Topological, geometric, and combinatorial aspects of metric thickenings

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Outline

Carathéodory subsets of moment curves and faces of orbitopes

simplicial metric thickenings

generalizations of the Borsuk–Ulam theorem

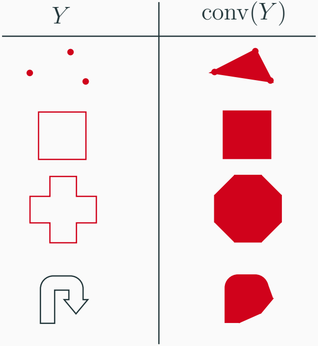
zeros of trigonometric polynomials

Background and related work

Let V denote a real vector space.

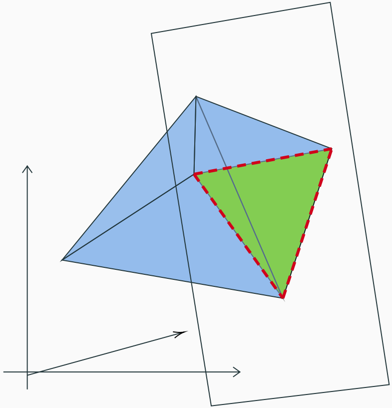
Given a subset $Y \subseteq V$, define the **convex hull of Y** to be

$$\mathrm{conv}(Y) := \left\{ \sum_{i=0}^{k} \lambda_i x_i \;\middle|\; k \geq 0,\ x_i \in Y,\ \lambda_i \in \mathbb{R}_{\geq 0},\ \sum_{i=0}^{k} \lambda_i = 1 \right\}.$$



Definition

A (proper) **face** of a convex body C is any nonempty intersection of C with an affine hyperplane P such that C is contained in a closed half-space bounded by P.



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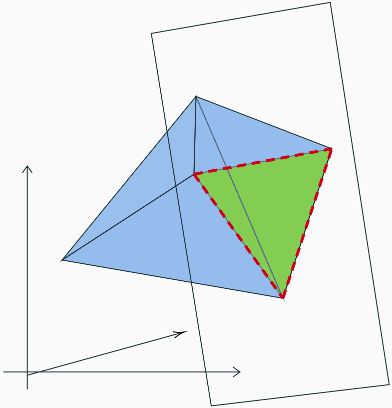


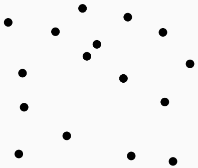
Equivalently, F is a proper face of C if and only if there exists an affine function

$$A(x) = \alpha_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n$$

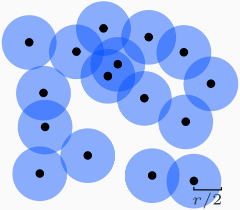
such that

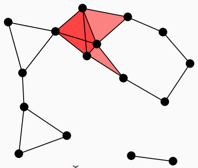
1. $A(x) = 0$ for all $x \in F$ and
2. $A(y) > 0$ for all $y \in C \setminus F$.

Simplicial complexes

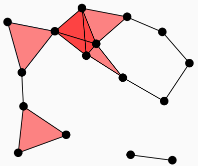


X





$\check{\mathrm{C}}(X;r)$



$\mathrm{VR}(X;r)$

Definition

Let X be a subset of a metric space (Y, d) and fix $r \geq 0$. The **open Čech simplicial complex of X at scale r**, denoted $\check{\mathrm{C}}_{<}(X; r)$, has X as its vertex set and a simplex $\sigma \subseteq X$ if and only if σ is nonempty, finite, and

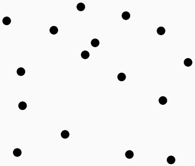
$$\bigcap_{v \in \sigma} B\left(v; \tfrac{1}{2}r\right) \neq \varnothing,$$

where $B\left(v; \frac{1}{2}r\right) \coloneqq \left\{y \in Y \mid d(v, y) < \frac{1}{2}r\right\} \subseteq Y$ denotes the open ball of radius $\frac{1}{2}r$ centered at v.

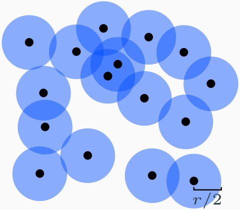
Definition

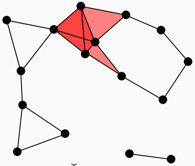
Let X be a metric space and fix $r \geq 0$. The **open Vietoris–Rips simplicial complex of X at scale** r, denoted $\mathrm{VR}_<(X;r)$, has X as its vertex set and a simplex $\sigma \subseteq X$ if and only if σ is nonempty, finite, and $\mathrm{diam}(\sigma) < r$.

Simplicial complexes



X





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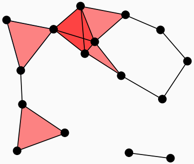

$\mathrm{VR}(X; r)$

Simplicial complex

Topology on a simplicial complex K?

We define

$$|K| := \left\{ \sum_{i=0}^{k} \lambda_i x_i \;\middle|\; k \geq 0, \{x_0, \ldots, x_k\} \in S(K), \lambda_i \in \mathbb{R}_{\geq 0}, \sum_{i=0}^{k} \lambda_i = 1 \right\}$$

- coherent topology
- metric topology
- simplicial metric thickening topology

These all coincide if K is *locally finite*.

Metric thickenings

Let X be a metric space. Let δ_x denote the Dirac delta mass at a point $x \in X$ and let $\mathcal{P}(X)$ denote the space of all Radon probability measures on X equipped with the 1-Wasserstein metric.

Then, X embeds $X \hookrightarrow \mathcal{P}(X)$ via $x \mapsto \delta_x$, and this is an isometry onto its image.

Definition

Let X be a metric space, and let K denote a simplicial complex with vertex set X. The **simplicial metric thickening** (or simply **metric thickening**) of K is defined to be the following submetric space of $\mathcal{P}(X)$,

$$K^{\mathrm{m}} := \left\{ \sum_{i=0}^{k} \lambda_i \delta_{x_i} \;\middle|\; k \geq 0, \{x_0, \ldots, x_k\} \in S(K), \lambda_i \in \mathbb{R}_{\geq 0}, \sum_{i=0}^{k} \lambda_i = 1 \right\},$$

equipped with the restriction of the 1-Wasserstein metric.

Conventions regarding spheres

Throughout, we equip the sphere S^n with the intrinsic metric in which great circles have circumference 2π.

Definition

Identify S^1 with $\mathbb{R}/2\pi\mathbb{Z}$. Let $a, b \in S^1$ with $a \neq b$. We define the **open circular arc** $(a, b)_{S^1}$ as follows:

The **closed circular arc** $[a, b]_{S^1}$ is defined analogously.

Trigonometric polynomials

A **trigonometric polynomial** is an expression of the form

$$p(t) = c + \sum_{j=1}^{n} \left(a_j \cos(jt) + b_j \sin(jt)\right),$$

inducing a map $S^1 \to \mathbb{R}$ under the identification $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Throughout, we assume all coefficients are real.

If $c = 0$, we call p a **homogeneous trigonometric polynomial**.

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If $c = 0$, we call p a **homogeneous trigonometric polynomial**.

The largest j for which $a_j \neq 0$ or $b_j \neq 0$ is the **degree** of p.

If $a_j = b_j = 0$ whenever j is even, then p is called a **raked trigonometric polynomial**.

The trigonometric moment curve

Definition

For $k \in \mathbb{N}$, the **trigonometric moment curve** $\mathrm{M}_{2k} \colon S^1 \to \mathbb{R}^{2k}$ is defined by

$$\mathrm{M}_{2k}(t) := (\cos(t), \sin(t), \cos(2t), \sin(2t), \ldots, \cos(kt), \sin(kt))^{\intercal}\,.$$

The **Carathéodory orbitopes** are defined by
$\mathcal{C}_{2k} := \mathrm{conv}(\mathrm{M}_{2k}(S^1)) \subseteq \mathbb{R}^{2k}$ for integers $k \geq 1$.

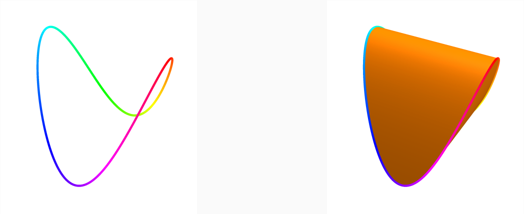

Figure 1: (Left) The image of the map $S^1 \to \mathbb{R}^3$ defined by $t \mapsto (\cos(t), \sin(t), \cos(2t))$. (Right) The convex hull of this set.

Theorem ([8, Corollary 5.4])

The proper faces of $\mathcal{C}_{2k}$ *are in inclusion-preserving bijection with sets of at most* k *points in* S^1*.*

Note that any $\{t_1, \ldots, t_m\} \subset S^1$ with $m \leq k$ must be disjoint from some open arc of length at least $\frac{2\pi}{k}$, and hence in some ball of S^1 of radius $r \leq \frac{(k-1)\pi}{k}$.

The centrally-symmetric trigonometric moment curve

Definition

For $k \in \mathbb{N}$, the **centrally symmetric trigonometric moment curve** (or **symmetric moment curve**) $\mathrm{SM}_{2k} \colon S^1 \to \mathbb{R}^{2k}$ is defined by

$$\mathrm{SM}_{2k}(t) \coloneqq \big(\cos t, \sin t, \cos 3t, \sin 3t, \ldots, \cos(2k-1)t, \sin(2k-1)t\big)^{\mathsf{T}}.$$

Note that SM_{2k} is centrally symmetric: $\mathrm{SM}_{2k}(t+\pi) = -\mathrm{SM}_{2k}(t)$.

The **Barvinok–Novik orbitopes** are defined by
$\mathcal{B}_{2k} \coloneqq \operatorname{conv}(\mathrm{SM}_{2k}(S^1)) \subseteq \mathbb{R}^{2k}$ for $k \geq 1$.

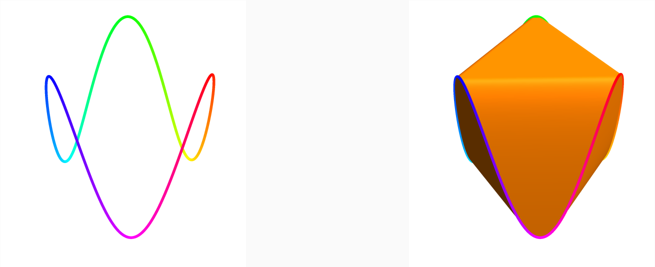

Figure 2: (Left) The image of the map $S^1 \to \mathbb{R}^3$ defined by $t \mapsto (\cos(t), \sin(t), \cos(3t))$. (Right) The convex hull of this set.

Theorem ([9, 10])

Every proper face of the Barvinok–Novik orbitope $\mathcal{B}_{2k}$ is a simplex such that the preimage of the vertex set of the simplex has diameter in S^1 at most $\frac{2\pi(k-1)}{2k-1}$.

A complete description of the faces of $\mathcal{B}_{2k}$ is currently unknown for $k > 2$.

The Borsuk–Ulam theorem

Given topological spaces X and Y equipped with $\mathbb{Z}/2\mathbb{Z}$-actions μ and ν respectively, we say a map $f\colon X \to Y$ is **odd** or **$\mathbb{Z}/2\mathbb{Z}$-equivariant** if $f \circ \mu = \nu \circ f$.

Throughout, we equip $\mathbb{R}^n$ and S^n with the standard antipodal $\mathbb{Z}/2\mathbb{Z}$-action.

The Borsuk–Ulam theorem

Theorem

Given a continuous map $f\colon S^n \to \mathbb{R}^n$*, there exists* $x_0 \in S^n$ *such that* $f(x_0) = f(-x_0)$.

– equivalently –

Theorem

Given a continuous odd map $f\colon S^n \to \mathbb{R}^n$*, there exists* $x_0 \in S^n$ *such that* $f(x_0) = \vec{0}$.

– equivalently –

Theorem

There does not exist a continuous odd map $S^n \to S^{n-1}$.

Corollaries of the Borsuk–Ulam theorem

Theorem (Stone–Tukey theorem for measures)

Let $\mu_1, \mu_2, \ldots, \mu_k$ be finite Borel measures in $\mathbb{R}^k$ such that every hyperplane has measure 0 for each of the μ_i. Then, there exists a hyperplane h such that

$$\mu_i(h^+) = \tfrac{1}{2}\mu_i(\mathbb{R}^k) \quad \textit{for} \quad i = 1, 2, \ldots, d,$$

where h^+ denotes one of the half-spaces defined by h.

Theorem (Lyusternik–Shnirel'man covering theorem)

For any cover $A_1, \ldots, A_{n+1}$ of the sphere S^n by $n+1$ sets such that the n sets $A_1, \ldots, A_n$ are each either open or closed, there is at least one set containing a pair of antipodal points.

Metric thickenings of the circle

Theorem ([1, Main Result])

There are homotopy equivalences

$$\check{\mathrm{C}}_{\leq}(S^1;r) \simeq \begin{cases} S^{2k-1} & \text{if } \frac{2\pi(k-1)}{k} < r < \frac{2\pi k}{k+1} \\ \bigvee^{\mathfrak{c}} S^{2(k-1)} & \text{if } r = \frac{2\pi(k-1)}{k}, \end{cases}$$

and

$$\mathrm{VR}_{\leq}(S^1;r) \simeq \begin{cases} S^{2k-1} & \text{if } \frac{2\pi(k-1)}{2k-1} < r < \frac{2\pi k}{2k+1} \\ \bigvee^{\mathfrak{c}} S^{2(k-1)} & \text{if } r = \frac{2\pi(k-1)}{2k-1}, \end{cases}$$

where $k = 1, 2, \ldots,$ *and where* $\mathfrak{c}$ *denotes the cardinality of the continuum.*

Čech and Vietoris–Rips metric thickenings of the circle

Conjecture

There are homotopy equivalences

$$\check{\mathrm{C}}^{\mathrm{m}}_{\leq}(S^1; r) \simeq S^{2k-1} \qquad \textit{if} \qquad \frac{2\pi(k-1)}{k} \leq r < \frac{2\pi k}{k+1},$$

and

$$\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1; r) \simeq S^{2k-1} \qquad \textit{if} \qquad \frac{2\pi(k-1)}{2k-1} \leq r < \frac{2\pi k}{2k+1},$$

where $k = 1, 2, \ldots$.

Metric thickenings of the circle

For what follows, we will focus on the Vietoris–Rips metric thickenings.

Metric thickenings of the circle

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Recall:

Theorem ([9, 10])

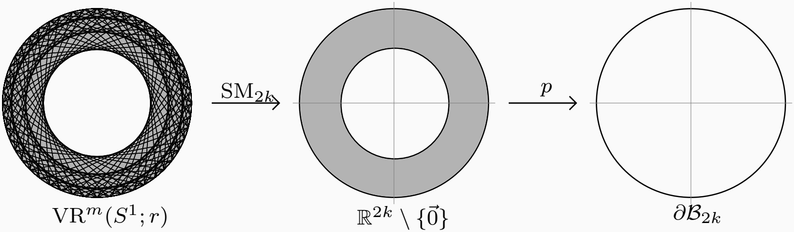
Every proper face of the Barvinok–Novik orbitope $\mathcal{B}_{2k}$ is a simplex such that the preimage of the vertex set of the simplex has diameter in S^1 at most $\frac{2\pi(k-1)}{2k-1}$.

Proof idea

At the appropriate scales r,

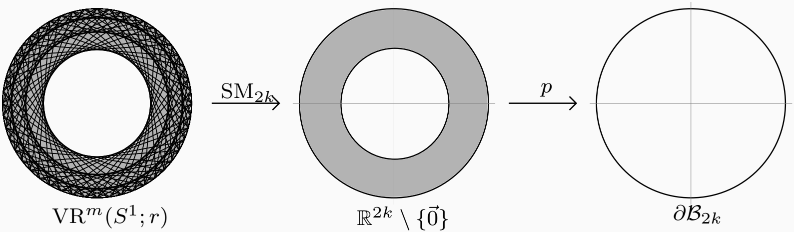
- extend the domain of SM_{2k} to $\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1; r)$
- radially project to the boundary of the corresponding orbitope (an odd-dimensional sphere)
- prove that the composition of these maps is a homotopy equivalence

Proof idea



The composition $\mathrm{VR}^{\mathrm{m}}(S^1; r) \xrightarrow{\mathrm{SM}_{2k}} \mathbb{R}^{2k} \setminus \{\vec{0}\} \xrightarrow{p} \partial\mathcal{B}_{2k}$, drawn in the case $k = 1$ (so, $0 \leq r < \frac{2\pi}{3}$).

Proof idea



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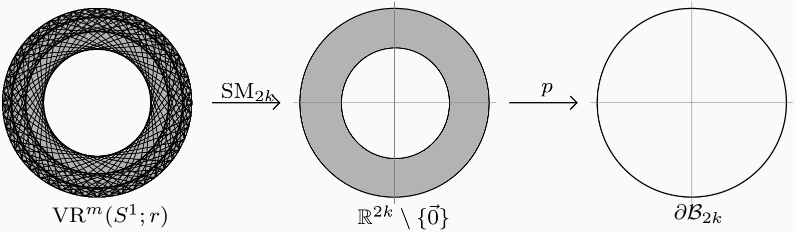
Let $\iota\colon \partial\mathcal{B}_{2k} \to \mathrm{VR}^{\mathrm{m}}(S^1; r)$ denote the inclusion,

$$\iota\colon \sum_i \lambda_i \mathrm{SM}_{2k}(x_i) \mapsto \sum_i \lambda_i \delta_{x_i}.$$

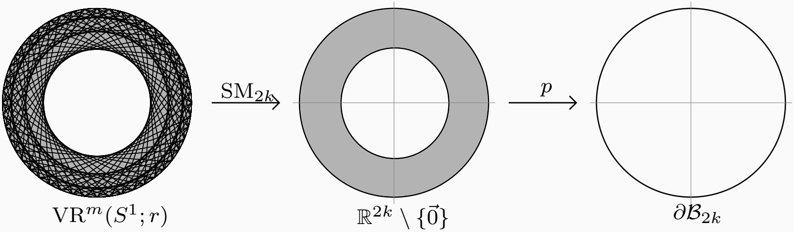
Proof idea



Ingredients:

- continuity: proved using the metric thickening topology

Proof idea



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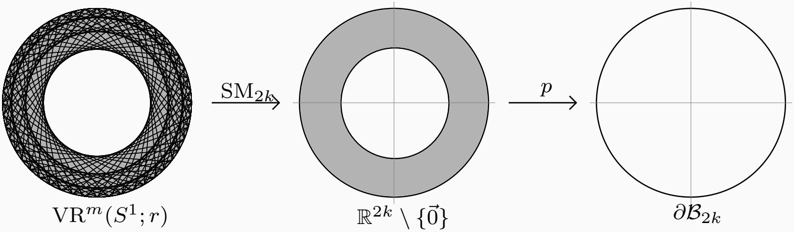
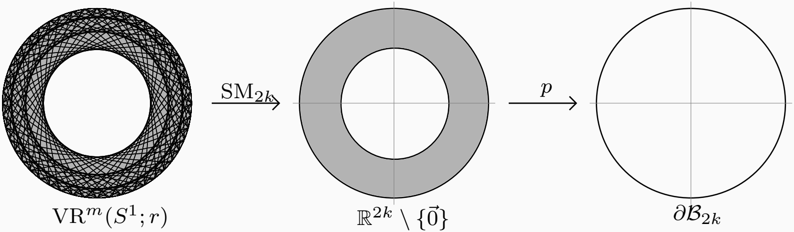


Ingredients:

- continuity: proved using the metric thickening topology ✓
- well-definedness: need $\mathrm{SM}_{2k}(\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1; r))$ to miss the origin for $\frac{2\pi(k-1)}{2k-1} \leq r < \frac{2\pi k}{2k+1}$

Proof idea



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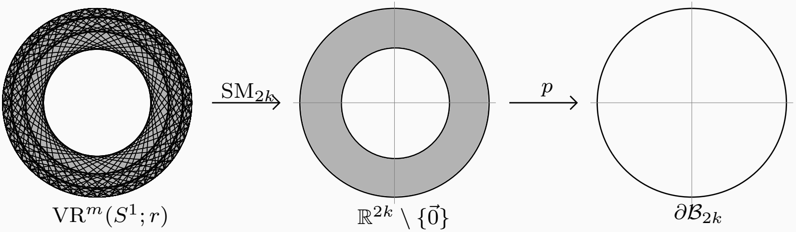

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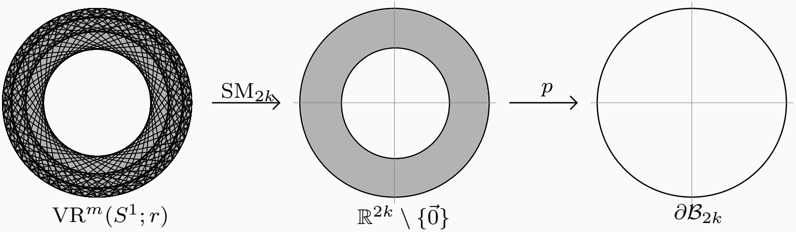
- homotopy equivalence:
  - note that $(p \circ \mathrm{SM}_{2k}) \circ \iota = \mathrm{id}_{\partial\mathcal{B}_{2k}}$ ✓

Proof idea



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- homotopy equivalence:
  - note that $(p \circ \mathrm{SM}_{2k}) \circ \iota = \mathrm{id}_{\partial\mathcal{B}_{2k}}$ ✓
  - need $\iota \circ (p \circ \mathrm{SM}_{2k}) \simeq \mathrm{id}_{\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1;r)}$ ...

Proof idea

Theorem

For $0 \leq r < \frac{2\pi}{3}$, $\iota \circ (p \circ \mathrm{SM}_2) \simeq \mathrm{id}_{\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1;r)}$ *by a linear homotopy.*

For $r = \frac{2\pi}{3}$, $\iota \circ (p \circ \mathrm{SM}_4) \simeq \mathrm{id}_{\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1;r)}$ *by a linear homotopy.*

Consequently, there is a homotopy equivalence

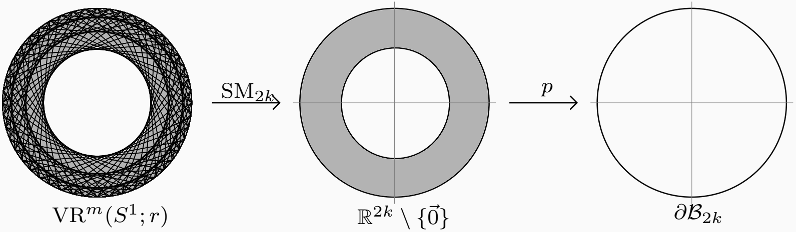
$$\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1;r) \simeq \begin{cases} S^1 & 0 \leq r < \frac{2\pi}{3} \\ S^3 & r = \frac{2\pi}{3}. \end{cases}$$

Proof idea

Difficulty at higher scales: five pages of combinatorial arguments to establish well-definedness at scale $r = \frac{2\pi}{3}$.

Michael Moy recently showed that a linear homotopy is not well-defined for $r > \frac{2\pi}{3}$. So, this proof technique will require a more complicated homotopy at higher scales.

Proof idea



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- homotopy equivalence:
  - note that $(p \circ \mathrm{SM}_{2k}) \circ \iota = \mathrm{id}_{\partial\mathcal{B}_{2k}}$ ✓
  - need $\iota \circ (p \circ \mathrm{SM}_{2k}) \simeq \mathrm{id}_{\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1;r)}$ (?)

Proof idea

The above is enough to prove that $\partial\mathcal{B}_{2k} \cong S^{2k-1}$ is a retract of $\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1; r)$ at the appropriate scales.

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Lemma

Similarly, for $\frac{2\pi(k-1)}{2k-1} \leq r < \frac{2\pi k}{2k+1}$*, the* $(2k-1)$*-dimensional homology, cohomology, and homotopy groups of* $\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1; r)$ *are nontrivial.*

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A similar argument using the Carathéodory orbitopes implies that $\partial\mathcal{C}_{2k} \cong S^{2k-1}$ is a retract of $\check{\mathrm{C}}^{\mathrm{m}}_{\leq}(S^1; r)$. Hence, for $\frac{2\pi(k-1)}{k} \leq r < \frac{2\pi k}{k+1}$, the $(2k-1)$-dimensional homology, cohomology, and homotopy groups of $\check{\mathrm{C}}^{\mathrm{m}}_{\leq}(S^1; r)$ are nontrivial.

Carathéodory subsets of moment curves and faces of orbitopes

Carathéodory subsets

Definition
Let $Y \subseteq \mathbb{R}^k$. We say $Y' \subseteq Y$ is a Carathéodory subset of Y if $\vec{0} \in \text{conv}(Y')$.

Motivation: we want to know when the image of the metric thickening misses the origin (so we can radially project to the boundary of the orbitope).

Carathéodory subsets of the trigonometric moment curve

The following is a corollary of a theorem due to Gilbert and Smyth [7].

Theorem

Let $X \subseteq S^1$ *be contained in a closed circular arc* $[a,b]_{S^1}$ *of length less than* L*. Then the convex hull* $\mathrm{conv}(\mathrm{M}_{2k}(X))$ *does not contain the origin* $\vec{0} \in \mathbb{R}^{2k}$ *if* $L = \frac{2\pi k}{k+1}$*, and this bound is sharp.*

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In particular, if $\mu \in \check{\mathrm{C}}^{\mathrm{m}}_{\leq}(S^1; r)$, then the support of μ is contained in a closed circular arc of length less than r; hence, the convex hull of $\mathrm{M}_{2k}(\check{\mathrm{C}}^{\mathrm{m}}_{\leq}(S^1; r))$ does not contain the origin when $r < \frac{2\pi k}{k+1}$.

Carathéodory subsets of the cs-trig. moment curve

Theorem

Let $X \subseteq S^1$ *be such that* $\mathrm{diam}(X) < D$. *Then the convex hull* $\mathrm{conv}(\mathrm{SM}_{2k}(X))$ *does not contain the origin* $\vec{0} \in \mathbb{R}^{2k}$ *if* $D = \frac{2\pi k}{2k+1}$, *and this bound is sharp.*

In particular, given $\mu \in \mathrm{VR}^{\mathrm{m}}_{\leq}(S^1; r)$, the support of μ has diameter at most r in S^1; hence, the convex hull of $\mathrm{SM}_{2k}(\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1; r))$ does not contain the origin when $r < \frac{2\pi k}{2k+1}$.

Proof sketch
If $\vec{0} \in \operatorname{conv}(\mathrm{SM}_{2k}(X))$, we may assume without loss of generality that $|X| \leq 2k+1$ by Carathéodory's theorem. So, suppose $\vec{0} = \sum_{i=0}^{2k} \lambda_i \mathrm{SM}_{2k}(t_i)$ for some convex coefficients $\{\lambda_i\}_i$.

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Define $\vec{\lambda} = (\lambda_0, \ldots, \lambda_{2k})^\intercal$. Then, $\mathbb{SM}_{2k}(\vec{t})\vec{\lambda} = \vec{0}$, where

$$\mathbb{SM}_{2k}(\vec{t}) := \begin{pmatrix} \cos(t_0) & \cos(t_1) & \ldots & \cos(t_{2k}) \\ \sin(t_0) & \sin(t_1) & \ldots & \sin(t_{2k}) \\ \cos(3t_0) & \cos(3t_1) & \ldots & \cos(3t_{2k}) \\ \sin(3t_0) & \sin(3t_1) & \ldots & \sin(3t_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ \cos((2k-1)t_0) & \cos((2k-1)t_1) & \ldots & \cos((2k-1)t_{2k}) \\ \sin((2k-1)t_0) & \sin((2k-1)t_1) & \ldots & \sin((2k-1)t_{2k}) \end{pmatrix}$$

Carathéodory subsets of the cs-trig. moment curve

Proof sketch.
So, if $\vec{0} \in \operatorname{conv}(\mathrm{SM}_{2k}(X))$, there exists a nonzero vector $\vec{\lambda}$ in the nullspace of $\mathbb{SM}_{2k}(\vec{t})$ such that all nonzero entries of $\vec{\lambda}$ are positive.

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So, we determine the nullspace of $\mathbb{SM}_{2k}(\vec{t})$ (it is one-dimensional) and establish a relationship between the configuration of points $t_0, \ldots, t_{2k} \in S^1$ and the generator of the nullspace.

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Last, we use a combinatorial argument to show that the vector generating the nullspace can not have strictly non-negative entries if $\operatorname{diam}(\{t_0, \ldots, t_{2k}\})$ is less than $\frac{2\pi k}{2k+1}$.

□

Faces of Carathéodory orbitopes

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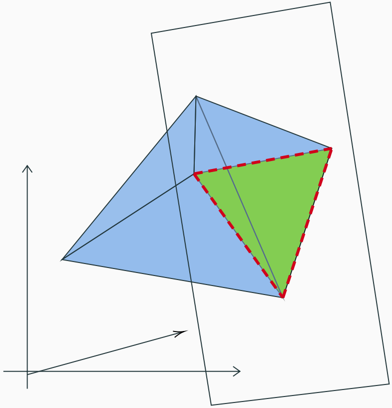
We found that other, analogous matrices are useful in different contexts.

We will consider matrices with columns consisting of points along the trigonometric moment curve and use them to describe the faces of the Carathéodory orbitopes.

Faces of Carathéodory orbitopes

Recall:



F is a proper face of C if and only if there exists an affine function

$$A(x) = \alpha_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n$$

such that

1. $A(x) = 0$ for all $x \in F$ and
2. $A(y) > 0$ for all $y \in C \setminus F$.

Faces of Carathéodory orbitopes

One may show: $F = \{\mathrm{M}_{2k}(t_1), \ldots, \mathrm{M}_{2k}(t_n)\}$ is the set of vertices of a proper face of $\mathcal{C}_{2k}$ if and only if there exists an affine function

$$\begin{aligned} A(t) &= \alpha_0 + \alpha_1 \cos(t) + \alpha_2 \sin(t) \cdots + \alpha_{n-1} \cos(kt) + \alpha_n \sin(kt) \\ &= c + \sum_{j=1}^{k} (a_j \cos(jt) + b_j \sin(jt)) \end{aligned}$$

such that

1. $A(t) = 0$ for all $t \in \{t_1, \ldots, t_n\}$ and
2. $A(s) > 0$ for all $s \in S^1 \setminus \{t_1, \ldots, t_n\}$.

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For any $s = (s_1, \ldots, s_{2k}) \in \mathbb{R}^{2k}$,

$$f_s(t) := \prod_{1 \leq j \leq 2k} \sin\left(\frac{s_j - t}{2}\right)$$

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Furthermore, we prove that these are all degree k trigonometric polynomials with $2k$ prescribed roots (counted with multiplicity).

Faces of Carathéodory orbitopes

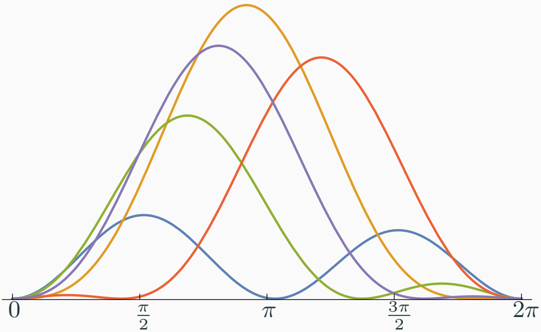

Figure 3: A set of non-negative trigonometric polynomials f_s of degree 2, each of which defines a 1-dimensional face on the boundary of the Carathéodory orbitope $\mathcal{C}_4$. The non-zero root of each polynomial has been chosen at random.

Faces of Barvinok–Novik orbitopes

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The roots of these polynomials are much harder to "control," so it is difficult to choose $s = (s_1, \dots, s_{2k})$ such that g_s is non-negative.

Experimentally, these g_s recover the faces of $\mathcal{B}_4$.

Faces of Barvinok–Novik orbitopes

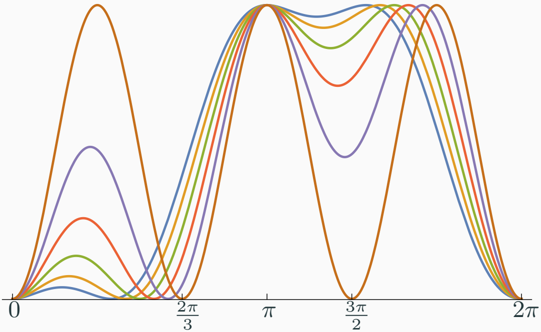


Figure 4: Non-negative trigonometric polynomials g_s of degree 3, each of which defines a face of the Barvinok–Novik orbitope $\mathcal{B}_4$. For clarity, each polynomial has been multiplied by a non-zero constant to achieve the same maximum value.

Generalizations of the Borsuk–Ulam theorem

The $\mathbb{Z}/2\mathbb{Z}$-index

At the appropriate scales, $\frac{2\pi(k-1)}{2k-1} \leq r < \frac{2\pi k}{2k+1}$, we have constructed continuous maps

$$\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1; r) \to \mathbb{R}^{2k} \setminus \{\vec{0}\} \to \partial\mathcal{B}_{2k} \to S^{2k-1}$$

and

$$S^{2k-1} \to \partial\mathcal{B}_{2k} \to \mathrm{VR}^{\mathrm{m}}_{\leq}(S^1; r).$$

Furthermore, these maps are all odd, or $\mathbb{Z}/2\mathbb{Z}$-equivariant.

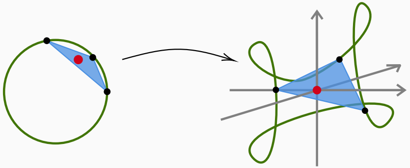
This proves that the $\mathbb{Z}/2\mathbb{Z}$-(co)index of $\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1; r)$ at these scales is $2k-1$.

Odd maps $S^1 \to \mathbb{R}^k$

Knowledge of the $\mathbb{Z}/2\mathbb{Z}$-index of $\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1; r)$ implies the following generalization of the Borsuk–Ulam theorem.

Theorem

If $f\colon S^1 \to \mathbb{R}^{2k+1}$ is odd and continuous, then there is a subset $X \subseteq S^1$ of diameter at most $\frac{2\pi k}{2k+1}$ such that $\mathrm{conv}(f(X))$ contains the origin, and this diameter bound is sharp.



Odd maps $S^n \to \mathbb{R}^k$

Theorem

If $f\colon S^n \to \mathbb{R}^{n+2}$ *is odd and continuous, then there is a subset* $X \subseteq S^n$ *of diameter at most* $\arccos(-1/(n+1))$ *such that* $\operatorname{conv}(f(X))$ *contains the origin, and this diameter bound is sharp.*

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Theorem

If $f\colon S^{2n-1} \to \mathbb{R}^{2kn+2n-1}$ *is odd and continuous, then there is a subset* $X \subseteq S^{2n-1}$ *of diameter at most* $\frac{2\pi k}{2k+1}$ *such that* $\operatorname{conv}(f(X))$ *contains the origin.*

Generalization of the ham sandwich theorem:

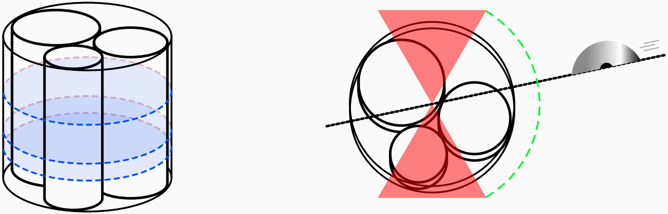

Figure 5: (Left) A bundle of three logs. Dashed blue lines indicate horizontal cuts. (Right) A vertical cut through the center of one slice of the log bundle. In this case, the saw blade is on a fixed pivot that can not swivel by an angle of more than $\frac{2\pi}{3}$.

Recall:

Theorem (Lyusternik–Shnirel'man covering theorem)

For any cover $A_1, \ldots, A_{n+1}$ of the sphere S^n by $n+1$ sets such that the n sets $A_1, \ldots, A_n$ are each either open or closed, there is at least one set containing a pair of antipodal points.

Generalization (in the case of the circle):

Theorem
For $k \geq 1$, suppose $A_1, \ldots, A_{k+1}$ is a cover of the sphere S^1 by $k+1$ sets such that the first k sets $A_1, \ldots, A_k$ are each open or closed. Furthermore, suppose that any subset of the circle of diameter at most $\frac{2\pi(k-1)}{2k-1}$ is contained in some subset A_i. Then, there is at least one set A_i containing a pair of antipodal points.

Definition

Given a set X and a collection of nonempty subsets $\mathcal{U} = \{U_\alpha \subseteq X \mid \alpha \in A\}$ for some index set A, we say $T \subseteq X$ is a **traversal** of $\mathcal{U}$ if $T \cap U_\alpha \neq \varnothing$ for all $\alpha \in A$.

Corollaries

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Theorem

Fix $k \geq 1$. For any collection of $k+1$ closed hemispheres $\mathcal{U} = \{H_1, \ldots, H_{k+1}\}$ of S^1, there exists a traversal of $\mathcal{U}$ of diameter at most $\frac{2\pi(k-1)}{2k-1}$.

Zeros of trigonometric polynomials

Zeros of trigonometric polynomials

Theorem (Gilbert and Smyth, [7, Corollary 1])

Let $[a, b]_{S^1} \subseteq S^1$ denote a closed circular arc of length less than $\frac{2\pi k}{k+1}$. Then, there is a homogeneous trigonometric polynomial of degree k that is positive on $[a, b]_{S^1}$. Moreover, no homogeneous trigonometric polynomial of degree at most k is positive on any subset that contains a closed circular arc of length $\frac{2\pi k}{k+1}$.

Zeros of trigonometric polynomials

Theorem

Let $X \subseteq S^1$ *be such that* $\operatorname{diam}(X) < \frac{2\pi k}{2k+1}$*. Then there is a raked homogeneous trigonometric polynomial of degree* $2k-1$ *that is positive on* X*. Moreover, no raked homogeneous trigonometric polynomial of degree at most* $2k-1$ *is positive on any subset that contains the vertices of a regular inscribed* $(2k+1)$*-gon.*

Lemma

Fix a list of odd continuous functions $f_i(t)\colon S^1 \to \mathbb{R}$ *for* $1 \leq i \leq 2k+1$*. Let* P *be the set of functions of the form* $p\colon S^1 \to \mathbb{R}$ *defined by* $p(t) = \sum_{j=1}^{2k+1} z_j f_j(t)$ *with* $z_j \in \mathbb{R}$*. Then there is a subset* $X \subseteq S^1$ *of diameter at most* $\frac{2\pi k}{2k+1}$ *such that no function in* P *is strictly positive on* X*.*

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Applies, for example, to functions of the form

$$p(t) = \sum_{j=1}^{k} a_j \cos(2j-1)t + \sum_{j=1}^{k} b_j \sin(2j-1)t.$$

Future work

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Thank you!