SM}_{2k}(s_0), \ldots, \text{SM}_{2k}(s_m)\}\right) = \text{Cone}\left(\{\text{SM}_{2k}(t_p)\}_{p \in I}\right)$$

Proof. The inclusion $\supseteq$ is clear; it remains to show the reverse direction $\subseteq$.

Assume, for the sake of contradiction, that there exists some vector

$$\vec{u} \in \text{Cone}\left(\{\text{SM}_{2k}(t_0), \ldots, \text{SM}_{2k}(t_n)\}\right) \cap \text{Cone}\left(\{\text{SM}_{2k}(s_1), \ldots, \text{SM}_{2k}(s_m)\}\right)$$

such that

$$\vec{u} \notin \text{Cone}\left(\{\text{SM}_{2k}(t_p)\}_{p \in I}\right).$$

In particular, we may write $\vec{u} = \sum_{i=0}^{n} \lambda_i \text{SM}_{2k}(t_i) = \sum_{j=0}^{m} \mu_j \text{SM}_{2k}(s_j)$ for some scalars $\lambda_i, \mu_j \geq 0$.

In the case that $\{s_0, \ldots, s_m\} \cap \{t_0, t_1, \ldots, t_n\} = \varnothing$, we obtain a contradiction by Lemma 4.4.5. Otherwise, suppose $|I| = M$ for some $0 < M < m$, and re-index as necessary so that $s_i = t_i$ for all $0 \leq i < M$. Since $\vec{u} \notin \text{Cone}(\{\text{SM}_{2k}(t_p)\}_{p \in I})$, it follows that $\lambda_i > 0$ for some $i \geq M$, and also $\mu_j > 0$ for some $j \geq M$. Next, define

$$\tilde{\lambda}_i = \begin{cases} \lambda_i - \mu_i & \text{if } 0 \leq i < M \text{ and } \lambda_i - \mu_i > 0 \\ 0 & \text{if } 0 \leq i < M \text{ and } \mu_i - \lambda_i \geq 0 \\ \lambda_i & \text{if } M \leq i \end{cases}$$

and

$$\tilde{\mu}_i = \begin{cases} \mu_i - \lambda_i & \text{if } 0 \leq i < M \text{ and } \mu_i - \lambda_i > 0 \\ 0 & \text{if } 0 \leq i < M \text{ and } \lambda_i - \mu_i \geq 0 \\ \mu_i & \text{if } M \leq i \end{cases}$$

and observe that we have $\sum_{i=0}^{n} \tilde{\lambda}_i \mathrm{SM}_{2k}(t_i) = \sum_{j=0}^{m} \tilde{\mu}_j \mathrm{SM}_{2k}(s_j)$. By removing terms with a zero coefficient and re-indexing as necessary, it follows that

$$\sum_{i=0}^{\tilde{n}} \tilde{\lambda}_i \mathrm{SM}_{2k}(t_i) = \sum_{j=0}^{\tilde{m}} \tilde{\mu}_j \mathrm{SM}_{2k}(s_j)$$

for some $\tilde{n} \geq n - M > 0$ and $\tilde{m} \geq m - M > 0$. Finally, we obtain a contradiction by Lemma 4.4.5. □

We now are ready for the proof of Theorem 4.4.1, which states if $x = \sum_{i=0}^{n} \lambda_i t_i \in \mathrm{VR}^m_{\leq}(S^1; \frac{2\pi}{3})$ with $\iota \circ (p \circ \mathrm{SM}_4)(x) = \sum_{j=0}^{m} \mu s_j$ for some $s_j \in S^1$, then $\mathrm{diam}(\{t_0, \ldots, t_n\} \cup \{s_0, \ldots, s_m\}) = \mathrm{diam}(\{t_0, \ldots, t_n\})$.

Proof of Theorem 4.4.1. Write $x = \sum_{i=0}^{n} \lambda_i t_i \in \mathrm{VR}^m_{\leq}(S^1; \frac{2\pi}{3})$ and observe that

$$\sum_{j=0}^{m} \mu \mathrm{SM}_4(s_i) = (p \circ \mathrm{SM}_4)(x) \in \mathrm{Cone}\left(\{\mathrm{SM}_4(t_0), \ldots, \mathrm{SM}_4(t_n)\}\right).$$

There are two cases:

1. $t_0, \ldots, t_n \in \Gamma$ for an arc $\Gamma = [t_0, t_n]_{S^1}$ of length less than or equal to $\frac{2\pi}{3}$, or

2. $n = 2$ and $t_i = t_0 + \frac{2\pi}{3} i$.

First, assume $\{t_0, \ldots, t_n\}$ are contained within an arc of length less than or equal to $\frac{2\pi}{3}$. In the case $n = 1$, note that $\mathrm{SM}_4(x) \in \partial \mathcal{B}_4$ and $\iota \circ (p \circ \mathrm{SM}_4)(x) = x$. Otherwise, assume $n \geq 2$. In light of Theorem 3.5.1, observe that the hypotheses of Corollary 4.4.6 are satisfied, unless $m = 1$ and $s_0, s_1 \in [t_0, t_n]$.

Finally, in the second case, $\mathrm{SM}_4(x) \in \partial \mathcal{B}_4$ by Theorem 3.5.1, and $\iota \circ (p \circ \mathrm{SM}_4)(x) = x$.

□

We have now reached the proof of our main result.

Proof of Theorem 4.1.1. As observed at the beginning of this section, it remains only to show that $\iota \circ (p \circ \mathrm{SM}_4) \simeq \mathrm{id}_{\mathrm{VR}^m_\leq(S^1;r)}$. Consider the linear homotopy $H \colon \mathrm{VR}^m_\leq(S^1;r) \times I \to \mathrm{VR}^m_\leq(S^1;r)$ defined by

$$H(x,t) = t[\iota \circ (p \circ \mathrm{SM}_4)](x) + (1-t)x.$$

Observe that H is well-defined by Theorem 4.4.1 and continuous by Lemma 3.8 of [2], with $H(\cdot,0) = \mathrm{id}_{\mathrm{VR}^m_\leq(S^1;r)}$ and $H(\cdot,1) = \iota \circ (p \circ \mathrm{SM}_4)$. □

We conjecture that a similar proof works in higher dimensions. There are two main obstacles. The first obstacle is knowing the facial structure of the Barvinok–Novik orbitopes $\mathcal{B}_{2k}$ for $k > 2$ (and in particular, knowing if they are simplicial or not). The second obstacle would be an analogue of Theorem 4.4.1 for larger r and k values.

Conjecture 4.4.7. *We conjecture that for* $\frac{2\pi k}{2k+1} \leq r < \frac{2\pi(k+1)}{2k+3}$*, the map* $p_{2k} \circ \mathrm{SM}_{2k} \colon \mathrm{VR}^m_\leq(S^1;r) \to \partial\mathcal{B}_{2k+2}$ *is a homotopy equivalence, and hence* $\mathrm{VR}^m_\leq(S^1;r) \simeq \partial\mathcal{B}_{2k+2} \cong S^{2k+1}$.

Chapter 5

Conclusion

In applications of persistent homology, Vietoris–Rips simplicial complexes provide a convenient method of associating a topological space to a dataset. As a step toward gaining a better understanding of the topological behavior of these complexes at large scale parameters, we consider the simplest manifold with nonzero homology: the circle. In this paper, we have shown that the Vietoris–Rips metric thickening of the circle achieves the homotopy type of the 3-sphere S^3 at scale parameter $r = \frac{2\pi}{3}$, in contrast to the infinite wedge-sum of 2-spheres attained by the ordinary Vietoris–Rips complex on the circle. In addition, we have employed a geometric method of proof, taking advantage of continuous maps afforded by the 1-Wasserstein metric, and revealing connections between Vietoris–Rips thickenings of the circle and the Barvinok-Novik orbitopes.

This work leads to a number of open questions. In particular, in light of Theorem 3.3.1, it seems reasonable to expect $\mathrm{VR}^m_{\leq}(S^1; r)$ to obtain the homotopy type of odd–dimensional spheres as r increases. As noted in Section 4.3, Vinzant's result would provide a continuous inclusion $\iota\colon \partial\mathcal{B}_{2k} \to \mathrm{VR}^m_{\leq}(S^1; r)$ for appropriate scale parameters if it were known that $\mathcal{B}_{2k}$ is simplicial. Then, it would remain to prove the homotopy equivalence $\iota \circ (p_{2k} \circ \mathrm{SM}_{2k}) \simeq \mathrm{id}_{\mathrm{VR}^m_{\leq}(S^1;r)}$, where a linear homotopy may again be well-defined given a generalization of Corollary 4.4.6 (or through other, more sophisticated methods).

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Appendix A

Metric thickenings of S^1 at small scale parameters

Following techniques similar to that of Chapter 4, we may deduce the following:

Theorem A.0.1. *Let r denote the side-length of an equilateral triangle inscribed in S^1. Then,* $\mathrm{VR}^m_<(S^1; r) \simeq S^1$.

Proof. We construct homotopy equivalences between $\mathrm{VR}^m_<(S^1; 2\pi/3)$ and $\partial\mathcal{B}_2 = S^1$ given by $p_2 \circ \mathrm{SM}_2$ and ι in the following diagram:

$$\mathrm{VR}^m_<(S^1; 2\pi/3) \xrightarrow{\mathrm{SM}_2} \mathbb{R}^2 \setminus \{\vec{0}\} \xrightarrow{p_2} \partial\mathcal{B}_2 \xrightarrow{\iota} \mathrm{VR}^m_<(S^1; 2\pi/3).$$

Observe that ι is well-defined and is continuous by an argument analogous to Lemma 4.3.1. Additionally, $\mathrm{SM}_2\colon \mathrm{VR}^m_<(S^1; 2\pi/3) \to \mathbb{R}^2$ is continuous by Lemma 5.2 of [2], and $p_2 \circ \mathrm{SM}_2$ is well-defined by Theorem 4.2.1. Finally, for $x = \sum_{i=0}^n \lambda_i t_i \in \mathrm{VR}^m_<(S^1; \frac{2\pi}{3})$, we may write $\iota \circ (p_2 \circ \mathrm{SM}_2)(x) = s_0 \in S^1$, and it is clear that $\mathrm{diam}(\{t_0, \ldots, t_n\} \cup \{s_0\}) = \mathrm{diam}(\{t_0, \ldots, t_n\})$. It follows as in the proof of Theorem 4.1.1 that ι and $p_2 \circ \mathrm{SM}_2$ are homotopy inverses.

$\square$