DISSERTATION

TOPOLOGICAL, GEOMETRIC, AND COMBINATORIAL ASPECTS OF METRIC THICKENINGS

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ABSTRACT

TOPOLOGICAL, GEOMETRIC, AND COMBINATORIAL ASPECTS OF METRIC THICKENINGS

The geometric realization of a simplicial complex equipped with the 1-Wasserstein metric of optimal transport is called a simplicial metric thickening. We describe relationships between these metric thickenings and topics in applied topology, convex geometry, and combinatorial topology. We give a geometric proof of the homotopy types of certain metric thickenings of the circle by constructing deformation retractions to the boundaries of orbitopes. We use combinatorial arguments to establish a sharp lower bound on the diameter of Carathéodory subsets of the centrally-symmetric version of the trigonometric moment curve. Topological information about metric thickenings allows us to give new generalizations of the Borsuk–Ulam theorem and a selection of its corollaries. Finally, we prove a centrally-symmetric analog of a result of Gilbert and Smyth about gaps between zeros of homogeneous trigonometric polynomials.

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DEDICATION

to my family

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Chapter 1

Introduction

Given a metric space X, one may embed X into $\mathscr{P}(X)$, the space of Radon probability measures on X, by mapping a point to the Dirac delta measure centered at that point. This embedding is an isometry onto its image. One may thicken this image inside $\mathscr{P}(X)$ by first taking convex combinations of points and then equipping the resulting set with the restriction of the 1-Wasserstein metric of optimal transport. In the case that these convex combinations are constrained to support sets of a simplicial complex on X such that all simplices have diameter bounded above by $r \geq 0$, the resulting space is a metric r-thickening of X in the sense of Gromov and is called a simplicial metric thickening.

This thesis is an attempt to expose and elucidate mathematical threads connecting simplicial metric thickenings and topics in applied topology and convex geometry, and, to a lesser extent, algebraic topology and topological combinatorics. Along these lines, we explain connections between Čech and Vietoris–Rips metric thickenings and convex bodies in Euclidean space, odd maps with large Carathéodory subsets, generalizations of the Borsuk–Ulam theorem, and quantitative results about the distribution of roots of trigonometric polynomials. We are hopeful the results in this thesis will demonstrate the utility of this multi-faceted approach to the study of metric thickenings.

In Chapter 2, we summarize background material and related work in applied and pure topology and convex geometry. We review Čech and Vietoris–Rips simplical complexes, simplicial metric thickenings defined by the optimal transport metric, reconstruction results for manifolds, and results about the persistent homology of manifolds. We fix notation and definitions from convex geometry and describe families of convex bodies, called orbitopes, that will play a key role in the sequel. We also recall the Borsuk–Ulam theorem and state a number of its corollaries; we give generalizations of this theorem and its corollaries in Chapter 5.

In Chapter 3, we consider Čech and Vietoris–Rips metric thickenings of the circle. We establish the homotopy types of these spaces, and certain subspaces, at particular scales in Theorems 3.1.3 and 3.2.3. Our proof technique involves mapping a metric thickening along the (symmetric) trigonometric moment curve into Euclidean space then projecting the image to a topological sphere given as the boundary of an orbitope.

Our work in Chapter 3 motivates the study of Carathéodory subsets of the (symmetric) trigonometric moment curve in Chapter 4. We say $Y \subseteq \mathbb{R}^k$ is a Carathéodory subset if the convex hull of Y contains the origin. Such subsets of the trigonometric moment curve were described by Gilbert and Smyth in [50]; we provide an analogous characterization of these subsets for the symmetric trigonometric moment curve in Theorem 4.2.1. Later in this chapter, we explicitly construct the Carathéodory coorbitope cone, that is, we give a formula for the coefficients of all degree k trigonometric polynomials with $2k$ prescribed roots counted with multiplicity in S^1.

We obtain generalizations of the Borsuk–Ulam theorem for maps $S^n \to \mathbb{R}^k$, in which the dimension of the codomain may exceed the dimension of the domain, in Chapter 5. These theorems follow from knowledge of the homotopy types of metric thickenings of spheres or from the existence of continuous $\mathbb{Z}/2\mathbb{Z}$-equivariant maps of spheres into and out of metric thickenings. These generalizations imply, in turn, generalizations of the Stone–Tukey (ham sandwich) theorem and the Lyusternik–Schnirel'man–Borsuk covering theorem.

In Chapter 6, we compile results about the zero sets of real trigonometric polynomials. Some of these results are used elsewhere in this thesis, while others are corollaries or re-phrasings of results proved in earlier chapters.

We conclude in Chapter 7 and describe the potential for future work along these lines.

Chapter 2

Background and related work

2.1 Applied topology

In applied topology, one interprets invariants of certain topological spaces associated to a metric space (X,d). Because the simplicial (co)homology of finite simplical complexes is amenable to direct computation, these topological spaces are commonly taken to be (the geometric realizations of) Čech or Vietoris–Rips simplicial complexes defined on X.

In this section, we recall basic notions of applied topology and topological data analysis.

Definition 2.1.1. A **simplicial complex** K consists of a pair of sets, $(V(K), S(K))$, called the set of **vertices** of K and the set of **simplices** of K, respectively, where $S(K)$ is a set of nonempty finite subsets of $V(K)$. Further, we require that $S(K)$ is closed under taking nonempty subsets (that is, $\tau \in S(K)$ whenever $\varnothing \neq \tau \subseteq \sigma \in S(K)$) and that $\{v\} \in S(K)$ whenever $v \in V(K)$. We allow the set of vertices $V(K)$ to have arbitrary cardinality.

Each element of $S(K)$ is called a **simplex**. Given a simplex $\sigma \in S(K)$, we define the **dimension** of σ to be $n = |\sigma| - 1$ and we say σ is a n**-simplex**. The simplicial complex obtained by restricting $S(K)$ to simplices to dimension at-most n is called the n**-skeleton** of K. Given simplices $\tau \subseteq \sigma \in S(K)$, we call τ a **face** of σ. We say a simplicial complex K is **locally finite** if, for each vertex $v \in V(K)$, we have $\{v\} \subseteq \sigma$ for only a finite number of simplices $\sigma \in S(K)$. The **empty simplicial complex** contains an empty set of vertices and an empty set of simplices.

A classic problem in algebraic topology is to build a topologically-accurate simplicial model of a more complicated space. From an open covering of a space X one may construct a certain simplicial complex, called the nerve complex of the covering, that recovers the homotopy type of X under favorable conditions. This so-called Nerve Lemma motivates the Čech complex, which is the nerve complex of a covering of a metric space by balls of a fixed radius. The Nerve Lemma and its applications are described in Subsection 2.1.2. Also see [19] regarding the introduction of the nerve complex by Aleksandrov in the late 1920s, and see [34, 45] regarding its use in the context of Čech (co)homology.

Definition 2.1.2. Let X be a subset of a metric space (Y, d) and fix $r \geq 0$. The **open Čech simplicial complex of X at scale r**, denoted $\check{\mathrm{C}}_{<}(X; r)$, has X as its vertex set and a simplex $\sigma \subseteq X$ if and only if σ is nonempty, finite, and

$$\bigcap_{v \in \sigma} B\left(v; \tfrac{1}{2} r\right) \neq \varnothing,$$

where $B\left(v; \frac{1}{2} r\right) := \left\{y \in Y \mid d(v, y) < \frac{1}{2} r\right\} \subseteq Y$ denotes the open ball of radius $\frac{1}{2} r$ centered at v.

The **closed Čech simplicial complex of X at scale r**, denoted $\check{\mathrm{C}}_{\leq}(X; r)$, is defined analogously by considering closed balls of radius $\frac{1}{2} r$. We may write $\check{\mathrm{C}}(X; r)$ when a statement holds for both $\check{\mathrm{C}}_{\leq}(X; r)$ and $\check{\mathrm{C}}_{<}(X; r)$.

In the case that $Y = X$, we call $\check{\mathrm{C}}(X; r)$ the **intrinsic Čech complex**. Otherwise, $\check{\mathrm{C}}(X; r)$ is called an **ambient Čech complex**.

By convention, we define $\check{\mathrm{C}}_{<}(X; 0)$ to be the empty simplicial complex, and we let $\check{\mathrm{C}}(X; \infty)$ denote the simplicial complex containing all nonempty finite subsets of X. Note that $\check{\mathrm{C}}_{\leq}(X; 0)$ consists of all singletons $\{v\} \subseteq X$ and no higher-dimensional simplices.

Remark 2.1.3. Given a metric space (X, d), we will assume $\check{\mathrm{C}}(X; r)$ denotes the intrinsic Čech complex unless specified otherwise.

An alternative approach to defining a topologically-accurate simplical model of a more general space was described by Leopold Vietoris in 1927 [99]. After being re-introduced by Elihu Rips in the context of the study of hyperbolic groups [53], these complexes are now commonly called Vietoris–Rips complexes. For a given metric space, a Vietoris–Rips complex may be defined in terms of the diameter of subsets of the space, although it exists more generally for an arbitrary cover of a topological space in analogy with the nerve complex.

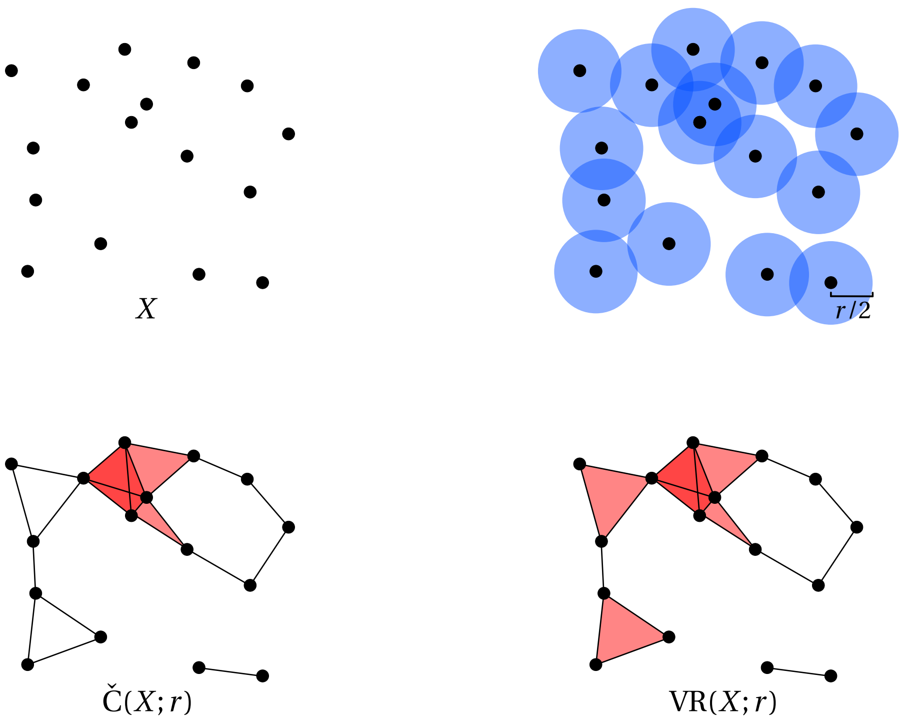


Figure 2.1.1: (Top left) A discrete subset $X \subset \mathbb{R}^2$. (Top right) Balls of radius $\frac{1}{2}r$ centered at the points of X. (Bottom left) The ambient Čech complex at scale r. (Bottom right) The Vietoris–Rips complex at scale r.

Definition 2.1.4. For a metric space (X,d) and subset $A \subseteq X$, we define the **diameter of** A by

$$\operatorname{diam}(A) := \sup_{x,y\in A} d(x,y).$$

Definition 2.1.5. Let X be a metric space and fix $r \geq 0$. The **open Vietoris–Rips simplicial complex of** X **at scale** r, denoted $\mathrm{VR}_<(X;r)$, has X as its vertex set and a simplex $\sigma \subseteq X$ if and only if σ is nonempty, finite, and $\operatorname{diam}(\sigma) < r$.

The **closed Vietoris–Rips simplicial complex of** X **at scale** r, denoted $\mathrm{VR}_\leq(X;r)$, is defined analogously by considering subsets of X of diameter at most r. We may write $\mathrm{VR}(X;r)$ when a statement holds for both $\mathrm{VR}_\leq(X;r)$ and $\mathrm{VR}_<(X;r)$.

By convention, we define $\mathrm{VR}_<(X;0)$ to be the empty simplicial complex, and we let $\mathrm{VR}(X;\infty)$ denote the simplicial complex containing all nonempty finite subsets of X. Note that $\mathrm{VR}_\leq(X;0)$ consists of all singletons $\{v\} \subseteq X$ and no higher-dimensional simplices.

Observe that the relationship $\check{\mathrm{C}}(X;r) \subseteq \mathrm{VR}(X;r) \subseteq \check{\mathrm{C}}(X;2r)$ holds for any metric space X.

While Čech complexes are perhaps the more classical objects of study in the context of (co)homology theories, they are computationally disadvantaged in practice. Unlike a Vietoris–Rips complex, which is entirely determined by its 1-skeleton, one must check many intersections to construct a Čech complex and must store information about its higher-skeleta.

Other simplicial complexes defined on metric spaces include witness complexes and flag complexes, for example [49, p. 29-30].

To a simplicial complex K we may associate a topological space called the geometric realization of K. As an intermediate step, for an integer $n \geq 0$ we define the **standard geometric n-simplex** to be the convex hull of the $(n+1)$ standard basis vectors in $\mathbb{R}^{n+1}$. Then, as a set, the geometric realization of K is defined to be

$$|K| := \left\{ \sum_{i=0}^{k} \lambda_i x_i \;\middle|\; k \geq 0,\ \{x_0, \ldots, x_k\} \in S(K),\ \lambda_i \in \mathbb{R}_{\geq 0},\ \sum_{i=0}^{k} \lambda_i = 1 \right\}.$$

Most commonly, this set is equipped with the weak topology with respect to the identification of shared boundaries of simplices, where each n-simplex $\sigma \in S(K)$ is identified with the geometric n-simplex with basis vectors labeled by vertices contained in σ. As a consequence of this definition, a map $f\colon |K| \to Y$ from the geometric realization of a simplicial complex to a space Y is continuous if and only if the restriction $f|_{|\sigma|}$ is continuous for each simplex $\sigma \in S(K)$ [76, Theorem 2, p. 290]. We define the **geometric realization of** K to be the set $|K|$ equipped with the weak topology, and we typically identify a simplicial complex with its geometric realization.

Alternatively, the set $|K|$ is sometimes given the **metric topology**, defined to be the coarsest topology such that all barycentric coordinate maps are continuous (see [57, Definition 2.1.1] or [90, Section 4.5], for example). If K is locally finite, then this space coincides with the geometric realization.

When the vertex set of K is itself a metric space X, as it is for a Čech or Vietoris–Rips complex, the set $|K|$ may alternatively be equipped with the topology induced by the 1-Wasserstein metric under the identification $x \leftrightarrow \delta_x$ for all $x \in X$. This space, which is defined precisely in Subsection 2.1.1, is called the **simplicial metric thickening** of K. As with the metric topology, the simplicial metric thickening coincides with the geometric realization whenever K is locally finite [3, Corollary 6.4].

A **filtration of topological spaces** is a functor from the poset $(\mathbb{R}, \leq)$ to the category of topological spaces. Explicitly, a filtration of topological spaces $\mathscr{U} = (U_*, f_{*,*})$ is a collection of pairs $(U_r, f_{r,s})$, one for each $r, s \in \mathbb{R}$ with $r \leq s$, such that U_r is a topological space, $f_{r,s} \colon U_r \to U_s$ is a continuous map, $f_{r,r} = \mathrm{id}_{U_r}$, and $f_{s,t} \circ f_{r,s} = f_{r,t}$ whenever $r \leq s \leq t$.

Vietoris–Rips and Čech complexes can be used to obtain filtrations of topological spaces from a given metric space X. For example, by defining spaces $U_r := \varnothing$ for $r < 0$ and $U_r := \mathrm{VR}_{\leq}(X; r)$ for $r \geq 0$, and inclusions $f_{r,s} := \iota_{r,s} \colon \mathrm{VR}_{\leq}(X; r) \hookrightarrow \mathrm{VR}_{\leq}(X; s)$ whenever $r \leq s$, one obtains the (closed) **Vietoris–Rips filtration** of X, denoted $\mathrm{VR}_{\leq}(X; *)$.

Similarly, a **filtration of vector spaces** is a functor from the poset $(\mathbb{R}, \leq)$ to the category of vector spaces. Here, continuous maps are replaced by linear maps.

Definition 2.1.6. Let $\mathscr{U} = (U_*, f_{*,*})$ be a filtration of topological spaces. By applying homology in degree k with coefficients in a field $\mathbb{F}$ to $\mathscr{U}$, we obtain a filtration of vector spaces $H_k(\mathscr{U}; \mathbb{F}) = H_k((U_*, f_{*,*}); \mathbb{F})$, called the **persistence module of $\mathscr{U}$ in degree k**. By convention, we may denote the resulting persistence module by $\mathrm{PH}_k(\mathscr{U}; \mathbb{F})$.

Definition 2.1.7. Given a field $\mathbb{F}$ and an interval λ of $\mathbb{R}$, let $I_\lambda(*)$ denote the **indecomposable persistence module** consisting of vector spaces $I_\lambda(r) = \mathbb{F}$ for $r \in \lambda$ and zero otherwise, and identity maps $f_{r,s} = \iota_{r,s}$ whenever $[r, s] \subseteq \lambda$ and zero otherwise.

Theorem 2.1.8 ([35, 36])**.** *If X is a totally bounded metric space, then there is a family of intervals Λ such that the persistence module induced by the Vietoris–Rips filtration decomposes as*

$$\mathrm{PH}_k(\mathrm{VR}(X;*);\mathbb{F}) = \bigoplus_{\lambda\in\Lambda} I_\lambda(*)$$

for any non-negative integer $k \geq 0$ and field $\mathbb{F}$.

While the notion of a totally bounded space exists more generally, a metric space X is **totally bounded** if there exists a finite cover of X by ε-balls for all $\varepsilon > 0$. For example, a subset of Euclidean space is totally bounded if and only if it is bounded.

The decomposition in Theorem 2.1.8 is called the **barcode decomposition** associated to the Vietoris–Rips filtration of X. An analogous barcode decomposition exists for other simplicial filtrations, including the Čech filtration, for example. The data of the barcode decomposition across all homological degrees is collectively referred to as the **persistent homology** of X.

Persistent homology is a common tool in applied topology with applications to data analysis [32, 48, 97], machine learning [15, 73], computer vision [7, 9, 10, 13], biology [97, 109], chemistry [77, 108], materials science [59, 82, 83, 85], medicine [25, 73, 86], neuroscience [38, 69], natural language processing [112], engineering [24, 87, 111], sensor networks [5, 14, 37, 40, 41, 65], statistics [30, 33], and fractal dimensions [6], for example.

The persistence module of a filtration $\mathscr{U}$ arising in practice is often **tame** in the sense that there exist only finitely many scales at which the homology $H_k(\mathscr{U};\mathbb{F})$ changes, and for each scale $r \in \mathbb{R}$, there exist only finitely many intervals $\lambda \in \Lambda$ such that $I_\lambda(r)$ is nonzero. In fact, a persistence module induced by the Vietoris–Rips or Čech filtration of a finite metric space is always tame. In these cases, it is often useful to display the collection of intervals Λ in the barcode decomposition as a **barcode diagram**. Figure 2.1.2 contains an example of the barcode diagrams associated to the 0- and 1-dimensional Vietoris–Rips persistent homology of a finite metric space.

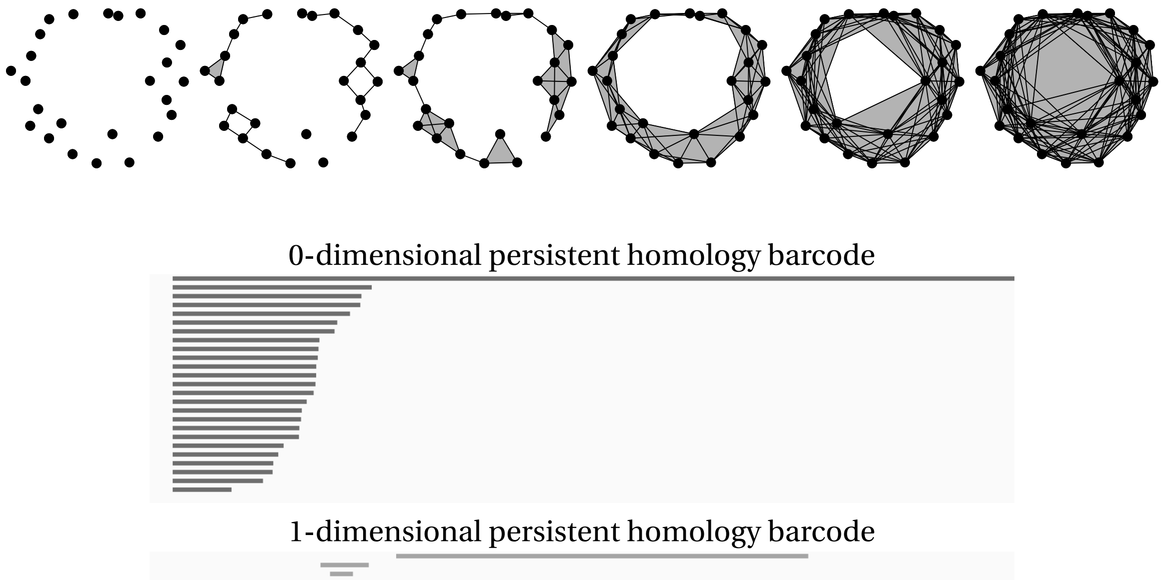


Figure 2.1.2: (Top) A discrete metric space and its Vietoris–Rips complexes at five different choices of scale. (Bottom) The 0-dimensional and 1-dimensional persistent homology intervals. The horizontal axis is the scale parameter r in the construction of the Vietoris–Rips complex.

There are a number of useful distances on the space of barcodes, including the Wasserstein distance and the bottleneck distance (see [43], for example). An important feature of Vietoris–Rips persistent homology is that the resulting barcodes are stable to distortions of the underlying metric space.

Definition 2.1.9. Let Z be a metric space and let X and Y be subsets of Z. The **Hausdorff distance** between X and Y is defined to be

$$d_{\mathrm{H}}(X,Y) := \inf\{\varepsilon > 0 \mid Y \subseteq X_\varepsilon \text{ and } X \subseteq Y_\varepsilon\},$$

where X_ε denotes the set of points of Z of distance at most ε from X, and similarly for Y_ε.

In other words, the Hausdorff distance is the infimum over all $\varepsilon > 0$ such that the (closed) ε-neighborhood of X contains Y, and vice versa. The Hausdorff distance between two subsets may be infinite, and it may be zero for a pair of distinct subsets; in this way, it defines an extended pseudometric on the powerset of a metric space.

The Hausdorff distance generalizes to a distance between any pair of metric spaces.

Definition 2.1.10. The **Gromov–Hausdorff** distance between metric spaces X and Y is defined by

$$d_{\mathrm{GH}}(X,Y) := \inf_{f,g} d_{\mathrm{H}}(f(X), g(Y)),$$

where the infimum is taken over all metric spaces Z and all pairs of isometric embeddings $f\colon X \to Z$ and $g\colon Y \to Z$.

Theorem 2.1.11 ([36])**.** *Fix an integer $k \geq 0$ and let X and Y be totally bounded metric spaces such that $d_{\mathrm{GH}}(X,Y) \leq \varepsilon$. Then, the bottleneck distance between the barcodes for* $\mathrm{PH}_k(\mathrm{VR}(X;*))$ *and* $\mathrm{PH}_k(\mathrm{VR}(Y;*))$ *is bounded above by* 2ε.

Hence, small distortions of a totally bounded metric space X, as measured in the Gromov–Hausdorff distance, can yield only small distortions of the barcodes of the Vietoris–Rips persistent homology of X, as measured in the bottleneck distance. Analogous results hold for other simplicial filtrations, including the Čech filtration, as well as for other distances on the space of barcodes.

2.1.1 Simplicial metric thickenings and optimal transport

When a metric space X is not finite, it is often impossible to equip $\check{\mathrm{C}}(X;r)$ or $\mathrm{VR}(X;r)$ with a metric without changing the homeomorphism type. In fact, a simplicial complex is metrizable if and only if it is locally finite [90, Proposition 4.2.16(2)]. This fact motivates the consideration of simplicial metric thickenings, which preserve a given metric on the vertex set of a simplicial complex. This definition was first introduced in [3]; see [11, 12, 16, 31, 80, 81] for related work on simplicial metric thickenings.

Let δ_x denote the Dirac delta mass at a point $x \in X$ and let $\mathscr{P}(X)$ denote the space of all Radon probability measures on X.

Definition 2.1.12 ([3])**.** Let X be a metric space, and let K denote a simplicial complex with vertex set X. The **simplicial metric thickening** (or simply **metric thickening**) of K is defined to be the following submetric space of $\mathscr{P}(X)$,

$$K^{\mathrm{m}} := \left\{ \sum_{i=0}^{k} \lambda_i \delta_{x_i} \;\middle|\; k \geq 0,\ \{x_0, \ldots, x_k\} \in S(K),\ \lambda_i \in \mathbb{R}_{\geq 0},\ \sum_{i=0}^{k} \lambda_i = 1 \right\},$$

equipped with the restriction of the 1-Wasserstein metric. Here, the superscript m denotes "metric" and is not a variable or parameter.

Given a measure $\mu = \sum_{i=0}^{k} \lambda_i \delta_{x_i} \in K^{\mathrm{m}}$, the **support** of μ, denoted $\mathrm{supp}(\mu)$, is the set of those x_i such that $\lambda_i > 0$. Under the identification $x \leftrightarrow \delta_x$, observe that the underlying set of K^{m} is equal to that of the geometric realization of K. The **Vietoris–Rips metric thickening** $\mathrm{VR}^{\mathrm{m}}(X;r)$ and the **Čech metric thickening** $\check{\mathrm{C}}^{\mathrm{m}}(X;r)$ are the special cases $K(X) = \mathrm{VR}(X;r)$ and $K(X) = \check{\mathrm{C}}(X;r)$, respectively. Throughout, we may simply write "metric thickening" when the distinction is unimportant, or if the underlying simplicial complex is clear through context.

The 1-Wasserstein metric is also called the Kantorovich, optimal transport, or earth mover's metric [98,100,101]; it provides a notion of distance between probability measures defined on a metric space. Although it exists more generally [44,63,64], the 1-Wasserstein metric on K^{m} can be defined as follows. Given $\mu, \mu' \in K^{\mathrm{m}}$ with $\mu = \sum_{i=0}^{k} \lambda_i \delta_{x_i}$ and $\mu' = \sum_{j=0}^{k'} \lambda'_j \delta_{x'_j}$, define a **matching** p between μ and μ' to be any collection of non-negative real numbers $\{p_{i,j}\}_{i,j}$ such that $\sum_{j=0}^{k'} p_{i,j} = \lambda_i$ and $\sum_{i=0}^{k} p_{i,j} = \lambda'_j$. Define the **cost of a matching** p to be $\sum_{i,j} p_{i,j} d(x_i, x'_j)$. The 1-**Wasserstein distance** between μ and μ', then, is the infimum over all matchings p between μ and μ' of the cost of p.

Note that $\mathrm{VR}^{\mathrm{m}}_{\leq}(X;0)$ is isometric to X. Furthermore, contrary to the situation for an arbitrary Vietoris–Rips complex, the embedding $X \hookrightarrow \mathrm{VR}^{\mathrm{m}}(X;r)$ into the Vietoris–Rips metric thickening given by $x \mapsto \delta_x$ is continuous and an isometry onto its image. For these reasons, we typically identify $x \in X$ with the measure $\delta_x \in \mathrm{VR}^{\mathrm{m}}(X;r)$ in the image of this embedding.

If M is a complete Riemannian manifold with curvature bounded from above and below, then $\mathrm{VR}^{\mathrm{m}}(M;r)$ is homotopy equivalent to M for r sufficiently small [3,17]; this is stated precisely in Theorem 2.1.28. This property provides an analogue of Hausmann's theorem (Theorem 2.1.23) for metric thickenings.

Similar results hold for Čech metric thickenings, including continuity of the inclusion and an analogue of Hausmann's theorem.

We remark that $\mathrm{VR}^{\mathrm{m}}_{\le}(X;r)$ is a **metric r-thickening** of X in the sense of Gromov [54], that is, the metric on $\mathrm{VR}^{\mathrm{m}}_{\le}(X;r)$ extends that of X and $d(\mu,x) \le r$ for all $\mu \in \mathrm{VR}^{\mathrm{m}}_{\le}(X;r)$ and $x \in X$. Similarly, $\check{\mathrm{C}}^{\mathrm{m}}_{\le}(X;r)$ is a metric r-thickening of X. In fact, any simplicial metric thickening $K^{\mathrm{m}}(X)$ such that each simplex has bounded diameter $r \ge 0$ is a metric r-thickening in this sense [3, Lemma 3.6].

2.1.2 Recovery of a metric space from a simplicial complex

In this subsection, we consider the problem of recovering the homotopy type of a metric space from its Čech or Vietoris–Rips simplicial complexes or metric thickenings.

A classical result in algebraic topology is the Nerve Theorem, which asserts that the homotopy type of a sufficiently nice space coincides with that of a certain nerve complex.

Definition 2.1.13. Let X be topological space, and let $\mathscr{U} = \{U_\alpha \mid \alpha \in V\}$ be an open cover of X for some index set V. The **nerve complex of $\mathscr{U}$** is a simplicial complex, denoted $\mathscr{N}(\mathscr{U})$, with vertex set V and a simplex for every finite set $\sigma \subseteq V$ such that $\cap_{\beta\in\sigma} U_\beta \neq \varnothing$.

Definition 2.1.14. For a metric space X and scale $r > 0$, we say a cover $\mathscr{U}$ of X satisfies the **Čech nerve condition** if $\mathscr{N}(\mathscr{U}) \simeq \check{\mathrm{C}}(X;r)$. Similarly, we say a cover $\mathscr{U}'$ of X satisfies the **Vietoris–Rips nerve condition** if $\mathscr{N}(\mathscr{U}') \simeq \mathrm{VR}(X;r)$.

Example 2.1.15. The open Čech complex of a metric space X at scale r is equal to the nerve of the cover of X by open balls of radius $r/2$. In other words, $\check{\mathrm{C}}_{<}(X;r) = \mathscr{N}(\mathscr{U})$ where $\mathscr{U} = \{B(x;r/2) \mid x \in X\}$. Hence, this cover satisfies the Čech nerve condition.

Analogously, a closed Čech complex is equal to the nerve of the cover of the space by closed balls of radius $r/2$.

A cover $\mathscr{U}$ of a space X such that all finite intersections of elements of $\mathscr{U}$ are either empty or contractible is said to be a **good cover** of X. We say a cover $\mathscr{U}$ of X is **numerable** if it admits a locally finite partition of unity on X subordinate to $\mathscr{U}$. We say a space is **paracompact** if every open cover of the space is numerable. Consequently, a closed cover of a paracompact space is numerable if the interiors of the elements of $\mathscr{U}$ cover the space. Furthermore, every metric space is paracompact.

Theorem 2.1.16 (Nerve Theorem)**.** *Let $\mathscr{U}$ be a good numerable cover of a space X. Then, X is homotopy equivalent to the nerve complex $\mathscr{N}(\mathscr{U})$.*

A proof of the Nerve Theorem is contained in [55, Corollary 4.G3], for example. The proof depends on the existence of a certain partition of unity; hence, the assumption that $\mathscr{U}$ is numerable can not be omitted.

The Nerve Theorem immediately implies the following.

Remark 2.1.17. Given a metric space X, if for some $r > 0$ there exists a good numerable cover of X that satisfies the Čech nerve condition, then $\check{\mathrm{C}}(X;r) \simeq X$. Analogously, if for some $r > 0$ there exists a good numerable cover of X that satisfies the Vietoris–Rips nerve condition, then $\mathrm{VR}(X;r) \simeq X$.

Example 2.1.15 and Remark 2.1.17 together imply the following reconstruction result for Čech complexes.

Lemma 2.1.18 (Čech Reconstruction Lemma)**.** *Given a metric space X, let $\mathscr{U}_r = \{B(x;r) \mid x \in X\}$ denote the cover of X by open balls of radius r. If the cover $\mathscr{U}_r$ is a good cover for some $r > 0$, then $\check{\mathrm{C}}_<(X;2r) \simeq X$.*

Analogous reconstruction results for Vietoris–Rips complexes are less common. Recently, Žiga Virk established the following sufficient condition for a cover to satisfy the Vietoris–Rips nerve condition (cf. Example 2.1.15).

Theorem 2.1.19 ([105, Theorem 3.9])**.** *For a metric space X and scale $r > 0$, let $\mathscr{U}$ be a cover of X such that every finite subset $\sigma \subseteq X$ is contained in an element of $\mathscr{U}$ if and only if* $\operatorname{diam}(\sigma) < r$ *(respectively,* $\operatorname{diam}(\sigma) \le r$*). Then, $\mathscr{U}$ satisfies the Vietoris–Rips nerve condition for* $\mathrm{VR}_<(X;r)$ *(respectively,* $\mathrm{VR}_\le(X;r)$*).*

For example, the nerve of the cover of X consisting of all subsets of diameter less than r is homotopy equivalent to $\mathrm{VR}_<(X;r)$.

Theorem 2.1.19 follows from an application of Dowker Duality [42], which itself is a consequence of the Nerve Theorem. In analogy with the Čech Reconstruction Lemma, Remark 2.1.17 and Theorem 2.1.19 together imply the following.

Lemma 2.1.20 (Vietoris–Rips Reconstruction Lemma, [105, Theorem 4.4])**.** *Given a metric space X and $r > 0$, suppose there exists a good numerable cover $\mathscr{U}$ of X such that every finite subset $\sigma \subseteq X$ is contained in an element of the cover if and only if* $\operatorname{diam}(\sigma) < r$ *(respectively,* $\operatorname{diam}(\sigma) \le r$*). Then,* $\mathrm{VR}_<(X;r) \simeq X$ *(respectively,* $\mathrm{VR}_\le(X;r) \simeq X$*).*

Hence, the homotopy type of a metric space X may be recovered by a Čech complex or a Vietoris–Rips complex provided a sufficiently nice cover of X exists.

Next, we consider the reconstruction of Riemannian manifolds and geodesic spaces.

Definition 2.1.21. We say a metric space X is a **geodesic space** if, for any pair of points $x, y \in X$, there exists a path $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$, $\gamma(1) = y$, and the length of $\gamma([0,1])$ is equal to $d(x,y)$. We will call such a path γ a **geodesic**. We say a geodesic γ is **arc-length parametrized** if $d(\gamma(s),\gamma(s')) = |s - s'|$ for all s and s' in the domain of γ.

Note that our definition of a geodesic is more restrictive than the concept of a geodesic in Riemannian geometry.

The following technical definition will play an important role in the recovery of a geodesic space by Vietoris–Rips complexes.

Definition 2.1.22. Given a geodesic space X, let $r(X)$ denote the supremum over all real numbers r satisfying the following conditions.

1. For all $x, y \in X$ with $d(x, y) < 2r$, there exists a unique geodesic from x to y,

2. if $x, y, z \in X$ with each of $d(x, y)$, $d(y, z)$, and $d(x, z)$ less than r, then any point t on the geodesic from x to y satisfies $d(t, z) \leq \max\{d(x, z), d(y, z)\}$, and

3. if γ and γ' are arc-length parameterized geodesics such that $\gamma(0) = \gamma'(0)$, and if $0 \leq s$, $s' < r$, and $0 \leq t < 1$, then $d(\gamma(ts), \gamma'(ts')) \leq d(\gamma(s), \gamma'(s'))$.

In [56], Hausmann proves that the homotopy type of the Vietoris–Rips complex defined on a geodesic space agrees with that of the underlying space for sufficiently small scales $r > 0$.

Theorem 2.1.23 (Hausmann's Theorem, [56])**.** *Let X be a geodesic space such that $r(X) > 0$. Then,* $\mathrm{VR}(X; r) \simeq X$ *whenever* $0 < r < r(X)$.

As remarked in [56], it is known that $r(X) > 0$ if M admits a strictly positive injectivity radius and an upper bound on its sectional curvature. In particular, every compact Riemannian manifold X satisfies $r(X) > 0$.

Corollary 2.1.24. *For a compact Riemannian manifold X, there exists $r_0 > 0$ such that* $\mathrm{VR}(X; r) \simeq X$ *whenever* $0 < r < r_0$.

Remark 2.1.25. To prove Theorem 2.1.23, Hausmann does not explicitly construct homotopy inverses $X \to \mathrm{VR}(X; r)$ and $\mathrm{VR}(X; r) \to X$. Instead, the axiom of choice is used to imply the existence of a total order on the points of X, and this total order is used to construct a map $T \colon \mathrm{VR}(X; r) \to X$. Then, T is shown to induce an isomorphism on all homology and homotopy groups at sufficiently small scales, and Whitehead's Theorem is invoked to conclude that T is a homotopy equivalence. One may expect that the inclusion $X \hookrightarrow \mathrm{VR}(X; r)$ defined by $\{x\} \mapsto x$ is a homotopy inverse of T; however, this map is not continuous.

In [105], Virk provides a simpler proof of Hausmann's Theorem.

In practice, one can not explicitly construct and compute with a Vietoris–Rips complex defined on all points of a manifold. The following result of Latschev provides a condition under which the homotopy type of a manifold is recoverable from only a (potentially finite) subset of the manifold.

Theorem 2.1.26 ([68, Theorem 1.1])**.** *Let X be a closed Riemannian manifold. Then, there exists $r_0 > 0$ such that, given $0 < r < r_0$, there exists $\delta_r > 0$ with $\mathrm{VR}(Y; r) \simeq X$ whenever $d_{GH}(X, Y) < \delta_r$.*

Adamaszek, Adams, and Frick provide a version of Hausmann's Theorem for Vietoris–Rips metric thickenings of manifolds in [3].

Definition 2.1.27. Given a complete Riemannian manifold X with sectional curvature bounds from above and below, let $\rho(X)$ denote the supremum over all real numbers ρ satisfying the following conditions.

1. For each $x \in X$, the geodesic ball $B(x; \rho)$ of radius ρ centered at x is convex, and
2. $\rho < \frac{1}{4}\pi\Delta^{-1/2}$, where Δ denotes the sectional curvature upper bound $K \leq \Delta$ for X.

Theorem 2.1.28 (Metric Hausmann's Theorem, [3, Theorem 4.2])**.** *Let X be a complete Riemannian manifold with sectional curvature bounds from above and below such that $\rho(X) > 0$. Then, $\mathrm{VR}^{\mathrm{m}}(X; r) \simeq X$ whenever $0 \leq r < \rho(X)$.*

Remark 2.1.29. Unlike Hausmann's proof of Theorem 2.1.23, the authors of [3] explicitly construct homotopy inverses $C\colon \mathrm{VR}^{\mathrm{m}}(X; r) \to X$ and $\iota\colon X \to \mathrm{VR}^{\mathrm{m}}(X; r)$. The first map sends a measure μ to its unique Karcher mean C_μ in X (see [100], for example), which is shown to be well-defined and continuous by the assumption that $r < \rho(X)$. The second map is simply the inclusion $X \hookrightarrow \mathrm{VR}^{\mathrm{m}}(X; r)$ defined by $x \mapsto \delta_x$. Notably, this inclusion is continuous with respect to the 1-Wasserstein metric, in contrast to the situation for the ordinary simplicial complex $\mathrm{VR}(X; r)$. Then, $C \circ \iota = \mathrm{id}_X$ and $\iota \circ C \simeq \mathrm{id}_{\mathrm{VR}^{\mathrm{m}}(X;r)}$ by a linear homotopy.

Another theorem along these lines, for subsets of Euclidean space, is provided by Adams and Mirth. As with the proof of Theorem 2.1.28, the authors construct a homotopy inverse to the now-continuous inclusion $X \hookrightarrow \mathrm{VR}^{\mathrm{m}}(X; r)$. This inverse is well-defined only for subsets of Euclidean space with positive reach.

Definition 2.1.30. Let $X \subseteq \mathbb{R}^n$ and let $Y = \{y \in \mathbb{R}^n \mid \exists x_1 \neq x_2 \text{ with } d(x_1, y) = d(x_2, y) = d(X, y)\}$. The **reach** of X is $\tau(X) := d(X, \overline{Y})$.

Theorem 2.1.31 ([17, Theorem 3.4])**.** *Let $X \subseteq \mathbb{R}^n$ such that $\tau(X) > 0$. Then,* $\mathrm{VR}^{\mathrm{m}}(X; r) \simeq X$ *whenever* $0 \le r < \tau$.

Hence, the homotopy type of any subset of Euclidean space with positive reach may be recovered by its Vietoris–Rips metric thickenings at sufficiently small scales.

2.1.3 The persistent homology of manifolds

In this subsection, we primarily restrict attention to simplical complexes and metric thickenings defined on manifolds. Throughout, spheres are equipped with the intrinsic geodesic metric in which great circles have circumference 2π.

The results of Hausmann (Theorem 2.1.23), Latschev (Theorem 2.1.26), Adamaszek, Adams, Frick, and Mirth (Theorem 2.1.28 and Theorem 2.1.31) describe conditions under which the homotopy type of a manifold is recoverable from a Vietoris–Rips complex or metric thickening for sufficiently small scales $r > 0$. However, much less is known about the topological behavior of these constructions at large scales, that is, scales at which the hypotheses of these theorems are no longer satisfied, even though large values of r arise naturally in applications of persistent homology [43]. However, more is known in the specific case when the underlying manifold is the circle. The following theorem from [2] is based on [1, 4].

Theorem 2.1.32 ([2, Main Result])**.** *There are homotopy equivalences*

$$\check{\mathrm{C}}_{\le}(S^1; r) \simeq \begin{cases} S^{2k-1} & \text{if } \frac{2\pi(k-1)}{k} < r < \frac{2\pi k}{k+1} \\ \bigvee^{\mathfrak{c}} S^{2(k-1)} & \text{if } r = \frac{2\pi(k-1)}{k}, \end{cases}$$

and

$$\mathrm{VR}_{\leq}(S^1;r) \simeq \begin{cases} S^{2k-1} & \text{if } \frac{2\pi(k-1)}{2k-1} < r < \frac{2\pi k}{2k+1} \\ \bigvee^{\mathfrak{c}} S^{2(k-1)} & \text{if } r = \frac{2\pi(k-1)}{2k-1}, \end{cases}$$

where $k = 1,2,\ldots$, *and where* $\mathfrak{c}$ *denotes the cardinality of the continuum.*

The proof of Theorem 2.1.32 depends on combinatorial arguments about the circle that do not generalize easily to higher spheres.

Hausmann's results in [56] imply that $\mathrm{VR}_{\leq}(S^n;r) \simeq S^n$ for all $0 < r < \frac{\pi}{2}$. In [70], Lim, Mémoli, and Okutan prove that this homotopy equivalence persists at scales beyond $\frac{\pi}{2}$.

Definition 2.1.33. Throughout, let $\Delta_n := \arccos\left(-\frac{1}{n+1}\right)$ denote the diameter of a regular inscribed $(n+1)$-simplex in S^n.

Lemma 2.1.34 ([70, Corollary 7.1 and Remark 7.6])**.** *There is a homotopy equivalence*

$$\mathrm{VR}_{<}(S^n;r) \simeq S^n \quad \text{for} \quad 0 < r < \Delta_n.$$

Furthermore, the homotopy type of $\mathrm{VR}_{<}(S^n;r)$ *changes at the scale* $r = \Delta_n$.

In the context of Theorem 2.1.32, note that $\Delta_1 = \arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$.

Lim, Mémoli, and Okutan establish the following homotopy equivalence for Vietoris–Rips complexes of the 2-sphere at certain scales beyond $r = \Delta_2$.

Theorem 2.1.35 ([70, Corollary 7.4])**.** *There is a homotopy equivalence*

$$\mathrm{VR}_{<}(S^2;r) \simeq S^2 * \frac{\mathrm{SO}(3)}{A_4} \quad \text{for} \quad \Delta_2 < r < \arccos\left(-\frac{1}{\sqrt{5}}\right).$$

Here, A_4 *denotes the tetrahedral group and* $\frac{\mathrm{SO}(3)}{A_4}$ *is a finite quotient of the symmetry group of the* 2*-sphere as described in [3].*

Both Lemma 2.1.34 and Theorem 2.1.35 are proved by establishing a homotopy equivalence between the Vietoris–Rips complex and an r-neighborhood of the Kuratowski embedding of the sphere.

Definition 2.1.36. Given a compact metric space (X, d_X), let $L^\infty(X)$ denote the Banach space of all bounded real-valued functions on X with the l^∞ norm. The embedding $K\colon X \to L^\infty(X)$ defined by $x \mapsto d_X(x,\cdot)$ is called the **Kuratowski embedding**.

Note that K is an isometric embedding, since $d_X(x,x') = ||d_X(x,\cdot) - d_X(x',\cdot)||_\infty$.

Theorem 2.1.37 ([70])**.** *Given a compact metric space X, let $B(K(X);r) \subseteq L^\infty(X)$ denote the open r-neighborhood of the Kuratowski embedding of X. Then,* $\mathrm{VR}_<(X;2r) \simeq B(K(X);r)$ *for all $r > 0$.*

The following lemma, due to M. Katz, together with Theorem 2.1.37, proves Theorem 2.1.35 and the first part of Lemma 2.1.34.

Lemma 2.1.38 ([61, Remark, p. 508], [62, Theorem 1.1])**.** *The neighborhood $B(K(S^n);r)$ of the Kuratowski embedding of a sphere S^n is homotopy equivalent to S^n for all $0 < r < \frac{1}{2}\Delta_n$. Furthermore, the neighborhood $B(K(S^2);r)$ of the Kuratowski embedding of the 2-sphere is homotopy equivalent to $\Sigma^3 \frac{\mathrm{SO}(3)}{A_4}$ for all $\frac{1}{2}\Delta_2 < r < \frac{1}{2}\arccos\left(-\frac{1}{\sqrt{5}}\right)$.*

In fact, Theorem 2.1.37 is a special case of the following much stronger and more general theorem due to Lim, Mémoli, and Okutan.

Theorem 2.1.39 ([70, Isomorphism Theorem])**.** *Let $\lambda\colon \mathsf{Met} \to \mathsf{PMet}$ be a metric homotopy pairing (for example, the Kuratowski functor). Then,* $\mathrm{B}_* \circ \lambda\colon \mathsf{Met} \to \mathsf{hTop}_*$ *is naturally isomorphic to* VR_{2*}.

In Theorem 2.1.39, the category Met consists of compact metric spaces and 1-Lipschitz maps. The category hTop_* consists of filtrations of topological spaces and homotopy classes of maps of filtrations. The category PMet consists of pairs of compact metric spaces, (X,E), where $X \hookrightarrow E$ isometrically, with 1-Lipschitz maps modulo an equivalence relation described in [70, Definition 14]. A **metric homotopy pairing** is any functor $\eta\colon \mathsf{Met} \to \mathsf{PMet}$ that is right-adjoint to the forgetful functor. The functor B_* sends a metric pair (X,E) with isometric embedding $\iota\colon X \hookrightarrow E$ to the filtration $B(\iota(X);*)$ of open neighborhoods of $\iota(X)$ in E. Finally, the functor VR_{2*} sends a metric space X to the filtration $\mathrm{VR}_<(X;2*)$ of open Vietoris–Rips complexes.

Theorem 2.1.39 may be used to imply a number of results about the persistence barcode associated to a manifold or to a more general metric space.

Theorem 2.1.40 ([70, Proposition 9.15])**.** *Let X be a closed, connected, n-dimensional Riemannian manifold. Then, the barcode decomposition of $\mathrm{PH}_n(\mathrm{VR}(X;*);\mathbb{F})$ contains $(0, 2\mathit{FillRad}(X)]$, where*

$$\mathbb{F} = \begin{cases} \mathbb{R} \text{ or } \mathbb{Q} & \text{if } X \text{ is orientable} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } X \text{ is not orientable,} \end{cases}$$

and this is the unique n-dimensional interval starting at 0.

Here, the **filling radius** $\mathrm{FillRad}(X)$ of a closed, connected, Riemannian manifold X was defined by Gromov [52] to be the infimal $r > 0$ such that the pushforward of the inclusion

$$\iota_r\colon X \hookrightarrow s_X^{-1}([0,r))$$

to n-dimensional homology annihilates the fundamental class $[X]$ of X, and where s_X denotes the sublevel set filtration of the distance function of the Kuratowski embedding of X,

$$s_X\colon L^\infty(X) \to \mathbb{R}_{\geq 0},\ f \mapsto \inf_{x\in X} ||d_X(x,\cdot) - f||_\infty.$$

For example, Katz proved [61, Theorem 2] that the filling radius of a sphere is $\mathrm{FillRad}(S^n) = \frac{1}{2}\Delta_n$.

Similar theorems along these lines obtained by the authors of [70] include a bound on the lengths of bars in the Vietoris–Rips persistent homology of a compact metric space in terms of a metric concept called the **spread** of the space (see [70, Definition 18] or [61, Lemma 1], for example), as well as a description of the kinds of bars that appear in terms of the inclusion or exclusion of their endpoints.

To date, the homotopy type of the Vietoris–Rips metric thickening of a sphere S^n is known up to and including the first scale at which the homotopy type changes (cf. Conjecture 3.2.1).

Theorem 2.1.41 ([3, Proposition 5.3 and Theorem 5.4])**.** *There are homotopy equivalences*

$$\mathrm{VR}^{\mathrm{m}}_{\leq}(S^n; r) \simeq \begin{cases} S^n & \text{if } 0 \leq r < \Delta_n \\ S^n * \frac{\mathrm{SO}(n+1)}{A_{n+2}} & \text{if } r = \Delta_n, \end{cases}$$

where $\frac{\mathrm{SO}(n+1)}{A_{n+2}}$ is a finite quotient of the symmetry group of the n-sphere, as described in [3].

Related papers include [47], which studies the 1-dimensional persistent homology of Čech and Vietoris–Rips complexes of metric graphs, [103] which extends this to geodesic spaces, [104] which studies approximations of Vietoris–Rips complexes by finite samples even at higher scale parameters, and [110] which applies Bestvina–Brady discrete Morse theory to Vietoris–Rips complexes.

2.2 Convex geometry and orbitopes

Convex geometry is the study of convex sets, especially convex polytopes and their facial structures [113]. In what follows, we let V denote a real vector space.

Given a subset $Y \subseteq V$, define

$$\mathrm{conv}(Y) := \left\{ \sum_{i=0}^{k} \lambda_i x_i \;\middle|\; k \geq 0,\ x_i \in Y,\ \lambda_i \geq 0,\ \sum_{i=0}^{k} \lambda_i = 1 \right\}$$

to be the **convex hull** of Y. For example, Figure 2.2.2 shows the convex hull of the image of the map $f\colon S^1 \to \mathbb{R}^3$ defined by $f(t) = (\cos(t), \sin(t), \cos(2t))$. Given any finite set $\{\lambda_0, \ldots, \lambda_k\} \subset \mathbb{R}$ such that $\lambda_i \geq 0$ for all i and $\sum_{i=0}^{k} \lambda_i = 1$, we say $\{\lambda_0, \ldots, \lambda_k\}$ is a collection of **convex coefficients**, and we call any expression of the form $\sum_{i=0}^{k} \lambda_i x_i$ a **convex combination** of the set $\{x_0, \ldots, x_k\}$.

Given a subset $Y \subseteq V$, define

$$\operatorname{cone}(Y) := \left\{ \sum_{i=0}^{k} \lambda_i x_i \;\middle|\; k \in \mathbb{N},\ x_i \in Y,\ \lambda_i \geq 0 \right\}$$

to be the **conical hull of** Y (or the **cone over** Y).

A subset $C \subseteq V$ is **convex** if, for any $x, y \in C$, the closed line segment with endpoints x and y is contained in C (that is, if $(1-t)x + ty \in C$ for all $t \in [0,1]$). Note that both the convex hull and the conical hull of a set $Y \subseteq V$ are convex.

Let $C \subseteq V$ be convex. We define a **face** of C to be any subset $F \subseteq C$ such that if $x, y \in C$ and $(1-t)x + ty \in F$ for all $t \in [0,1]$, then $x, y \in F$. We define an **exposed face** of C to be any intersection of C with an affine hyperplane P such that C is contained in a closed half-space bounded by P (and P is called a **supporting hyperplane** in this case). Because any face of C is contained in an exposed face, we will not distinguish between faces and exposed faces when the distinction is unimportant. Note that both $\varnothing$ and C are faces of C; a nonempty face that is a proper subset of C is called a **proper face** of C.

An extrinsic description of a convex body $C \subseteq V$ is given in terms of its support function.

Definition 2.2.1. Given a subset $Y \subseteq V$, we define the **support function** of Y by

$$m_Y \colon V^* \to \mathbb{R}, \quad l \mapsto \sup\{l(x) \mid x \in Y\}.$$

With this definition, a convex body C consists of the points $x \in V$ such that $l(x) \leq m_C(l)$ for every $l \in V^*$. Given any subset $Y \subseteq V$, we call the polar body

$$Y^\circ := \{l \in V^* \mid m_Y(l) \leq 1\}$$

the **dual of** Y. One may check that Y° is convex, even if Y is not.

Let $Y \subseteq \mathbb{R}^k$ be a set in Euclidean space. Carathéodory's theorem states that if the convex hull of Y contains the origin, then there is a subset $Y' \subseteq Y$ whose convex hull also contains the origin and such that $|Y'| \leq k+1$.

Definition 2.2.2. Given $Y \subseteq \mathbb{R}^n$, we say $Y' \subseteq Y$ is a **Carathéodory subset of** Y if the convex hull of Y' contains the origin.

2.2.1 Conventions regarding spheres and circular arcs

Throughout, we equip the sphere S^n with the intrinsic metric in which great circles have circumference 2π. With easy modifications, all stated results hold when S^n is instead equipped with the restriction of the Euclidean metric on $\mathbb{R}^{n+1}$.

Identify S^1 with $\mathbb{R}/2\pi\mathbb{Z}$. For $t \in [0, 2\pi)$, we will identify t with the coset $t + 2\pi\mathbb{Z}$ and write $t \in S^1$. Conversely, given a coset $t + 2\pi\mathbb{Z}$, we will assume $t \in [0, 2\pi)$ unless specified otherwise. We will use the following notation to specify circular arcs.

Definition 2.2.3. Let $a, b \in S^1$ with $a \neq b$. We define the **open circular arc**

$$(a,b)_{S^1} := \begin{cases} \{t \in \mathbb{R} \mid a < t < b\} \mod 2\pi\mathbb{Z} & \text{if } a < b \\ \{t \in \mathbb{R} \mid a < t < b + 2\pi\} \mod 2\pi\mathbb{Z} & \text{if } b < a. \end{cases}$$

The **closed circular arc** $[a,b]_{S^1}$ is defined analogously.

2.2.2 Trigonometric polynomials

Trigonometric polynomials play a special role in the definition and study of certain convex bodies that we consider in Subsections 2.2.6 and 2.2.7. A **trigonometric polynomial** is an expression of the form

$$p(t) = c + \sum_{j=1}^{k} \left(a_j \cos(jt) + b_j \sin(jt)\right),$$

inducing a map $S^1 \to \mathbb{R}$ under the identification $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Throughout, we assume all coefficients are real. In the case that $c = 0$, we call p a **homogeneous trigonometric polynomial**. The set $S \subseteq \{1, \ldots, k\}$ of integers j with $a_j \neq 0$ or $b_j \neq 0$ is called the **spectrum of** p, and the largest integer in S is the **degree of** p. The spectrum of p constrains the set of roots of p; for example, if p is homogeneous of degree n then it has a root on any closed circular arc of length $\frac{2\pi k}{k+1}$; see [21, 50]. Kozma and Oravecz in [67] give upper bounds on the length of an arc where a trigonometric polynomial with spectrum bounded away from zero (that is, $S \subseteq [j, k]$) is nonzero. If the spectrum of p consists only of odd integers, then p is called a **raked trigonometric polynomial**.

2.2.3 The trigonometric moment curve

Definition 2.2.4. For $k \geq 1$, the **trigonometric moment curve** $\mathrm{M}_{2k} \colon S^1 \to \mathbb{R}^{2k}$ is defined by

$$\mathrm{M}_{2k}(t) := (\cos(t), \sin(t), \cos(2t), \sin(2t), \ldots, \cos(kt), \sin(kt))^\intercal.$$

Here, we identify the domain S^1 with $\mathbb{R}/2\pi\mathbb{Z}$. A related map is the **moment curve** $\gamma_k \colon \mathbb{R} \to \mathbb{R}^k$, which is defined by $\gamma_k(t) := (t, t^2, \ldots, t^k)^\intercal$. In [46], Gale shows that the facial lattices of the convex bodies $\mathrm{conv}(\gamma_{2k}(\mathbb{R}))$ and $\mathrm{conv}(\mathrm{M}_{2k}(S^1))$ are equivalent for all $k \geq 1$.

2.2.4 The centrally symmetric trigonometric moment curve

The centrally symmetric trigonometric moment curve is analogous to the trigonometric moment curve, with the additional property that it is symmetric under the involution $x \mapsto -x$.

Definition 2.2.5. For $k \geq 1$, the **centrally symmetric trigonometric moment curve** (or **symmetric moment curve**) $\mathrm{SM}_{2k} \colon S^1 \to \mathbb{R}^{2k}$ is defined by

$$\mathrm{SM}_{2k}(t) := \big(\cos t, \sin t, \cos 3t, \sin 3t, \ldots, \cos(2k-1)t, \sin(2k-1)t\big)^\intercal.$$

Again, we identify the domain S^1 with $\mathbb{R}/2\pi\mathbb{Z}$. Since $\mathrm{SM}_{2k}(t+\pi) = -\mathrm{SM}_{2k}(t)$, we say that SM_{2k} is centrally symmetric about the origin. Interestingly, this curve is closely related to the multidimensional scaling embedding $S^1 \hookrightarrow \mathbb{R}^{2k}$ of the geodesic circle [8, 26, 60, 106]; multidimensional scaling is a way to map a metric space into Euclidean space in a way that distorts the metric (in some sense) as little as possible.

2.2.5 Orbitopes

Recall that a **real representation of a group** G is a group homomorphism $\rho\colon G \to \mathrm{GL}(V)$, where V is a finite dimensional real vector space. For $g \in G$ and $v \in V$, we define $g \cdot v := \rho(g)(v)$.

Definition 2.2.6. An **orbitope** is the convex hull of an orbit of a compact algebraic group G acting linearly on a finite dimensional real vector space V. In particular, the orbitope of G with respect to a vector $v \in V$ is the convex body

$$\operatorname{conv}(G \cdot v) := \operatorname{conv}\{g \cdot v \mid g \in G\} \subseteq V.$$

In this definition, we assume that the representation ρ of G is clear from context.

An orbitope is a convex semialgebraic set, and the orbit itself is a real algebraic variety. Hence, orbitopes are prototypical objects of study in the field of convex algebraic geometry. We refer the reader to [91] for numerous examples of orbitopes and an exposition of their properties from the perspectives of convexity, algebraic geometry, and optimization.

The following definition will allow us to better understand the boundaries of orbitopes.

Definition 2.2.7. Given an orbitope $\mathscr{O} = \operatorname{conv}(G \cdot v)$ with $v \in V$, recall that the support function $m_{\mathscr{O}}\colon V^* \to \mathbb{R}$ is defined by $m_{\mathscr{O}}(l) = \sup\{l(x) \mid x \in \mathscr{O}\}$. Then, the polar body

$$\mathscr{O}^\circ := \{l \in V^* \mid m_{\mathscr{O}}(l) \leq 1\}$$

is called the **coorbitope** of $\mathscr{O}$. The **coorbitope cone** of $\mathscr{O}$ is defined to be

$$\widehat{\mathscr{O}^\circ} := \{(r, l) \in \mathbb{R} \oplus V^* \mid m_{\mathscr{O}}(l) \leq r\}.$$

In what follows, we will restrict attention to orbitopes $\operatorname{conv}(G \cdot v)$ such that the orbit $G \cdot v$ coincides with the curve M_{2k} or SM_{2k} as defined in Subsections 2.2.3 and 2.2.4, respectively. Further, we will observe that elements of the resulting coorbitope cones correspond precisely to the trigonometric polynomials defining faces of these convex bodies.

2.2.6 Carathéodory orbitopes

The **Carathéodory orbitopes** are defined by $\mathscr{C}_{2k} \coloneqq \operatorname{conv}(\mathrm{M}_{2k}(S^1)) \subseteq \mathbb{R}^{2k}$ for integers $k \geq 1$. Note that the boundary of $\mathscr{C}_{2k}$ is homemorphic to the sphere S^{2k-1}.

This convex body is not the convex hull of a finite set of points; it is an orbitope instead of a polytope [91].

To realize $\mathscr{C}_{2k}$ as an orbitope in the sense of Definition 2.2.6, first recall that the irreducible representations ρ_n of SO(2) are indexed by integers $n \geq 0$. We let ρ_0 denote the trivial representation. When $n > 0$, the representation ρ_n acts on $\mathbb{R}^2$ and sends a rotation matrix to its n^{th} power, that is,

$$\rho_n \colon \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \mapsto \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}^n = \begin{pmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{pmatrix}. \tag{2.1}$$

Then, by defining the direct sum of representations

$$\rho \coloneqq \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_k,$$

the Carathéodory orbitope $\mathscr{C}_{2k}$ is the convex hull of the orbit $\mathrm{SO}(2) \cdot (1,0)^k$ under the action of ρ on $(\mathbb{R}^2)^k \cong \mathbb{R}^{2k}$.

Remark 2.2.8. Given a vector $z \in \mathbb{R}^{2k}$, observe that the inner product of z and $\mathrm{M}_{2k}(t)$ is a homogeneous trigonometric polynomial of degree $2k-1$. By considering the definition of a face of a convex body in terms of a supporting hyperplane, we observe that the image of $\{t_1, \ldots, t_n\} \subset S^1$ is the vertex set of a proper face $\operatorname{conv}(\{\mathrm{M}_{2k}(t_1), \ldots, \mathrm{M}_{2k}(t_n)\})$ of $\mathscr{C}_{2k}$ if and only if there exists a trigonometric polynomial $p \not\equiv 0$ of degree at most k such that p is either non-negative or non-positive on S^1 and $p(t_i) = 0$ for all $1 \le i \le n$. For brevity, we will say that such a polynomial p "defines the face" $\operatorname{conv}(\{\mathrm{M}_{2k}(t_1), \dots, \mathrm{M}_{2k}(t_n)\})$ of $\mathscr{C}_{2k}$.

Figure 2.2.1: Examples of non-negative trigonometric polynomials defining faces of Carathéodory orbitopes. A formula for the coefficient vectors of all such polynomials in given in Section 4.3.

On the other hand, given $(c, \langle v, -\rangle) \in \mathbb{R} \oplus (\mathbb{R}^{2k})^*$, observe

$$\begin{aligned}
\widehat{\mathscr{C}^\circ_{2k}} \ni (c, \langle v, -\rangle) &\iff m_{\mathscr{C}_{2k}}(\langle v, -\rangle) \le c \\
&\iff 0 \le c - \max_{\sum_i \lambda_i \mathrm{M}_{2k}(t_i) \in \mathscr{C}_{2k}} \langle v, \Sigma_i \lambda_i \mathrm{M}_{2k}(t_i)\rangle \\
&\iff 0 \le c + \min_{\sum_i \lambda_i \mathrm{M}_{2k}(t_i) \in \mathscr{C}_{2k}} \langle -v, \Sigma_i \lambda_i \mathrm{M}_{2k}(t_i)\rangle \\
&\iff 0 \le c + \min_{t \in S^1} \langle -v, \mathrm{M}_{2k}(t)\rangle \\
&\iff 0 \le c + \langle -v, \mathrm{M}_{2k}(t)\rangle \quad \text{for all } t \in S^1.
\end{aligned}$$

Note that the second line follows from compactness of $\mathscr{C}_{2k}$. Hence, $(c, \langle v, -\rangle) \in \widehat{\mathscr{C}^\circ_{2k}}$ if and only if the trigonometric polynomial $c + \langle -v, \mathrm{M}_{2k}(t)\rangle$ is non-negative on S^1. For that reason, given $v = (-a_1, -b_1, \dots, -a_k, -b_k)$, we make the identification $c + \langle -v, \mathrm{M}_{2k}(t)\rangle \leftrightarrow (c, a_1, b_1, \dots, a_k, b_k)$ and write

$$\widehat{\mathscr{C}^\circ_{2k}} \cong \left\{ (c, a_1, b_1, \dots, a_k, b_k) \in \mathbb{R}^{2k+1} \;\middle|\; 0 \le c + \sum_{j=1}^{k} (a_j \cos(jt) + b_j \sin(jt)) \right\}.$$

In this way, the coorbitope cone $\widehat{\mathscr{C}_{2k}^{\circ}}$ consists precisely of the coefficient vectors of the trigonometric polynomials defining faces of $\mathscr{C}_{2k}$, with all strictly positive (or strictly negative) polynomials corresponding to the empty face $\varnothing$ and with the zero polynomial corresponding to the maximal face $\mathscr{C}_{2k}$. In Section 4.3, we will explicitly construct the vectors that generate the coorbitope cone $\widehat{\mathscr{C}_{2k}^{\circ}}$.

Theorem 2.2.9 ([91, Corollary 5.4])**.** *The proper faces of $\mathscr{C}_{2k}$ are in inclusion-preserving bijection with sets of at most k points in S^1.*

In particular, any point in a proper face of $\mathscr{C}_{2k}$ can be expressed as a convex combination $\sum_{i=1}^{m} \lambda_i \mathrm{M}_{2k}(t_i)$ with $t_i \in S^1$ and $m \leq k$, and vice versa. Note that any $\{t_1, \ldots, t_m\} \subset S^1$ with $m \leq k$ must be disjoint from some open arc of length at least $\frac{2\pi}{k}$, and hence in some ball of S^1 of radius $r \leq \frac{\pi(k-1)}{k}$. In this way, the facial structure of Carathédory orbitopes is related to Čech simplicial complexes defined on S^1 at scale $r \geq \frac{2\pi(k-1)}{k}$. We will explore this connection more thoroughly in Chapter 3.

Figure 2.2.2: (Left) The image of the map $S^1 \to \mathbb{R}^3$ defined by $t \mapsto (\cos(t), \sin(t), \cos(2t))$. (Right) The convex hull of this set.

The convex hull displayed in Figure 2.2.2 is not a Carathéodory orbitope; rather, it is the convex hull of a three-dimensional projection of the curve M_4. Observe that some faces of $\mathscr{C}_4$ are visible, however.

2.2.7 Barvinok–Novik orbitopes

The **Barvinok–Novik orbitopes** [23] are defined by $\mathscr{B}_{2k} := \mathrm{conv}(\mathrm{SM}_{2k}(S^1)) \subseteq \mathbb{R}^{2k}$ for $k \geq 1$. Note that the boundary of $\mathscr{B}_{2k}$ is homeomorphic to the sphere S^{2k-1}.

As with the Carathéodory orbitopes $\mathscr{C}_{2k}$, we can realize $\mathscr{B}_{2k}$ as an orbitope in the sense of Definition 2.2.6. In this case, the representation ρ of SO(2) is the direct sum of representations $\rho = \rho_1 \oplus \rho_3 \oplus \rho_5 \oplus \cdots \oplus \rho_{2k-1}$ with each ρ_i as defined in Equation 2.1.

Remark 2.2.10. Given a vector $z \in \mathbb{R}^{2k}$, observe that the inner product of z and $\mathrm{SM}_{2k}(t)$ is a raked homogeneous trigonometric polynomial of degree $2k-1$. Hence, we observe that $\{t_1, \ldots, t_n\} \subset S^1$ defines a proper face $\mathrm{conv}(\{\mathrm{SM}_{2k}(t_1), \ldots, \mathrm{SM}_{2k}(t_n)\})$ of $\mathscr{B}_{2k}$ if and only if there exists a raked trigonometric polynomial $p \not\equiv 0$ of degree at most $2k-1$ such that p is non-negative on S^1 and $p(t_i) = 0$ for all $1 \leq i \leq n$.

In analogy with the Carathéodory orbitopes (cf. Remark 2.2.8), the trigonometric polynomials defining faces of $\mathscr{B}_{2k}$ are precisely those whose coefficient vectors belong to the coorbitope cone $\widehat{\mathscr{B}^\circ_{2k}}$.

The faces of $\mathscr{B}_{2k}$ are known for $k = 2$; a subset of these faces is visible in Figure 2.2.3 (which is a subset of $\mathbb{R}^3$, not $\mathbb{R}^4$).

Theorem 2.2.11 ([23, 94])**.** *The proper faces of $\mathscr{B}_4$ are*

- *the 0-dimensional faces (vertices)* $\mathrm{SM}_4(t)$ *for $t \in S^1$,*
- *the 1-dimensional faces (edges)* $\mathrm{conv}(\mathrm{SM}_4(\{t_1, t_2\}))$ *where $t_1 \neq t_2$ are the edges of an arc of S^1 of length at most $\frac{2\pi}{3}$, and*
- *the 2-dimensional faces (triangles)* $\mathrm{conv}(\mathrm{SM}_4(\{t, t+\frac{2\pi}{3}, t+\frac{4\pi}{3}\}))$ *for $t \in S^1$.*

Though the precise facial structure of the Barvinok–Novik orbitope $\mathscr{B}_{2k}$ is currently unknown for $k > 2$, certain neighborliness results have been established [22]. Sinn has shown that the orbitopes are simplicial [93]. Additionally, Vinzant proved that the edges of the boundary of $\mathscr{B}_{2k}$ consist of all line segments $\mathrm{conv}(\mathrm{SM}_{2k}(\{t_0, t_1\}))$ with $d(t_0, t_1) \leq \frac{2\pi(k-1)}{2k-1}$ [102]. In other words, the edges of $\mathscr{B}_{2k}$ are the same as the edges of $\mathrm{VR}(S^1; \frac{2\pi(k-1)}{2k-1})$. The following is an immediate corollary of the work of Sinn and Vinzant.

Theorem 2.2.12 ([93, 102])**.** *Every proper face of the Barvinok–Novik orbitope $\mathscr{B}_{2k}$ is a simplex such that the preimage of the vertex set of the simplex has diameter in S^1 at most $\frac{2\pi(k-1)}{2k-1}$.*

Because every proper face of $\mathscr{B}_{2k}$ is a simplex in $\mathbb{R}^{2k}$, note that every proper face is defined by at most $2k-1$ points in S^1. Furthermore, the diameter bound on the preimage of vertex sets provides a relationship between the facial structure of Barvinok–Novik orbitopes and Vietoris–Rips simplicial complexes defined on S^1 at scale $r \geq \frac{2\pi(k-1)}{2k-1}$. We will explore this connection more thoroughly in Chapter 3.

Figure 2.2.3: (Left) The image of the map $S^1 \to \mathbb{R}^3$ defined by $t \mapsto (\cos(t), \sin(t), \cos(3t))$. (Right) The convex hull of this set.

The convex hull displayed in Figure 2.2.3 is not a Barvinok–Novik orbitope; rather, it is the convex hull of a three-dimensional projection of the curve SM_4. Note that some faces of $\mathscr{B}_4$ are visible, however.

Next, we prove that the maximal-dimensional simplices belonging to $\partial\mathscr{B}_{2k}$ are precisely those defined by sets of $2k-1$ equally-spaced points in S^1. This characterization is stated precisely in Lemma 2.2.17, which will be used to obtain a stronger version of Theorem 5.1.3 in Section 5.1. We will make use of the following theorem about the structure of roots of raked trigonometric polynomials due to Barvinok, Lee, and Novik.

Theorem 2.2.13 (Theorem 3.1.4 of [22])**.** *Let $f(t)$ be a raked trigonometric polynomial, not constantly zero, of degree at most $2k-1$, let $t_1,\dots,t_n \in S^1$ be distinct roots of f in S^1, and let $m_1,\dots,m_n$ be their multiplicities. Suppose $t_1,\dots,t_n$ lie in an arc $\Gamma \subset S^1$ of length less than π, that $\sum_{i=1}^n m_i = 2k$, and that $t^* \in S^1 \setminus \Gamma$ is yet another root of f. Then, $t^* \in \Gamma + \pi$.*

For notational convenience, we make the following definition. Recall Definition 2.2.3 regarding circular arcs.

Definition 2.2.14. Given $\{t_0,\dots,t_k\} \subset S^1$, define

$$\chi(t_i) := \left|\left\{t_j \mid t_j \in (t_i, t_i+\pi)_{S^1}\right\}\right|.$$

The following two technical lemmas together describe a relationship between the diameter of a finite set of points in S^1 and their relative configuration. This relationship will allow us to apply Theorem 2.2.13 to the proof of Lemma 2.2.17.

Lemma 2.2.15. *For $k \geq 1$, let $X = \{t_0,\dots,t_{2k-2}\} \subseteq S^1$ be given with no two points equal or antipodal. Then, $\sum_{i=0}^{2k-2} \chi(t_i) = (k-1)(2k-1)$.*

Proof. Since no two points are equal or antipodal, note that $t_j \in (t_i, t_i+\pi)_{S^1}$ if and only if $t_i \notin (t_j, t_j+\pi)_{S^1}$. Hence, $\sum_{i=0}^{2k-2} \chi(t_i) = \binom{2k-1}{2} = (k-1)(2k-1)$. □

Lemma 2.2.16. *For $k \geq 1$, let $X = \{t_0,\dots,t_{2k-2}\} \subseteq S^1$ be given with no two points equal or antipodal. If $\chi(t_i) = k-1$ for all i, then $\mathrm{diam}(X) \geq \frac{2\pi(k-1)}{2k-1}$. Further, if $\mathrm{diam}(X) = \frac{2\pi(k-1)}{2k-1}$, then the points of X form the vertices of a regular inscribed $(2k-1)$-gon.*

Proof. Define $t_{2k-1} := t_0$ and let ℓ_i denote the length of $(t_i, t_{i+1})_{S^1}$ for all i. Because $\chi(t_i) = k = \chi(t_{i+1})$, it follows that there exists exactly one point t_j in the arc $(t_i+\pi, t_{i+1}+\pi)_{S^1}$. Further, because the function $f\colon (t_i+\pi, t_{i+1}+\pi)_{S^1} \to \mathbb{R}$ defined by $f(t) = \max\{d(t,t_i), d(t,t_{i+1})\}$ is minimized at the midpoint of $(t_i+\pi, t_{i+1}+\pi)_{S^1}$, it follows that $\mathrm{diam}(X) \geq \pi - \frac{\ell_i}{2}$. On the other hand, because there are $2k-1$ consecutive pairs of points t_i, t_{i+1}, we must have $\ell_j \leq \frac{2\pi}{2k-1}$ for some $0 \leq j \leq 2k$. Hence $\mathrm{diam}(X) \geq \pi - \frac{\pi}{2k-1} = \frac{2\pi(k-1)}{2k-1}$.

Next, assume $\operatorname{diam}(X) = \frac{2\pi(k-1)}{2k-1}$. As before, $\operatorname{diam}(X) \geq \pi - \frac{\ell_i}{2}$, and it follows that $\ell_i \geq \frac{2\pi}{2k-1}$ for all i. Finally, because $\sum_{i=0}^{2k-2} \ell_i = 2\pi$, we have $\ell_i = \frac{2\pi}{2k-1}$ for all i. □

Finally, we are prepared to prove Lemma 2.2.17, which characterizes the maximal dimensional proper faces of the Barvinok–Novik orbitope $\mathscr{B}_{2k}$.

Lemma 2.2.17. *For $k \geq 1$, distinct and non-antipodal points $\{t_0, \ldots, t_{2k-2}\} \subset S^1$ define a $(2k-2)$-simplex on the boundary of B_{2k} if and only if they form the vertices of a regular inscribed $(2k-1)$-gon.*

Proof. The backwards direction, at least, has been proven in [22] using the polynomial $1 - \cos(2k-1)t$.

For the forwards direction, suppose points $t_0 < t_1 < \cdots < t_{2k-2}$ on S^1 are given such that $\{t_0, \ldots, t_{2k-2}\}$ define the vertices of a $(2k-2)$-simplex on the boundary of B_{2k}. Suppose, for the sake of contradiction, that these points are not equally spaced. It follows from Lemma 2.2.17 that $\operatorname{diam}(\{t_0, \ldots, t_{2k-2}\}) \leq \frac{2\pi(k-1)}{2k-1}$. Hence, there must exist $0 \leq j \leq 2k-2$ such that $\chi(t_j) \neq k-1$ by Lemma 2.2.16. Since $\sum_{i=0}^{2k-2} \chi(t_i) = \sum_{i=0}^{2k-2} k-1 = (k-1)(2k-1)$ is constant by Lemma 2.2.15, there must exist some $0 \leq i \leq 2k-2$ such that $\chi(t_i) \geq k$. Reordering points sequentially, we may assume $\chi(t_0) \geq k$. Therefore, $t_0, \ldots, t_{k-1}$ belong to the arc $\Gamma = [t_0, t_{k-1}]_{S^1}$ of length less than π. Next, observe that $\{t_0, \ldots, t_{2k-2}\}$ are each roots of multiplicity $m_i = 2$ of a raked trigonometric polynomial of degree $2k-1$. Further, $\sum_{i=0}^{k-1} m_i = \sum_{i=0}^{k-1} 2 = 2k$, and we may apply Theorem 2.2.13 to conclude that the roots $t_k, t_{k+1}, \ldots, t_{2k-2}$ must belong to the arc $\Gamma + \pi$, contradicting the fact that $\chi(t_0) \geq k$. □

2.2.8 Vandermonde matrices and related matrices

A recurring computational tool in our study of the Carathéodory and Barvinok–Novik orbitopes is the Vandermonde matrix. This matrix has a particularly simple determinant, and by converting trigonometric functions into complex exponential form, we are able to reduce certain matrices to Vandermonde (or near-Vandermonde) matrices in order to compute their determinants.

Definition 2.2.18. A *Vandermonde matrix* is an $n \times n$ matrix of the form

$$V = \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix}.$$

The determinant of the above matrix is $\det(V) = \prod_{1 \le i < j \le n}(a_j - a_i)$; see for example [88, Section 2.8.1].

The following matrices and their (sub)determinants will also be useful in our study of the Carathéodory and Barvinok–Novik orbitopes.

Definition 2.2.19. For integers $k \geq 1$, let $\vec{t} = (t_0, \ldots, t_{2k}) \in \mathbb{R}^{2k+1}$ and define the $(2k+1) \times (2k+1)$ matrices

$$\mathbb{M}^1_{2k}(\vec{t}) = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \mathrm{M}_{2k}(t_0) & \mathrm{M}_{2k}(t_1) & \ldots & \mathrm{M}_{2k}(t_{2k}) \end{bmatrix}$$

and

$$\mathbb{SM}^1_{2k}(\vec{t}) = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \mathrm{SM}_{2k}(t_0) & \mathrm{SM}_{2k}(t_1) & \ldots & \mathrm{SM}_{2k}(t_{2k}) \end{bmatrix}.$$

Similarly, define the $2k \times (2k+1)$ matrices

$$\mathbb{M}_{2k}(\vec{t}) = \begin{bmatrix} \mathrm{M}_{2k}(t_0) & \mathrm{M}_{2k}(t_1) & \mathrm{M}_{2k}(t_2) & \ldots & \mathrm{M}_{2k}(t_{2k}) \end{bmatrix}$$

and

$$\mathbb{SM}_{2k}(\vec{t}) = \begin{bmatrix} \mathrm{SM}_{2k}(t_0) & \mathrm{SM}_{2k}(t_1) & \mathrm{SM}_{2k}(t_2) & \ldots & \mathrm{SM}_{2k}(t_{2k}) \end{bmatrix}.$$

Here, we consider $\mathrm{M}_{2k}(t)$ and $\mathrm{SM}_{2k}(t)$ to be written as column vectors.

2.3 The Borsuk–Ulam theorem

The Borsuk–Ulam theorem is one of the most important theorems in elementary algebraic topology. Its power lies in the numerous generalizations, applications, and equivalent formulations of the theorem that have been used to establish many connections between algebraic topology, combinatorics, and discrete geometry. For example, Lovász's proof of the Kneser conjecture [72] from 1978, in which the Borsuk–Ulam theorem is used to prove a statement about the chromatic number of certain planar graphs, was one of the first major results in the field of topological combinatorics. In this section, we outline common formulations of the Borsuk–Ulam theorem and describe some of its well-known corollaries and equivalent statements.

Karol Borsuk's original paper [27] was published in 1933 and contains three equivalent versions of the Borsuk–Ulam theorem. In fact, a set covering theorem (Theorem 2.3.6), from which the Borsuk–Ulam theorem may be deduced, was published slightly earlier by Lyusternik and Schnirel'man [74] in 1930. The only known reference to Stanisław Ulam's involvement with the theorem is the following footnote that appears in Borsuk's paper [27, p. 178].

[7]) Dieser Satz wurde als Vermutung von St. Ulam aufgestellt.

Figure 2.3.1: Translated, this footnote reads "This theorem was posed as a conjecture by St. Ulam."

Since its publication, hundreds of papers containing new proofs, generalizations, and applications of the Borsuk–Ulam theorem have appeared; see [95] for a survey of such results. See [79] for an introductory exposition of combinatorial and geometric results that follow from an application of the Borsuk–Ulam theorem and related techniques.

The following is one of the most common formulations of the Borsuk–Ulam theorem.

Theorem 2.3.1 (Borsuk–Ulam theorem)**.** *Given a continuous map $f\colon S^n \to \mathbb{R}^n$, there exists $x_0 \in S^n$ such that $f(x_0) = f(-x_0)$.*

We say a map $f\colon S^n \to \mathbb{R}^k$ is **odd** or **centrally symmetric** if $f(-x) = -f(x)$ for all $x \in S^n$. More generally, given topological spaces X and Y equipped with $\mathbb{Z}/2\mathbb{Z}$-actions μ and ν respectively, we say a map $f\colon X \to Y$ is **odd** or $\mathbb{Z}/2\mathbb{Z}$**-equivariant** if $f \circ \mu = \nu \circ f$. Throughout, we equip $\mathbb{R}^n$ and S^n with the standard antipodal $\mathbb{Z}/2\mathbb{Z}$-action.

The following two theorems are equivalent formulations of the Borsuk–Ulam theorem for continuous odd maps.

Theorem 2.3.2 (Borsuk–Ulam theorem)**.** *Given a continuous odd map $f\colon S^n \to \mathbb{R}^n$, there exists $x_0 \in S^n$ such that $f(x_0) = \vec{0}$.*

Theorem 2.3.3 (Borsuk–Ulam theorem)**.** *There does not exist a continuous odd map $S^n \to S^{n-1}$.*

Lemma 2.3.4. *Theorems 2.3.1, 2.3.2, and 2.3.3 are equivalent.*

Proof. (2.3.1 $\Longrightarrow$ 2.3.2) If f is odd, then $\vec{0} = f(x_0) - f(-x_0) = f(x_0) + f(x_0)$.

(2.3.2 $\Longrightarrow$ 2.3.1) Observe that the map defined by $g(x) := f(x) - f(-x)$ is continuous and odd.

(2.3.2 $\Longrightarrow$ 2.3.3) Such a map $S^n \to S^{n-1} \subseteq \mathbb{R}^n \setminus \{\vec{0}\}$ would contradict Theorem 2.3.2.

(2.3.3 $\Longrightarrow$ 2.3.2) Assuming a continuous odd map $f\colon S^n \to \mathbb{R}^n \setminus \{\vec{0}\}$ exists, one obtains a continuous odd map $\frac{f}{\|f\|}\colon S^n \to S^{n-1}$ contradicting Theorem 2.3.3. □

While the proof of Lemma 2.3.4 is straightforward, the proof of the Borsuk–Ulam theorem itself is more difficult. For brevity, we do not include a proof and instead refer the reader to [29, 55, 79, 84], for example.

2.3.1 Corollaries of the Borsuk–Ulam theorem

The Borsuk–Ulam theorem has a vast number of corollaries. We include a few notable examples here to indicate how the Borsuk–Ulam theorem is applied in the proof.

Theorem 2.3.5 (Stone–Tukey theorem for measures)**.** *Let $\mu_1, \mu_2, \ldots, \mu_k$ be finite Borel measures in $\mathbb{R}^k$ such that every hyperplane has measure 0 for each of the μ_i. Then, there exists a hyperplane h such that*

$$\mu_i(h^+) = \tfrac{1}{2}\mu_i(\mathbb{R}^k) \quad \textit{for} \quad i = 1, 2, \ldots, d,$$

where h^+ denotes one of the half-spaces defined by h.

Proof sketch. We follow the proof given in [79, Theorem 3.1.1]. Let $\vec{u} = (u_0, u_1, \ldots, u_k)$ be a point of the sphere S^k. We assign $\vec{u}$ to the subset of $\mathbb{R}^k$ defined by

$$h^+(\vec{u}) \coloneqq \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid u_1 x_1 + u_2 x_2 + \cdots + u_k x_k \leq u_0\}.$$

Note that $h^+((1,0,\ldots,0)) = \mathbb{R}^d$ and $h^+((-1,0,\ldots,0)) = \varnothing$; otherwise, if one of the components $u_1, \ldots, u_k$ is nonzero, $h^+(\vec{u})$ is a half-space of $\mathbb{R}^d$. In the case that $h^+(\vec{u})$ is a half-space, observe that antipodal points of S^k correspond to opposite half-spaces.

Next, we define a map $f\colon S^k \to \mathbb{R}^k$ with components

$$f_i(\vec{u}) \coloneqq \mu_i(h^+(\vec{u})).$$

An application of Lebesgue's dominated convergence theorem (see [89, Theorem 1.34], for example) shows that f is continuous.

Finally, the Borsuk–Ulam theorem implies $f(\vec{u}_0) = f(-\vec{u}_0)$ for some $\vec{u}_0$. It cannot happen that $f((1,0,\ldots,0)) = f((-1,0,\ldots,0))$, so $h^+(\vec{u}_0)$ is indeed the desired half-space. $\square$

The Stone–Tukey theorem is also affectionately known as the "ham sandwich" theorem: a portion of ham, cheese, and bread in the plane may be bisected with a single cut. We give generalizations of this theorem, in which the number of measures may exceed the ambient dimension, in Subsection 5.3.1.

A version of the following set-covering theorem first appeared in [74]. We give a proof of this theorem that follows from the Borsuk–Ulam theorem and remark that, in fact, the two theorems are equivalent.

Theorem 2.3.6 (Lyusternik–Shnirel'man–Borsuk covering theorem)**.** *For any cover $A_1, \ldots, A_{n+1}$ of the sphere S^n by $n+1$ sets such that the n sets $A_1, \ldots, A_n$ are each either open or closed, there is at least one set containing a pair of antipodal points.*

Proof. We follow the proof given in [18]. Let the cover $A_1, \dots, A_{n+1}$ be given and assume, for the sake of contradiction, that no set A_i contains antipodal points. Define a map $f\colon S^n \to \mathbb{R}^n$ by

$$f(x) := (\operatorname{dist}(x, A_1), \dots, \operatorname{dist}(x, A_n)).$$

This map is clearly continuous, so the Borsuk–Ulam theorem implies $f(x_0) = f(-x_0)$ for some $x_0 \in S^n$. Since A_{n+1} does not contain antipodal points, there must be some $1 \le i \le n$ such that at least one of x_0 and $-x_0$ is contained in A_i. Relabel points as necessary so that $x_0 \in A_i$. In particular, $f(x_0) = f(-x_0)$ implies that both $\operatorname{dist}(x_0, A_i) = 0$ and $\operatorname{dist}(-x_0, A_i) = 0$.

Now, if A_i is closed, then $\operatorname{dist}(-x_0, A_i) = 0$ implies that $-x_0 \in A_i$, contradicting the assumption that no set in the cover contains antipodal points.

On the other hand, if A_i is open, then $\operatorname{dist}(-x_0, A_i) = 0$ implies that $-x_0$ belongs to the closure $\overline{A_i} \subseteq S^n \setminus (-A_i)$. This contradicts the fact that x_0 belongs to A_i. □

We give a generalization of this theorem, in which the number of sets in the cover may exceed the dimension of the sphere by more than one, in Subsection 5.3.2.

We conclude this section by sketching a proof of Kneser's conjecture. Toward that end, we recall the definition of the chromatic number of a graph.

Definition 2.3.7. Given a simplicial complex G with vertex set $V(G)$ and simplex set $S(G)$, we call G a **graph** if $|\sigma| \le 2$ whenever $\sigma \in S(G)$. In this context, we call each vertex in $V(G)$ a **node** of G and we call each 1-simplex in $S(G)$ an **edge** of G.

Definition 2.3.8. For integers $n \ge k \ge 1$, the **Kneser graph** $K(n,k)$ has

- vertex set consisting of all subsets of $\{1, \dots, n\}$ of cardinality k, called k**-sets**, and
- simplex set such that $\{A, B\}$ is an edge of the graph if and only if $A \cap B = \varnothing$.

Definition 2.3.9. For a graph G, let a function $c\colon V(G) \to \{1, \dots, m\}$ be called an m**-coloring of** G if $c(v_i) \ne c(v_j)$ whenever $\{v_i, v_j\} \in S(G)$. Then, the **chromatic number of** G, denoted $\chi(G)$, is defined to be the minimum over all integers $m \ge 0$ such that a m-coloring exists.

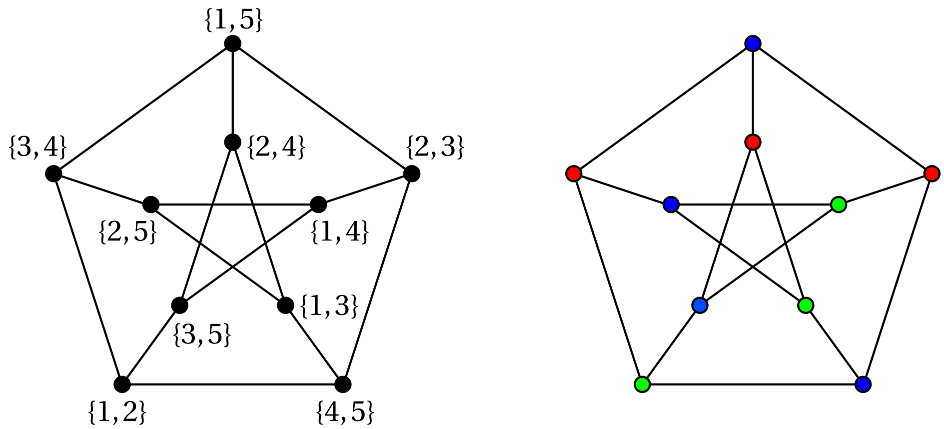


Figure 2.3.2: A 3-coloring of the Petersen graph $K(5,2)$.

Remark 2.3.10. Observe that the Kneser graph $K(n,k)$ has no edges if $n < 2k$. Hence, we will assume $n \geq 2k$ and consider the graphs $K(2k+d,k)$, for integers $k \geq 1$ and $d \geq 0$, for the remainder of this section.

A $(d+2)$-coloring of any Kneser graph $K(2k+d,k)$ is easy to construct. For example, define

$$c(v) := \min\{\min(v), d+2\}$$

for all vertices v of $K(2k+d,k)$. Then, if two sets v and v' are assigned the same color $c(v) = c(v') = i < d+2$, they must both contain i and cannot be disjoint. Otherwise, if two vertices w and w' are assigned the color $c(w) = c(w') = d+2$, then they are both contained in the set $\{d+2,\ldots,2k+d\}$, which has only $2k-1$ elements; hence, they cannot be disjoint either. Consequently, $\chi(K(2k+d,k)) \leq d+2$.

In 1955, Kneser conjectured that, in fact, $\chi(K(2k+d,k)) = d+2$ for all integers $k \geq 1$ and $d \geq 0$ [66]. Twenty-three years later, László Lovász used the Borsuk–Ulam theorem to give the first proof of Kneser's conjecture [72]. Since then, numerous alternative proofs have been found, including a simple proof by then-undergraduate student Joshua Greene in 2002 [51]. In [78], Matoušek gives a purely combinatorial proof of Kneser's conjecture that follows from an application of Tucker's lemma, which itself has a combinatorial proof; however, all other known proofs contain topological arguments.

Theorem 2.3.11 (Kneser–Lovász conjecture)**.** *For all integers $k \geq 1$ and $d \geq 0$, we have*

$$\chi(K(2k+d,k)) = d+2.$$

The following proof of Theorem 2.3.11 is due to Greene [51].

Proof. First, let us identify the elements of $\{1,\ldots,2k+d\}$ with $2k+d$ points $X \subseteq S^{d+1}$ in general position, meaning no $d+2$ points lie in any equatorial d-sphere. We proceed by contradiction and assume that there exists a $(d+1)$-coloring of $K(2k+d,k)$. Define sets $A_1,\ldots,A_{d+1} \subseteq S^{d+1}$ such that $x \in A_i$ if there exists at least one k-set of color i contained in the open hemisphere

$$H(x) := \{y \in S^{d+1} \mid x^{\mathsf{T}} y > 0\}.$$

Finally, we define $A_{d+2} = S^{d+1} \setminus (A_1 \cup \cdots \cup A_{d+1})$. Now, by the Lyusternik–Shnirel'man–Borsuk covering theorem (Theorem 2.3.6), there exists $1 \leq i \leq d+2$ such that A_i contains antipodal points x and $-x$.

If $i \leq d+1$, then there exist two disjoint k-sets with the same color, one in the open hemisphere $H(x)$ and the other in the open hemisphere $H(-x)$, in contradiction with the definition of a coloring of $K(2k+d,k)$.

On the other hand, if $i = d+2$, then both $H(x)$ and $H(-x)$ contain at most $k-1$ points of X, and it follows that the complement $S^{d+1} \setminus (H(x) \cup H(-x))$ contains at least $d+2$ points of X, in contradiction with the assumption that the points of X are in general position. $\square$

Chapter 3

Metric thickenings of the circle

In this chapter, we establish connections between the Carathéodory and Barvinok–Novik orbitopes and certain metric thickenings of the circle. Toward understanding the homotopy types of the Vietoris–Rips and Čech metric thickenings of the circle, we first establish homeomorphisms between subspaces of these metric thickenings and odd-dimensional spheres S^{2k-1}.

3.1 Carathéodory and Barvinok–Novik metric thickenings

Definition 3.1.1. For an integer $k \geq 1$, let $\mathrm{CA}(k)$ denote the simplicial complex with vertex set S^1 and as simplices all nonempty finite sets $\{x_0, \ldots, x_n\} \subseteq S^1$ such that $\mathrm{conv}(\{\mathrm{M}(x_0), \ldots, \mathrm{M}(x_n)\})$ is a face of the Carathéodory orbitope $\mathscr{C}_{2k}$.

Similarly, for an integer $k \geq 1$, let $\mathrm{BN}(k)$ denote the simplicial complex with vertex set S^1 and as simplices all nonempty finite sets $\{x_0, \ldots, x_n\} \subseteq S^1$ such that $\mathrm{conv}(\{\mathrm{SM}(x_0), \ldots, \mathrm{SM}(x_n)\})$ is a face of the Barvinok–Novik orbitope $\mathscr{B}_{2k}$.

We define the **Carathéodory and Barvinok–Novik metric thickenings** to be $\mathrm{CA}^{\mathrm{m}}(k)$ and $\mathrm{BN}^{\mathrm{m}}(k)$, respectively.

Remark 3.1.2. By Theorem 2.2.9, note that $\mathrm{CA}^{\mathrm{m}}(k)$ is a submetric space of $\check{\mathrm{C}}^{\mathrm{m}}_{\leq}(S^1; r)$ for $r \geq \frac{2\pi(k-1)}{k}$. Similarly, by Theorem 2.2.12, note that $\mathrm{BN}^{\mathrm{m}}(k)$ is a submetric space of $\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1; r)$ for $r \geq \frac{2\pi(k-1)}{2k-1}$.

The Carathéodory and the Barvinok–Novik metric thickenings are homeomorphic to the boundaries of the corresponding orbitopes, that is, to odd-dimensional spheres.

Theorem 3.1.3. *For all integers $k \geq 1$, there are homeomorphisms* $\mathrm{CA}^{\mathrm{m}}(k) \cong \partial\mathscr{C}_{2k} \cong S^{2k-1}$ *and* $\mathrm{BN}^{\mathrm{m}}(k) \cong \partial\mathscr{B}_{2k} \cong S^{2k-1}$.

To prove this theorem, we require an intermediate lemma. While this lemma may be stated in more generality, we give a version that is notationally convenient for what follows.

Lemma 3.1.4. *Let X be a compact metric space, and let $f\colon X \to \mathbb{R}^k$ be continuous and bounded. Let $\mathrm{K}^{\mathrm{m}}(X)$ denote the metric thickening of a simplicial complex $K(X)$ with vertex set X. Define a map $\iota\colon \partial\mathrm{conv}(f(X)) \to \mathrm{K}^{\mathrm{m}}(X)$ by $\iota\left(\sum_i \lambda_i f(t_i)\right) = \sum_i \lambda_i \delta_{t_i}$. Then, ι is continuous whenever ι is well-defined.*

Proof. Suppose ι is well-defined, that is, suppose every point of $\partial\mathrm{conv}(f(X))$ can be uniquely expressed as a convex combination $\sum_i \lambda_i f(t_i)$ with all λ_i nonzero, and suppose further that $\sum_i \lambda_i \delta_{t_i} \in \mathrm{K}^{\mathrm{m}}(X)$ whenever $\sum_i \lambda_i f(t_i) \in \partial\mathrm{conv}(f(X))$. Extend $f\colon X \to \mathbb{R}^k$ to a map $f\colon \mathrm{K}^{\mathrm{m}}(X) \to \mathbb{R}^k$ by declaring $f\left(\sum_i \lambda_i \delta_{t_i}\right) = \sum_i \lambda_i f(t_i)$. Here, the sum on the left-hand side defines a measure in $\mathrm{K}^{\mathrm{m}}(X)$ as a convex combination of Dirac delta functions at the points $t_i \in X$, and the sum on the right-hand side is a convex combination of vectors in $\mathbb{R}^k$ belonging to a face of $\mathrm{conv}(f(X))$. By [3, Lemma 5.2], this extension of f is continuous.

We will show that

$$f|_{\iota(\partial\mathrm{conv}(f(X)))}\colon \iota(\partial\mathrm{conv}(f(X))) \to \partial\mathrm{conv}(f(X))$$

is a bijective continuous function from a compact space to a Hausdorff space. It will then follow from [20, Theorem 3.7] that $f|_{\iota(\partial\mathrm{conv}(f(X)))}$ is a homeomorphism with a continuous inverse $\iota\colon \partial\mathrm{conv}(f(X)) \to \iota(\partial\mathrm{conv}(f(X)))$. Therefore, $\iota\colon \partial\mathrm{conv}(f(X)) \to \mathrm{K}^{\mathrm{m}}(X)$ is continuous.

The fact that $f|_{\iota(\partial\mathrm{conv}(f(X)))}$ is bijective follows from the definition of the extension of f to the metric thickening. The space $\partial\mathrm{conv}(f(X))$ is Hausdorff since it inherits the subspace topology from Euclidean space. Last, to see that $\iota(\partial\mathrm{conv}(f(X)))$ is compact, we note that $\iota(\partial\mathrm{conv}(f(X)))$ is a closed subset of $\mathscr{P}(X)$, the space of all Radon probability measures on X equipped with the Wasserstein metric. Since X is compact, it follows that $\mathscr{P}(X)$ is compact by [101, Remark 6.19], and therefore $\iota(\partial\mathrm{conv}(f(X)))$ is compact as a closed subset of a compact space. □

Proof of Theorem 3.1.3. As in the proof of Lemma 3.1.4, we extend both $\mathrm{M}_{2k}\colon S^1 \to \partial\mathscr{C}_{2k}$ and $\mathrm{SM}_{2k}\colon S^1 \to \partial\mathscr{B}_{2k}$ to continuous maps $\mathrm{M}_{2k}\colon \mathrm{CA}^{\mathrm{m}}(k) \to \partial\mathscr{C}_{2k}$ and $\mathrm{SM}_{2k}\colon \mathrm{BN}^{\mathrm{m}}(k) \to \partial\mathscr{C}_{2k}$, respectively. Because M_{2k} and SM_{2k} are continuous and bounded maps, these extensions are continuous by [3, Lemma 5.2]. Furthermore, observe that M_{2k} and SM_{2k} are bijective with inverses $\iota\colon \partial\mathscr{C}_{2k} \to \mathrm{CA}^{\mathrm{m}}(k)$ and $\iota\colon \partial\mathscr{B}_{2k} \to \mathrm{BN}^{\mathrm{m}}(k)$, respectively, and Lemma 3.1.4 proves that these maps are continuous. □

3.2 Čech and Vietoris–Rips metric thickenings of the circle

Recall (see Remark 3.1.2) that the Carathéodry and the Barvinok–Novik metric thickenings are sub-metric spaces of the Čech and the Vietoris–Rips metric thickenings of the circle at certain scales, respectively. Indeed, they are proper subsets: $\mathrm{CA}^{\mathrm{m}}(k)$ and $\mathrm{BN}^{\mathrm{m}}(k)$ contain only simplices of bounded dimension for all k, whereas $\check{\mathrm{C}}^{\mathrm{m}}(S^1;r)$ and $\mathrm{VR}^{\mathrm{m}}(S^1;r)$ contain simplices of arbitrarily high dimension for all $r > 0$. However, we conjecture that these additional high-dimensional simplices do not affect the homotopy type of the metric thickening.

Conjecture 3.2.1. *Both the Čech and the Vietoris–Rips metric thickenings of the circle are homotopy equivalent to odd-dimensional spheres. In particular,*

- $\check{\mathrm{C}}^{\mathrm{m}}_{\le}(S^1;r) \simeq \mathrm{CA}^{\mathrm{m}}(k) \cong \partial\mathscr{C}_{2k} \cong S^{2k-1}$ *whenever* $\frac{2\pi(k-1)}{k} \le r < \frac{2\pi k}{k+1}$.
- $\mathrm{VR}^{\mathrm{m}}_{\le}(S^1;r) \simeq \mathrm{BN}^{\mathrm{m}}(k) \cong \partial\mathscr{B}_{2k} \cong S^{2k-1}$ *whenever* $\frac{2\pi(k-1)}{2k-1} \le r < \frac{2\pi k}{2k+1}$.

As evidence for Conjecture 3.2.1, we will give a geometric proof in the case $k = 1$, and we will show that this geometric argument extends to the case $k = 2$ for the Vietoris–Rips thickening at scale $r = \frac{2\pi}{3}$ (establishing the homotopy equivalences $\check{\mathrm{C}}^{\mathrm{m}}_{\le}(S^1;r) \simeq S^1$ for $0 \le r < \pi$, $\mathrm{VR}^{\mathrm{m}}_{\le}(S^1;r) \simeq S^1$ for $0 \le r < \frac{2\pi}{3}$, and $\mathrm{VR}^{\mathrm{m}}_{\le}(S^1;\frac{2\pi}{3}) \simeq S^3$). We will also demonstrate that the $(2k-1)$-dimensional homology, cohomology, and homotopy groups of $\check{\mathrm{C}}^{\mathrm{m}}_{\le}(S^1;r)$ and $\mathrm{VR}^{\mathrm{m}}_{\le}(S^1;r)$ are nontrivial at the scales $\frac{2\pi(k-1)}{k} \le r < \frac{2\pi k}{k+1}$ and $\frac{2\pi(k-1)}{2k-1} \le r < \frac{2\pi k}{2k+1}$, respectively. Finally, we will state a geometric condition related to the support sets of certain faces of the Carathéodory and the Barvinok–Novik orbitopes (Question 3.2.5) that would be sufficient to prove the conjecture for all k.

We remark that both $\check{\mathrm{C}}^{\mathrm{m}}_{\le}(S^1;2\pi)$ and $\mathrm{VR}^{\mathrm{m}}_{\le}(S^1;\pi)$ are contractible, and are therefore homotopy equivalent to S^∞, in analogy with the simplicial complexes $\check{\mathrm{C}}_{\le}(S^1;2\pi)$ and $\mathrm{VR}_{\le}(S^1;\pi)$ [2]. The following lemma implies, more generally, that for any metric space X of finite diameter, both $\check{\mathrm{C}}^{\mathrm{m}}_{\leq}(X; 2\cdot\mathrm{diam}(X))$ and $\mathrm{VR}^{\mathrm{m}}_{\leq}(X; \mathrm{diam}(X))$ are contractible.

Lemma 3.2.2. *Suppose K is a simplical complex with vertex set X, a metric space, such that every simplex of K is a face of a simplex containing the vertex x_0 for some fixed $x_0 \in X$. Then, the simplicial metric thickening K^m is contractible.*

Proof. We will prove that K^{m} deformation retracts onto $\{\delta_{x_0}\}$. Indeed, let $f\colon K^{\mathrm{m}} \to \{\delta_{x_0}\}$ denote the constant map, and let $\iota\colon \{\delta_{x_0}\} \hookrightarrow K^{\mathrm{m}}$ denote the inclusion. It is clear that f and ι are continuous, and that $f\circ\iota = \mathrm{id}_{\{\delta_{x_0}\}}$. Furthermore, for any $\mu = \sum_i \lambda_i \delta_{x_i} \in K^{\mathrm{m}}$ and $t \in [0,1]$, note that $(1-t)\sum_i \lambda_i\delta_{x_i} + t\delta_{x_0} \in K^{\mathrm{m}}$. Hence, the linear homotopy $H\colon K^{\mathrm{m}} \times I \to K^{\mathrm{m}}$ defined by

$$H(\mu, t) = (1-t)\mu + t[\iota\circ f(\mu)]$$

is well-defined. Finally, H is continuous by [3, Lemma 3.8], and we have $\iota\circ f \simeq \mathrm{id}_{K^{\mathrm{m}}}$. $\square$

In particular, both $\check{\mathrm{C}}^{\mathrm{m}}_{\leq}(S^n; 2\pi)$ and $\mathrm{VR}^{\mathrm{m}}_{\leq}(S^n; \pi)$ are contractible for all integers $n \geq 1$.

An analogous version of Lemma 3.2.2 is true for simplicial complexes. For example, the cone over a simplicial complex is contractible.

3.2.1 Outline of the proof technique

Toward proving Conjecture 3.2.1, we consider a geometric proof technique in which the Čech or Vietoris–Rips metric thickening is first mapped to Euclidean space along the moment curve or the symmetric moment curve, respectively. We then "project away" all high dimensional simplices that do not contribute to the homotopy type of the metric thickening by radially projecting the image of the thickening to the boundary of a Carathéodory or Barvinok–Novik orbitope. Finally, we include the boundary of the corresponding orbitope back into the metric thickening. We prove that the composition of mapping to Euclidean space followed by radial projection is a homotopy equivalence between the metric thickening and the boundary of the orbitope (that is, an odd-dimensional sphere) at certain scales, and conjecture that this map is, in fact, a homotopy equivalence at all scales.

This technique is analogous to the "kernel trick" of machine learning, in which a data set is mapped into a higher dimensional space to illuminate its underlying structure [92]. For example, SM_{2k} maps any regular polygon in S^1 with $2k+1$ vertices to a regular $2k$-*simplex* in $\mathbb{R}^{2k}$ in which all sides have equal length [23], recovering the "true" geometry of the simplex. Finding appropriate generalizations for higher spheres would pave the way for future work developing analogous maps for broader classes of manifolds, thereby illuminating the topology of their Čech and Vietoris–Rips complexes.

In more detail, we will build the following sequence of maps for all integers $k \geq 1$:

$$\check{\mathrm{C}}^{\mathrm{m}}_{\leq}(S^1;r) \xrightarrow{\mathrm{M}_{2k}} \mathbb{R}^{2k} \setminus \{\vec{0}\} \xrightarrow{p} \partial\mathscr{C}_{2k} \xrightarrow{\iota} \check{\mathrm{C}}^{\mathrm{m}}_{\leq}(S^1;r) \qquad \text{for} \qquad \tfrac{2\pi(k-1)}{k} \leq r < \tfrac{2\pi k}{k+1},$$

$$\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1;r) \xrightarrow{\mathrm{SM}_{2k}} \mathbb{R}^{2k} \setminus \{\vec{0}\} \xrightarrow{p} \partial\mathscr{B}_{2k} \xrightarrow{\iota} \mathrm{VR}^{\mathrm{m}}_{\leq}(S^1;r) \qquad \text{for} \qquad \tfrac{2\pi(k-1)}{2k-1} \leq r < \tfrac{2\pi k}{2k+1}.$$

Our construction will proceed as outlined below.

1. Subsection 3.2.2: We extend the domain of M_{2k} and SM_{2k} to the simplicial metric thickenings $\check{\mathrm{C}}^{\mathrm{m}}_{\leq}(S^1;r)$ and $\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1;r)$, respectively, and we define the radial projection maps $p\colon \mathbb{R}^{2k} \setminus \{\vec{0}\} \to \partial\mathscr{C}_{2k}$ and $p\colon \mathbb{R}^{2k} \setminus \{\vec{0}\} \to \partial\mathscr{B}_{2k}$. We note that the compositions $p \circ \mathrm{M}_{2k}$ and $p \circ \mathrm{SM}_{2k}$ are well-defined since the first maps miss the origin; this is proved in Sections 4.1 and 4.2.

2. Subsection 3.2.3: We define the inclusions $\iota\colon \partial\mathscr{C}_{2k} \to \check{\mathrm{C}}^{\mathrm{m}}_{\leq}(S^1;r)$ and $\iota\colon \partial\mathscr{B}_{2k} \to \mathrm{VR}^{\mathrm{m}}_{\leq}(S^1;r)$. Since $(p \circ \mathrm{M}_{2k}) \circ \iota = \mathrm{id}_{\partial\mathscr{C}_{2k}}$ and $(p \circ \mathrm{SM}_{2k}) \circ \iota = \mathrm{id}_{\partial\mathscr{B}_{2k}}$, we obtain that the $(2k-1)$-dimensional homology, cohomology, and homotopy groups of $\check{\mathrm{C}}^{\mathrm{m}}_{\leq}(S^1;r)$ and $\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1;r)$ are nontrivial.

3. Subsection 3.2.4: We prove that $\iota \circ (p \circ \mathrm{M}_{2k}) \simeq \mathrm{id}_{\check{\mathrm{C}}^{\mathrm{m}}_{\leq}(S^1;r)}$ and $\iota \circ (p \circ \mathrm{SM}_{2k}) \simeq \mathrm{id}_{\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1;r)}$ at certain scales $r \geq 0$ and for certain k. More generally, we conjecture that ι is a homotopy equivalence with homotopy inverse $p \circ \mathrm{M}_{2k}$ or $p \circ \mathrm{SM}_{2k}$ at *all* scales, and we give a geo-

metric condition related to the Carathéodory and Barvinok–Novik orbitopes that would be sufficient to prove this conjecture (Question 3.2.5).

When $k=1$, this proof is easy to interpret. The maps M_2 and SM_2 map the spaces $\check{\mathrm{C}}^{\mathrm{m}}_{\leq}(S^1;r)$ and $\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1;r)$, respectively, to an annulus missing the origin in $\mathbb{R}^2$ (see Figure 3.2.1). Then, the map p radially projects the annulus to its outer circle, and the map ι includes the circle back into $\check{\mathrm{C}}^{\mathrm{m}}_{\leq}(S^1;r)$ or $\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1;r)$.

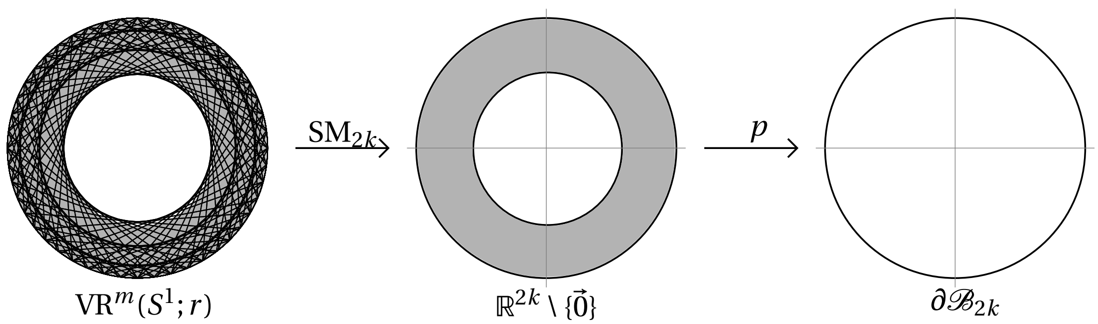


Figure 3.2.1: The composition of maps $\mathrm{VR}^{\mathrm{m}}(S^1;r)\xrightarrow{\mathrm{SM}_{2k}}\mathbb{R}^{2k}\setminus\{\vec{0}\}\xrightarrow{p}\partial\mathscr{B}_{2k}$, drawn in the case $k=1$.

As a result of step (3) of the construction outlined above, we obtain the following theorem.

Theorem 3.2.3. *There is a homotopy equivalence*

$$\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1;r)\simeq\begin{cases} S^1 & 0\leq r<\frac{2\pi}{3} \\ S^3 & r=\frac{2\pi}{3}. \end{cases}$$

Note that $\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1;\frac{2\pi}{3})\not\simeq\mathrm{VR}_{\leq}(S^1;\frac{2\pi}{3})\simeq\bigvee^{\mathfrak{c}}S^2$. In fact, the analogous inclusion $S^1\hookrightarrow\mathrm{VR}_{\leq}(S^1;r)$ into the simplicial complex is not continuous.

3.2.2 Map from the metric thickening to the boundary of an orbitope

Because $\mathscr{C}_{2k}$ and $\mathscr{B}_{2k}$ are convex bodies containing the origin in their interiors, each ray emanating from the origin intersects either $\partial\mathscr{C}_{2k}$ or $\partial\mathscr{B}_{2k}$ exactly once. Hence, the radial projection maps $p\colon\mathbb{R}^{2k}\setminus\{\vec{0}\}\to\partial\mathscr{C}_{2k}\simeq S^{2k-1}$ and $p\colon\mathbb{R}^{2k}\setminus\{\vec{0}\}\to\partial\mathscr{B}_{2k}\simeq S^{2k-1}$ are well-defined.

Next, we extend $\mathrm{M}_{2k}\colon S^1 \to \mathbb{R}^{2k}$ and $\mathrm{SM}_{2k}\colon S^1 \to \mathbb{R}^{2k}$ to maps $\mathrm{M}_{2k}\colon \check{\mathrm{C}}^{\mathrm{m}}_{\le}(S^1;r) \to \mathbb{R}^{2k}$ and $\mathrm{SM}_{2k}\colon \mathrm{VR}^{\mathrm{m}}_{\le}(S^1;r) \to \mathbb{R}^{2k}$, respectively, by extending linearly across simplices. That is, we declare $\mathrm{M}_{2k}\left(\sum_i \lambda_i \delta_{t_i}\right) = \sum_i \lambda_i \mathrm{M}_{2k}(t_i)$ and $\mathrm{SM}_{2k}\left(\sum_i \lambda_i \delta_{t_i}\right) = \sum_i \lambda_i \mathrm{SM}_{2k}(t_i)$. Here, the sum on the left-hand side defines a measure as a convex combination of Dirac delta functions at the points $t_i \in S^1$, whereas the sum on the right-hand side is a convex combination of vectors in $\mathbb{R}^{2k}$. Because M_{2k} and SM_{2k} restricted to S^1 are both continuous and bounded, Lemma 5.2 of [3] proves that their extensions to the metric thickenings are continuous.

Finally, we will prove in Sections 4.1 and 4.2 that $\vec{0} \notin \mathrm{M}_{2k}\left(\check{\mathrm{C}}^{\mathrm{m}}_{\le}(S^1;r)\right)$ whenever $r < \frac{2\pi k}{k+1}$ (Theorem 4.1.1), and similarly that $\vec{0} \notin \mathrm{SM}_{2k}\left(\mathrm{VR}^{\mathrm{m}}_{\le}(S^1;r)\right)$ whenever $r < \frac{2\pi k}{2k+1}$ (Theorem 4.2.1). Hence, the compositions $p \circ \mathrm{M}_{2k}$ and $p \circ \mathrm{SM}_{2k}$, mapping a metric thickening to the boundary of an orbitope in $\mathbb{R}^{2k}$, are well-defined at the appropriate scales.

3.2.3 Map from the boundary of an orbitope to the metric thickening

For $r \ge \frac{2\pi(k-1)}{k}$, define $\iota\colon \partial\mathscr{C}_{2k} \to \check{\mathrm{C}}^{\mathrm{m}}_{\le}(S^1;r)$ as follows. Given a point $\sum_i \lambda_i \mathrm{M}_{2k}(t_i) \in \partial\mathscr{C}_{2k}$ with $\lambda_i > 0$ for all i, let $\iota\left(\sum_i \lambda_i \mathrm{M}_{2k}(t_i)\right) = \sum_i \lambda_i \delta_{t_i}$. By Theorem 2.2.9, the preimage of the vertex set of any proper face of $\mathscr{C}_{2k}$ is contained in a ball of radius at most $\frac{\pi(k-1)}{k}$ in S^1; hence, ι is well-defined.

Similarly, for $r \ge \frac{2\pi(k-1)}{2k-1}$, define $\iota\colon \partial\mathscr{B}_{2k} \to \mathrm{VR}^{\mathrm{m}}(S^1;r)$ by $\iota\left(\sum_i \lambda_i \mathrm{SM}_{2k}(t_i)\right) = \sum_i \lambda_i \delta_{t_i}$ whenever $\sum_i \lambda_i \mathrm{SM}_{2k}(t_i) \in \partial\mathscr{B}_{2k}$ with $\lambda_i > 0$ for all i. By Theorem 2.2.12, the preimage of the vertex set of any proper face of $\mathscr{B}_{2k}$ has diameter at most $\frac{2\pi(k-1)}{2k-1}$ in S^1; hence, ι is well-defined.

By Lemma 3.1.4, both definitions of ι above are continuous.

We can now give the following corollary of Theorem 4.1.1 and Theorem 4.2.1, which gives partial information about the topology of Čech and Vietoris–Rips metric thickenings of the circle at all scales.

Corollary 3.2.4. *For $\frac{2\pi(k-1)}{k} \le r < \frac{2\pi k}{k+1}$, the $(2k-1)$-dimensional homology, cohomology, and homotopy groups of $\check{\mathrm{C}}^{\mathrm{m}}_{\le}(S^1;r)$ are nontrivial.*

Similarly, for $\frac{2\pi(k-1)}{2k-1} \le r < \frac{2\pi k}{2k+1}$, *the* $(2k-1)$*-dimensional homology, cohomology, and homotopy groups of* $\mathrm{VR}^{\mathrm{m}}_{\le}(S^1; r)$ *are nontrivial.*

Proof. For these ranges of r values, Theorems 4.1.1 and 4.2.1 imply that $(p \circ \mathrm{M}_{2k}) \circ \iota = \mathrm{id}_{\partial \mathscr{C}_{2k}}$ and $(p \circ \mathrm{SM}_{2k}) \circ \iota = \mathrm{id}_{\partial \mathscr{B}_{2k}}$, respectively. Hence, $\partial \mathscr{C}_{2k} \cong S^{2k-1}$ is a retract of $\check{\mathrm{C}}^{\mathrm{m}}_{\le}(S^1; r)$ and $\partial \mathscr{B}_{2k} \cong S^{2k-1}$ is a retract of $\mathrm{VR}^{\mathrm{m}}_{\le}(S^1; r)$ at these scales. □

3.2.4 Show ι is a homotopy equivalence

We conjecture that the compositions $\iota \circ p \circ \mathrm{M}_{2k}$ and $\iota \circ p \circ \mathrm{SM}_{2k}$ have a controllable effect on the diameter of any measure in the Čech and Vietoris–Rips thickenings of the circle, respectively. If true, this geometric condition would be sufficient to establish the homotopy type of these metric thickenings at all scales.

Question 3.2.5. Given $\nu \in \check{\mathrm{C}}^{\mathrm{m}}_{\le}(S^1; r)$ with $\frac{2\pi(k-1)}{k} \le r < \frac{2\pi k}{k+1}$, is it true that

$$\mathrm{supp}(\nu) \cup \mathrm{supp}(\iota \circ p \circ \mathrm{M}_{2k}(\nu)) \subset B(x; r)$$

for some $x \in S^1$ (where $B(x; r)$ denotes the closed ball of radius r)?

Similarly, given $\mu \in \mathrm{VR}^{\mathrm{m}}_{\le}(S^1; r)$ with $\frac{2\pi(k-1)}{2k-1} \le r < \frac{2\pi k}{2k+1}$, is it true that

$$\mathrm{diam}(\mathrm{supp}(\mu)) = \mathrm{diam}(\mathrm{supp}(\mu) \cup \mathrm{supp}(\iota \circ p \circ \mathrm{SM}_{2k}(\mu)))?$$

Theorem 3.2.6. *An affirmative answer to Question 3.2.5 would imply Conjecture 3.2.1.*

Proof. For these ranges of r values, we observed in the proof of Corollary 3.2.4 that $(p \circ \mathrm{M}_{2k}) \circ \iota = \mathrm{id}_{\partial \mathscr{C}_{2k}}$ and $(p \circ \mathrm{SM}_{2k}) \circ \iota = \mathrm{id}_{\partial \mathscr{B}_{2k}}$. Hence, it remains to show that $\iota \circ (p \circ \mathrm{M}_{2k}) \simeq \mathrm{id}_{\check{\mathrm{C}}^{\mathrm{m}}_{\le}(S^1; r)}$ and $\iota \circ (p \circ \mathrm{SM}_{2k}) \simeq \mathrm{id}_{\mathrm{VR}^{\mathrm{m}}_{\le}(S^1; r)}$. Consider the linear homotopies $G \colon \check{\mathrm{C}}^{\mathrm{m}}_{\le}(S^1; r) \times I \to \check{\mathrm{C}}^{\mathrm{m}}_{\le}(S^1; r)$ and $H \colon \mathrm{VR}^{\mathrm{m}}_{\le}(S^1; r) \times I \to \mathrm{VR}^{\mathrm{m}}_{\le}(S^1; r)$ defined by

$$G(\nu, t) = (1-t)\nu + t[\iota \circ (p \circ \mathrm{M}_{2k})(\nu)]$$

and

$$H(\mu, t) = (1-t)\mu + t[\iota \circ (p \circ \mathrm{SM}_{2k})(\mu)].$$

Question 3.2.5 would imply that both G and H are well-defined, and hence also continuous by Lemma 3.8 of [3]. □

Lemma 3.2.7. *There are homotopy equivalences*

$$\check{\mathrm{C}}^{\mathrm{m}}_{\leq}(S^1; r) \simeq S^1 \quad \textit{for} \quad 0 \leq r < \pi,$$

$$\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1; r) \simeq S^1 \quad \textit{for} \quad 0 \leq r < \tfrac{2\pi}{3}.$$

Proof. Observe that Question 3.2.5 is true at these scales. □

The remainder of this subsection is devoted to proving that Question 3.2.5 is true for the Vietoris–Rips thickening at scale $r = \frac{2\pi}{3}$, and hence $\mathrm{VR}^{\mathrm{m}}(S^1; \frac{2\pi}{3}) \simeq S^3$. Toward that end, we first describe a number of intermediate geometric lemmas.

The first such lemma, Farkas' Lemma, characterizes when a vector lies in the convex cone generated by a set of vectors. Let $\mathbb{R}^+ = \{t \in \mathbb{R} \mid t \geq 0\}$.

Lemma 3.2.8 (Farkas' Lemma [28])**.** *Let $A \in \mathbb{R}^{m\times n}$, let $a_i \in \mathbb{R}^m$ for $1 \leq i \leq n$ denote the columns of A, and let $v \in \mathbb{R}^m$. Then, exactly one of the following is true:*

1. *There exists $x \in (\mathbb{R}^+)^n$ such that $Ax = v$.*

2. *There exists $y \in \mathbb{R}^m$ such that $a_i^\intercal y \geq 0$ for all i and $v^\intercal y < 0$.*

Case (1) of Farkas' Lemma is equivalent to $v \in \mathrm{cone}(\{a_1, \ldots, a_n\})$, and case (2) is equivalent to $v \notin \mathrm{cone}(\{a_1, \ldots, a_n\})$. Hence, we can use this lemma to study how cones intersect.

Lemma 3.2.9. *Let $u_0, \ldots, u_n, v_0, \ldots, v_k \in \mathbb{R}^m$. If there exists some $y \in \mathbb{R}^m$ such that $u_i^\intercal y \geq 0$ for $0 \leq i \leq n$ and $v_i^\intercal y < 0$ for $0 \leq i \leq k$, then* $\mathrm{cone}(\{u_0, \ldots, u_n\}) \cap \mathrm{cone}(\{v_0, \ldots, v_k\}) = \vec{0}$.

Proof. Suppose such a vector $y \in \mathbb{R}^m$ exists, and let $\vec{0} \neq v = \sum_{i=0}^{k} \lambda_i v_i \in \operatorname{cone}(\{v_0, \dots, v_k\})$. Then, because there exists some $0 \le j \le k$ with $\lambda_j > 0$, we have $v^\intercal y = \sum_{i=0}^{k} \lambda_i v_i^\intercal y \le \lambda_j v_j^\intercal y < 0$. Hence, by Lemma 3.2.8, v is not contained in the convex cone generated by $\{u_0, \dots, u_n\}$. □

The next lemma, which is a corollary of [22, Theorem 3.1.2], will allow us to construct a vector satisfying the hypotheses of Lemma 3.2.9 given certain configurations of points along the curve SM_{2k}.

Lemma 3.2.10. *Fix an integer $k \ge 0$ and distinct $v_1, \dots, v_{2k-1} \in S^1$ with no two points antipodal. Let $u_1, \dots, u_{4k-2}$ denote the set of points $\{v_1, \dots, v_{2k-1}\} \cup \{v_1 + \pi, \dots, v_{2k-1} + \pi\}$ labeled in counterclockwise order such that $u_1 = v_1$. Then, there exists a raked homogeneous trigonometric polynomial f of degree $2k-1$ such that $f(u_i) = 0$ for $1 \le i \le 4k-2$. Further,* $\operatorname{sign}(f(t)) = (-1)^i$ *for $t \in (u_i, u_{i+1})_{S^1}$, where we define $u_{4k-1} = u_1$.*

The proof of Lemma 3.2.10 is contained in Chapter 6.

Remark 3.2.11. In the setting of Lemma 3.2.10, observe that there exists a vector $y \in \mathbb{R}^{2k}$ such that $(\mathrm{SM}_{2k}(u_i))^\intercal y = 0$ for all i. Further, $\operatorname{sign}\big((\mathrm{SM}_{2k}(t))^\intercal y\big) = (-1)^i$ for $t \in (u_i, u_{i+1})_{S^1}$, where we define $u_{4k-1} = u_1$.

Proposition 3.2.12. *Let distinct $t_1, \dots, t_n \in S^1$ be in counterclockwise order and contained in an arc $[t_1, t_n]_{S^1}$ of length at most $\frac{2\pi}{3}$. Let distinct $s_1, \dots, s_m \in S^1$ be such that* $\operatorname{conv}(\mathrm{SM}_4(\{s_1, \dots, s_m\}))$ *is a face of $\mathscr{B}_4$, and $\{s_1, \dots, s_m\} \not\subseteq [t_1, t_n]_{S^1}$. Then*

$$\operatorname{cone}(\mathrm{SM}_4(\{s_1, \dots, s_m\})) \cap \operatorname{cone}(\mathrm{SM}_4(\{t_1, \dots, t_n\})) = \operatorname{cone}(\mathrm{SM}_4(\{s_1, \dots, s_m\} \cap \{t_1, \dots, t_n\})).$$

For the above proposition we agree $\operatorname{cone}(\varnothing) = \vec{0}$.

Proof. Throughout, for convenience, consider points $\mathrm{SM}_4(t) \in \mathbb{R}^4$ to be written as column vectors. In light of the known facial structure of $\mathscr{B}_4$ (Theorem 2.2.11), it follows that $m \le 3$. Hence, there are three cases:

(i) The sets $\{s_1, \dots, s_m\}$ and $\{t_1, \dots, t_n\}$ are disjoint.

(ii) The sets $\{s_1, \dots, s_m\}$ and $\{t_1, \dots, t_n\}$ contain one point of intersection. In this case, $m \in \{2, 3\}$, that is, $\{s_1, \dots, s_m\}$ determines an edge or an equilateral triangle in $\partial\mathscr{B}_4$.

(iii) The sets $\{s_1, \dots, s_m\}$ and $\{t_1, \dots, t_n\}$ contain two points of intersection. In this case, $m = 3$, the points $\{s_1, s_2, s_3\}$ determine an equilateral triangle in $\partial\mathscr{B}_4$, $\{s_1, s_2, s_3\} \cap \{t_1, \dots, t_n\} = \{t_1, t_n\}$, and the length of $(t_1, t_n)_{S^1}$ is $\frac{2\pi}{3}$.

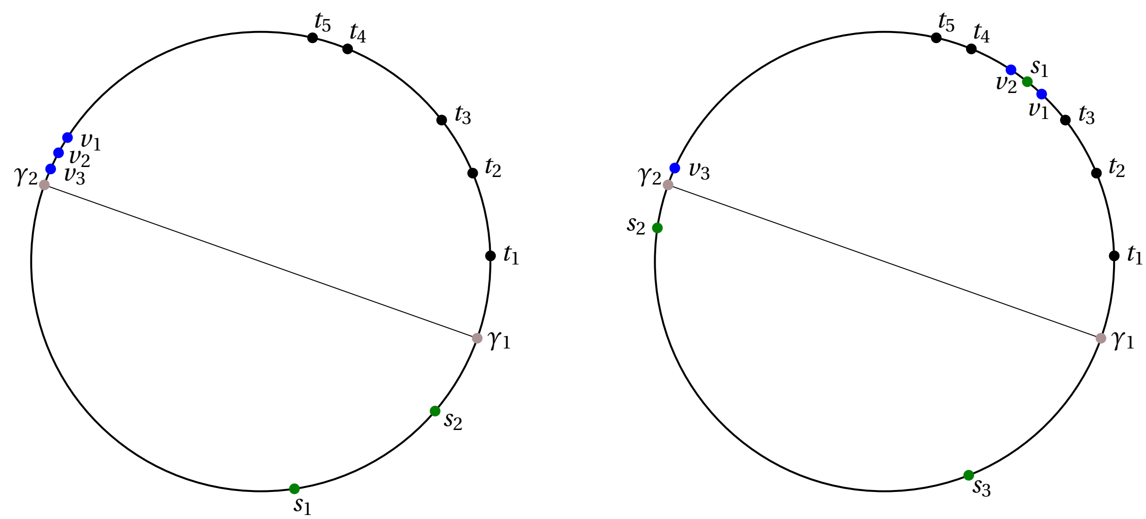
The proof will proceed as follows. First, we consider the case that $\{t_1, \dots, t_n\}$ and $\{s_1, \dots, s_m\}$ are disjoint and apply Lemma 3.2.9 to prove that the resulting cones in $\mathbb{R}^4$ must be disjoint. Then, we will generalize this argument to allow for intersections and consider the remaining two cases.

Toward that end, suppose $\{s_1, \dots, s_m\} \cap \{t_1, \dots, t_n\} = \varnothing$ and note, by Lemma 3.2.9, that it is sufficient to find $y \in \mathbb{R}^4$ such that $(\mathrm{SM}_4(t_i))^\intercal y \geq 0$ for $1 \leq i \leq n$ and $(\mathrm{SM}_4(s_i))^\intercal y < 0$ for $1 \leq i \leq m$. To define such a vector y, fix points $v_1, v_2, v_3 \in S^1$ as follows. By the assumptions on the configuration of the points $\{s_1, \dots, s_m\}$, observe there must exist an arc $\Gamma = (\gamma_1, \gamma_2)_{S^1}$ of length π such that

- $[t_1, t_n]_{S^1} \subseteq \Gamma$,
- $\{s_1, \dots, s_m\} \cap \{\gamma_1, \gamma_2\} = \varnothing$, and
- $|\{s_1, \dots, s_m\} \cap \Gamma| = N$ for $N \leq 1$.

Indeed, to see that we can arrange $N \leq 2$, note that if $m = 3$ then $\{s_1, s_2, s_3\}$ are the vertices of an equilateral triangle, and hence not in an arc of length π. To see that we can arrange $N \leq 1$, note that if $m = 2$, then since one of the s_i points is outside $[t_1, t_n]_{S^1}$, we can choose Γ so that the same s_i point is also outside Γ.

If $N = 0$, define $v_1 = \gamma_2 - \delta$, with $\delta > 0$ small enough such that both $(v_1 + \pi, v_1)_{S^1} \cap \{s_1, \dots, s_m\} = \varnothing$ and $(v_1 + \pi, v_1)_{S^1} \cap \{t_1, \dots, t_n\} = \{t_1, \dots, t_n\}$. Then, define v_2 and v_3 so that v_1, v_2, v_3, and γ_2



An example of points $\{t_1, \ldots, t_5\}$ and $\{s_1, s_2\}$ in S^1 in the case $N = 0$.

An example of points $\{t_1, \ldots, t_5\}$ and $\{s_1, s_2, s_3\}$ in S^1 in the case $N = 1$.

Figure 3.2.2: An example of points $\{\gamma_1, \gamma_2\}$ and $\{v_1, \ldots, v_3\}$, as defined in the proof of Proposition 3.2.12, that are used to construct a vector satisfying the hypotheses of Lemma 3.2.9.

appear in counterclockwise order. See Figure 3.2.2 (left) for an example of such a configuration of points in S^1.

If $N = 1$, assume without loss of generality that $\Gamma \cap \{s_1, \ldots, s_m\} = \{s_1\}$. Then, define $v_1 = s_1 - \varepsilon$ and $v_2 = s_1 + \varepsilon$. Choose $\varepsilon > 0$ small enough such that $(v_1, v_2)_{S^1}$ does not contain any point in $\{t_1, t_2, \ldots, t_n, \gamma_1, \gamma_2\}$ and furthermore so that $(v_1 + \pi, v_2 + \pi)_{S^1} \cap \{s_1, \ldots, s_m\} = \varnothing$. Such points must exist because no two elements of $\{s_1, \ldots, s_m\}$ are antipodal. Finally, define $v_3 = \gamma_2 - \delta$, with $\delta > 0$ small enough such that both $(v_3 + \pi, v_3)_{S^1} \cap \{s_1, \ldots, s_m\} = \{s_1\}$ and $(v_3 + \pi, v_3)_{S^1} \cap \{t_1, \ldots, t_n\} = \{t_1, \ldots, t_n\}$. See Figure 3.2.2 (right).

Now, apply Remark 3.2.11 to obtain $y \in \mathbb{R}^4$ such that for $t \notin \{v_1, v_2, v_3\} \cup \{v_1 + \pi, v_2 + \pi, v_3 + \pi\}$, we have

$$\operatorname{sign}\left((\mathrm{SM}_4(t))^\top y\right) = \operatorname{sign}\left(\prod_{1 \le l \le 3} \sin(v_l - t)\right) = (-1)^{\rho(t)},$$

where $\rho(t) = |\{v_l \mid v_l \in (t + \pi, t)_{S^1}, 1 \le l \le 3\}|$. When we consider the case $t = t_i$ for $1 \le i \le n$, we note by construction that $\rho(t_i)$ is even for each t_i, and so $(\mathrm{SM}_4(t_i))^\top y \ge 0$ for $1 \le i \le n$.

On the other hand, in the case $N = 0$, we note that $\rho(s_i) = 3$ and $\operatorname{sign}\left((\mathrm{SM}_{2k}(s_i))^{\top} y\right) = -1$ for $1 \le i \le m$. Finally, in the case $N = 1$, note that $\rho(s_1) = 1$ and $\operatorname{sign}\left((\mathrm{SM}_{2k}(s_1))^{\top} y\right) = -1$. Further, the pair $\{v_1, v_2\}$ has zero net effect on the parity of $\rho(s_i)$ for $2 \le i \le m$ by the fact that $(v_1+\pi, v_2+\pi)_{S^1} \cap \{s_1,\ldots,s_m\} = \varnothing$. Hence, $\operatorname{sign}\left((\mathrm{SM}_{2k}(s_i))^{\top} y\right) = -1$ for $2 \le i \le m$.

This concludes the proof of case (i) that $\operatorname{cone}(\mathrm{SM}_4(\{s_1,\ldots,s_m\})) \cap \operatorname{cone}(\mathrm{SM}_4(\{t_1,\ldots,t_n\})) = \{\vec{0}\}$ when $\{s_1,\ldots,s_m\} \cap \{t_1,\ldots,t_n\} = \varnothing$.

Next, consider case (ii). Assume without loss of generality that $\{s_1,\ldots,s_m\} \cap \{t_1,\ldots,t_n\} = \{s_1\}$, and write $s_1 = t_\alpha$ for some $1 \le \alpha \le n$. Given $\vec{u} \in \operatorname{cone}(\mathrm{SM}_4(\{t_1,\ldots,t_n\})) \cap \operatorname{cone}(\mathrm{SM}_4(\{s_1,\ldots,s_m\}))$, write $\vec{u} = \sum_{i=1}^{n} \lambda_i \mathrm{SM}_4(t_i) = \sum_{j=1}^{m} \kappa_j \mathrm{SM}_4(s_j)$ for some non-negative scalars λ_i, κ_j. To show $\vec{u} \in \operatorname{cone}(\mathrm{SM}_4(t_\alpha))$, observe that it is sufficient to prove $\lambda_i = 0$ for all $i \in \{1,\ldots,n\} \setminus \alpha$. We consider the possibilities $\lambda_\alpha \ge \kappa_1$ and $\lambda_\alpha < \kappa_1$ separately.

If $\lambda_\alpha \ge \kappa_1$, then

$$\vec{u} - \kappa_1 \mathrm{SM}_4(s_1) = (\lambda_\alpha - \kappa_1)\mathrm{SM}_4(t_\alpha) + \sum_{i \in \{1,\ldots,n\} \setminus \alpha} \lambda_i \mathrm{SM}_4(t_i) = \sum_{j=2}^{m} \kappa_j \mathrm{SM}_4(s_j).$$

It follows that $\vec{u} - \kappa_1 \mathrm{SM}_4(s_1) \in \operatorname{cone}(\mathrm{SM}_4(\{t_1,\ldots,t_n\})) \cap \operatorname{cone}(\mathrm{SM}_4(\{s_2,,\ldots,s_m\}))$. Hence, because $\{t_1,\ldots,t_n\} \cap \{s_2,\ldots,s_m\} = \varnothing$, we have obtained a configuration of points satisfying the hypotheses of case (i) of this proof. Therefore, $\vec{u} - \kappa_1 \mathrm{SM}_4(s_1) = \vec{0}$, and by Corollary 3.2.13 of case (i) below, it follows that $\lambda_\alpha = \kappa_1$ and $\lambda_i = 0$ for all $i \in \{1,\ldots,n\} \setminus \alpha$.

If $\lambda_\alpha < \kappa_1$, then $\vec{u} - \lambda_\alpha \mathrm{SM}_4(t_\alpha) = \sum_{i \in \{1,\ldots,n\} \setminus \alpha} \lambda_i \mathrm{SM}_4(t_i) = \sum_{j=2}^{m} \kappa_j \mathrm{SM}_4(s_j) - \lambda_\alpha \mathrm{SM}_4(t_\alpha)$. That is,

$$\vec{u} - \lambda_\alpha \mathrm{SM}_4(t_\alpha) = (\kappa_1 - \lambda_\alpha)\mathrm{SM}_4(s_1) + \sum_{j=2}^{m} \kappa_j \mathrm{SM}_4(s_j).$$

As before, because $(\{t_1,\ldots,t_n\} \setminus \{t_\alpha\}) \cap \{s_1,\ldots,s_m\} = \varnothing$, we have obtained a configuration of points satisfying the hypotheses of case (i) of this proof. Hence $\vec{u} - \lambda_\alpha \mathrm{SM}_4(t_\alpha) = \vec{0}$, and by Corollary 3.2.13 of case (i), it follows that $\lambda_i = 0$ for all $i \in \{1,\ldots,n\} \setminus \alpha$. This concludes the proof for case (ii).

Last, observe that case (iii) follows by a similar trick: by rewriting a vector $\vec{u}$ contained in the intersection of both cones, we may obtain a configuration of points satisfying the hypotheses of case (i) or case (ii). $\square$

The following is a corollary of only case (i) in the proof of Proposition 3.2.12; indeed it is used in the proof of case (ii).

Corollary 3.2.13. *Let distinct $t_1, \ldots, t_n \in S^1$ be in counterclockwise order and contained in an arc $[t_1, t_n]_{S^1}$ of length at most $\frac{2\pi}{3}$. If $\sum_{i=1}^{n} \lambda_i \mathrm{SM}_4(t_i) = \vec{0}$ with $\lambda_i \geq 0$, then $\lambda_i = 0$ for all $1 \leq i \leq n$.*

Proof. The claim is obvious in the case $n = 1$. Otherwise, because $\mathrm{SM}_4(-t) = -\mathrm{SM}_4(t)$, we may write $\sum_{i=1}^{n-1} \lambda_i \mathrm{SM}_4(t_i) = \lambda_n \mathrm{SM}_4(-t_n)$, with $-t_n \notin [t_1, t_n]_{S^1}$. With $s_1 = -t_n$, observe that the hypotheses of case (i) of Proposition 3.2.12 are satisfied, implying

$$\mathrm{cone}\,(\mathrm{SM}_4(\{t_1, \ldots, t_{n-1}\})) \cap \mathrm{cone}\,(\mathrm{SM}_4(-t_n)) = \vec{0}$$

Since $\lambda_n \mathrm{SM}_4(-t_n)$ is in this intersection of cones, this implies $\lambda_n = 0$. Hence $\sum_{i=1}^{n-1} \lambda_i \mathrm{SM}_4(t_i) = \vec{0}$, and we may proceed iteratively to conclude $\lambda_i = 0$ for all i. $\square$

We are now ready to prove that the "diameter non-increasing" result in Question 3.2.5 is true for $\mu \in \mathrm{VR}^{\mathrm{m}}_{\leq}(S^1; \frac{2\pi}{3})$.

Proposition 3.2.14. *For $\mu \in \mathrm{VR}^{\mathrm{m}}(S^1; \frac{2\pi}{3})$, we have*

$$\mathrm{diam}(\mathrm{supp}(\mu)) = \mathrm{diam}(\mathrm{supp}(\mu) \cup \mathrm{supp}(\iota \circ p \circ \mathrm{SM}_4(\mu))).$$

Proof. Let $\mu = \sum_{i=1}^{n} \lambda_i \delta_{t_i} \in \mathrm{VR}^{\mathrm{m}}(S^1; \frac{2\pi}{3})$ for $t_i \in S^1$ and $\lambda_i > 0$ with $\sum_i \lambda_i = 1$. There are two cases. If $\{t_1, \ldots, t_n\}$ are in counterclockwise order and belong to an arc of length at most $\frac{2\pi}{3}$, then Proposition 3.2.12 implies that $\mathrm{supp}(\iota \circ p \circ \mathrm{SM}_4(\mu)) \subseteq [t_1, t_n]_{S^1}$, and hence

$$\mathrm{diam}(\mathrm{supp}(\mu)) = \mathrm{diam}(\mathrm{supp}(\mu) \cup \mathrm{supp}(\iota \circ p \circ \mathrm{SM}_4(\mu))).$$

Otherwise, $n = 3$ and $\{t_1, t_2, t_3\}$ form the vertices of an equilateral triangle. In this case, we have $\iota \circ p \circ \mathrm{SM}_4(\mu) = \mu$ in light of Theorem 2.2.11. $\square$

Hence, Question 3.2.5 is true for $\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1; r)$ for $0 \leq r \leq \frac{2\pi}{3}$, and this concludes the proof of Theorem 3.2.3.

Chapter 4

Carathéodory subsets of moment curves and the faces of Carathéodory and Barvinok–Novik orbitopes

Let $Y \subseteq \mathbb{R}^k$ be a set in Euclidean space. Recall that we say $Y' \subseteq Y$ is a Carathéodory subset of Y if the convex hull of Y' contains the origin.

In this chapter, we describe the Carathéodory subsets of the trigonometric moment curve, $\mathrm{M}_{2k}(S^1)$, and of the centrally symmetric trigonometric moment curve, $\mathrm{SM}_{2k}(S^1)$. These results (Theorem 4.1.1 and Theorem 4.2.1) imply that the maps $p \circ \mathrm{M}_{2k}\colon \check{\mathrm{C}}^{\mathrm{m}}_{\leq}(S^1; r) \to \partial\mathscr{C}_{2k}$ and $p \circ \mathrm{SM}_{2k}\colon \mathrm{VR}^{\mathrm{m}}_{\leq}(S^1; r) \to \partial\mathscr{B}_{2k}$ from the Čech and Vietoris–Rips thickenings to the Carathéodory and Barvinok–Novik orbitopes considered in Section 3.2.2 are well-defined at the appropriate scales, that is, at scales $r < \frac{k\pi}{k+1}$ and $r < \frac{2\pi k}{2k+1}$, respectively.

Following our discussion of Carathéodory subsets of these moment curves, we explicitly construct the vectors generating the coorbitope cone of the Carathéodory orbitope $\mathscr{C}_{2k}$ in Section 4.3.

4.1 Carathéodory subsets of the trigonometric moment curve

The following corollary of Theorem 6.0.2, due to Gilbert and Smyth [50], gives lower bounds on Carathéodory subsets (Definition 2.2.2) of the trigonometric moment curve in terms of the lengths of arcs of S^1.

Theorem 4.1.1. *Let $X \subseteq S^1$ be contained in a closed circular arc $[a, b]_{S^1}$ of length less than L. Then the convex hull $\mathrm{conv}(\mathrm{M}_{2k}(X))$ does not contain the origin $\vec{0} \in \mathbb{R}^{2k}$ if $L = \frac{2\pi k}{k+1}$, and this bound is sharp.*

Proof. We follow the proof of [4, Corollary 7.3]. By Theorem 6.0.2, there exists a homogeneous trigonometric polynomial p of degree k that is positive on $[a, b]_{S^1}$. Writing $p(t) := z^\intercal \mathrm{M}_{2k}(t)$ for some $z \in \mathbb{R}^{2k}$, observe that the hyperplane $H_z = \{x \in \mathbb{R}^{2k} \mid z^\top x = 0\}$ separates $\mathrm{M}_{2k}([a,b]_{S^1})$ from the origin.

Sharpness of the bound $\frac{2\pi k}{k+1}$ follows directly from the second half of Theorem 6.0.2. □

Observe that Theorem 4.1.1 describes Carathéodory subsets of the trigonometric moment curve in terms of their preimages. In particular, if $\mu \in \check{\mathrm{C}}^{\mathrm{m}}_{\leq}(S^1; r)$, then the support of μ is contained in a closed circular arc of length less than r; hence, the convex hull of $\mathrm{M}_{2k}(\check{\mathrm{C}}^{\mathrm{m}}_{\leq}(S^1; r))$ does not contain the origin when $r < \frac{2\pi k}{k+1}$.

4.2 Carathéodory subsets of the centrally symmetric trigonometric moment curve

In analogy with Theorem 4.1.1, we give lower bounds on Carathéodory subsets of the symmetric moment curve in terms of the diameter of subsets of S^1.

Theorem 4.2.1. *Let $X \subseteq S^1$ be such that* $\mathrm{diam}(X) < D$. *Then the convex hull* $\mathrm{conv}(\mathrm{SM}_{2k}(X))$ *does not contain the origin* $\vec{0} \in \mathbb{R}^{2k}$ *if* $D = \frac{2\pi k}{2k+1}$, *and this bound is sharp.*

In particular, given $\mu \in \mathrm{VR}^{\mathrm{m}}_{\leq}(S^1; r)$, the support of μ has diameter at most r in S^1; hence, the convex hull of $\mathrm{SM}_{2k}(\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1; r))$ does not contain the origin when $r < \frac{2\pi k}{2k+1}$.

The remainder of this section is devoted to proving Theorem 4.2.1 and many of the arguments contained therein first appeared in [31].

4.2.1 The proof of Theorem 4.2.1

First, observe that we may restrict attention to subsets of $\mathrm{SM}_{2k}(S^1)$ of size at most $2k+1$ by Carathéodory's theorem. Suppose $X = \{t_0, \ldots, t_{2k}\} \subseteq S^1$ is such that the origin is contained in the convex hull of $\{\mathrm{SM}_{2k}(t_0), \ldots, \mathrm{SM}_{2k}(t_{2k})\}$. Then, there exist scalars $\lambda_i \geq 0$ such that $\vec{0} = \sum_{i=0}^{2k} \lambda_i \mathrm{SM}_{2k}(t_i)$ and $\sum_{i=0}^{2k} \lambda_i = 1$. In this way, we obtain a system of $2k$ equations

$$\sum_{i=0}^{2k} \lambda_i \cos(n t_i) = 0 \quad \text{and} \quad \sum_{i=0}^{2k} \lambda_i \sin(n t_i) = 0 \quad \text{for} \quad n = 1, 3, \ldots, 2k-1. \tag{4.1}$$

We therefore fix $\vec{t} = (t_0, \dots, t_{2k}) \in \mathbb{R}^{2k+1}$ and consider $\mathbb{SM}_{2k}(\vec{t})$, that is, the $2k \times (2k+1)$ matrix

$$\mathbb{SM}_{2k}(\vec{t}) = \begin{pmatrix} \cos(t_0) & \cos(t_1) & \dots & \cos(t_{2k}) \\ \sin(t_0) & \sin(t_1) & \dots & \sin(t_{2k}) \\ \cos(3t_0) & \cos(3t_1) & \dots & \cos(3t_{2k}) \\ \sin(3t_0) & \sin(3t_1) & \dots & \sin(3t_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ \cos((2k-1)t_0) & \cos((2k-1)t_1) & \dots & \cos((2k-1)t_{2k}) \\ \sin((2k-1)t_0) & \sin((2k-1)t_1) & \dots & \sin((2k-1)t_{2k}) \end{pmatrix}$$

as defined in Subsection 2.2.8. With this notation, observe that we may rewrite Equation 4.1 as $\mathbb{SM}_{2k}(\vec{t})\vec{\lambda} = \vec{0}$ for $\vec{\lambda} = (\lambda_0, \dots, \lambda_{2k})^\intercal$. Hence, to prove Theorem 4.2.1, we build toward describing the nullspace of $\mathbb{SM}_{2k}(\vec{t})$, which we complete in Lemma 4.2.4.

Lemma 4.2.2. *Let $\mathbb{SM}_{2k,0}(\vec{t})$ denote the $2k \times 2k$ matrix obtained by removing the column containing the vector $\mathrm{SM}_{2k}(t_0)$ from the matrix $\mathbb{SM}_{2k}(\vec{t})$. Then, $\det(\mathbb{SM}_{2k,0}(\vec{t})) = \kappa \prod_{1 \le j < l \le 2k} \sin(t_l - t_j)$ for some nonzero constant κ depending only on k.*

We would like to thank Harrison Chapman for the insights behind the proof of Lemma 4.2.2. The idea of the proof is to perform elementary row and column operations to obtain a Vandermonde matrix. In addition to the general case, the simpler case $k = 2$ of this proof is written out in more detail in [31].

Proof of Lemma 4.2.2. To ease notation, write $M := \mathbb{SM}_{2k,0}(\vec{t})$. We will perform elementary row and column operations on M to obtain a Vandermonde matrix. For a function $f : \mathbb{R} \to \mathbb{C}$ and $\underline{t} = (t_1, t_2, \dots, t_{2k})^\intercal \in \mathbb{R}^{2k}$, let us write

$$f(\underline{t}) := (f(t_1), f(t_2), \dots, f(t_{2k}))^\intercal \in \mathbb{C}^{2k}.$$

Since

$$M = \begin{pmatrix} \cos(t_1) & \cos(t_2) & \dots & \cos(t_{2k}) \\ \sin(t_1) & \sin(t_2) & \dots & \sin(t_{2k}) \\ \cos(3t_1) & \cos(3t_2) & \dots & \cos(3t_{2k}) \\ \sin(3t_1) & \sin(3t_2) & \dots & \sin(3t_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ \cos((2k-1)t_1) & \cos((2k-1)t_2) & \dots & \cos((2k-1)t_{2k}) \\ \sin((2k-1)t_1) & \sin((2k-1)t_2) & \dots & \sin((2k-1)t_{2k}) \end{pmatrix},$$

we have

$$\begin{aligned} \det(M) &= \det\left(M^{\mathsf{T}}\right) \\ &= \det\begin{pmatrix} \cos(\underline{t}) & \sin(\underline{t}) & \cos(3\underline{t}) & \sin(3\underline{t}) & \cdots & \cos((2k-1)\underline{t}) & \sin((2k-1)\underline{t}) \end{pmatrix} \\ &= \det\begin{pmatrix} \frac{e^{i\underline{t}}+e^{-i\underline{t}}}{2} & \frac{e^{i\underline{t}}-e^{-i\underline{t}}}{2i} & \frac{e^{3i\underline{t}}+e^{-3i\underline{t}}}{2} & \frac{e^{3i\underline{t}}-e^{-3i\underline{t}}}{2i} & \cdots & \frac{e^{(2k-1)i\underline{t}}+e^{-(2k-1)i\underline{t}}}{2} & \frac{e^{(2k-1)i\underline{t}}-e^{-(2k-1)i\underline{t}}}{2i} \end{pmatrix} \\ &= \frac{1}{2^{2k}}(-i)^k \det\begin{pmatrix} e^{i\underline{t}}+e^{-i\underline{t}} & e^{i\underline{t}}-e^{-i\underline{t}} & \cdots & e^{(2k-1)i\underline{t}}+e^{-(2k-1)i\underline{t}} & e^{(2k-1)i\underline{t}}-e^{-(2k-1)i\underline{t}} \end{pmatrix}. \end{aligned}$$

Next, let C_j denote the j-th column of the above matrix. For $j = 1, 3, \dots, 2k-1$, perform the column operations $C_j \mapsto C_j + C_{j+1}$, and then after each C_j has been updated, perform the column operations $C_{j+1} \mapsto C_{j+1} - \frac{1}{2}C_j$. It follows that

$$\begin{aligned} \det(M) &= \frac{1}{2^{2k}}(-i)^k \det\begin{pmatrix} 2e^{i\underline{t}} & -e^{-i\underline{t}} & 2e^{3i\underline{t}} & -e^{-3i\underline{t}} & \cdots & 2e^{(2k-1)i\underline{t}} & -e^{-(2k-1)i\underline{t}} \end{pmatrix} \\ &= \frac{i^k}{2^k} \det\begin{pmatrix} e^{i\underline{t}} & e^{-i\underline{t}} & e^{3i\underline{t}} & e^{-3i\underline{t}} & \cdots & e^{(2k-1)i\underline{t}} & e^{-(2k-1)i\underline{t}} \end{pmatrix} \end{aligned}$$

by factoring out column multiples. Defining $\omega := e^{-(2k-1)i(t_1+t_2+\cdots+t_{2k})}$, we may factor $e^{-(2k-1)it_j}$ from row j to obtain

$$\begin{aligned}\det(A) &= \frac{i^k}{2^k}\omega \det\begin{pmatrix} e^{((2k-1)+1)i\underline{t}} & e^{((2k-1)-1)i\underline{t}} & \cdots & e^{((2k-1)+(2k-1))i\underline{t}} & e^{((2k-1)-(2k-1))i\underline{t}}\end{pmatrix}\\ &= \frac{i^k}{2^k}\omega \det\begin{pmatrix} e^{2ki\underline{t}} & e^{(2k-2)i\underline{t}} & e^{(2k+2)i\underline{t}} & e^{(2k-4)i\underline{t}} & \cdots & e^{2(2k-1)i\underline{t}} & \underline{1}\end{pmatrix},\end{aligned}$$

where $\underline{1} \in \mathbb{R}^{2k}$ denotes the vector of all 1's. After re-ordering rows by a permutation σ and taking the determinant of the resulting Vandermonde matrix, we have

$$\begin{aligned}\det(M) &= \operatorname{sign}(\sigma)\frac{i^k}{2^k}\omega \det\begin{pmatrix} \underline{1} & e^{2i\underline{t}} & e^{4i\underline{t}} & \cdots & e^{(2(2k-1))i\underline{t}}\end{pmatrix}\\ &= \operatorname{sign}(\sigma)\frac{i^k}{2^k}\omega \prod_{1\le j<l\le 2k}\left(e^{2it_l} - e^{2it_j}\right).\end{aligned}$$

Finally, note $\omega = \prod_{1\le j<l\le 2k} e^{-i(t_l+t_j)}$ and multiply each term $\left(e^{2it_l} - e^{2it_j}\right)$ above by the factor $e^{-i(t_l+t_j)}$ extracted from ω to obtain

$$\begin{aligned}\det(M) &= \operatorname{sign}(\sigma)\frac{i^k}{2^k}\prod_{1\le j<l\le 2k}\left(e^{i(t_l-t_j)} - e^{-i(t_l-t_j)}\right)\\ &= \operatorname{sign}(\sigma)\frac{i^k}{2^k}\prod_{1\le j<l\le 2k} 2i\sin(t_l-t_j) = \kappa \prod_{1\le j<l\le 2k}\sin(t_l-t_j)\end{aligned}$$

where $\kappa = \operatorname{sign}(\sigma)\frac{i^k}{2^k}(2i)^{2k^2-k} = \operatorname{sign}(\sigma)i^{2k^2}2^{2k(k-1)} = \operatorname{sign}(\sigma)2^{2k(k-1)}$. □

The following corollary is immediate.

Corollary 4.2.3. *For* $0 \le i \le 2k$, *let* $\mathbb{SM}_{2k,i}(\vec{t})$ *denote the* $2k\times 2k$ *matrix obtained by removing the column containing the vector* $\mathrm{SM}_{2k}(t_i)$ *from the matrix* $\mathbb{SM}_{2k}(\vec{t})$*. Then*

$$\det\left(\mathbb{SM}_{2k,i}(\vec{t})\right) = \kappa \prod_{\substack{0\le j<l\le 2k\\ j,l\ne i}} \sin(t_l-t_j),$$

for some nonzero constant κ *depending only on* k.

Lemma 4.2.4. *If no two points $t_0, t_1, \dots, t_{2k} \in S^1$ are equal or antipodal, then the nullspace of the matrix $\mathbb{SM}_{2k}(\vec{t})$ is one-dimensional and is spanned by $\vec{\lambda} = (\lambda_0, \lambda_1, \dots, \lambda_{2k})^\intercal$, where*

$$\lambda_i = (-1)^i \prod_{\substack{0 \le j < l \le 2k \\ j,l \neq i}} \sin(t_l - t_j).$$

Proof. Because $\mathbb{SM}_{2k}(\vec{t})$ has $2k$ rows and $2k+1$ columns, it has nullity at least one. Further, by Corollary 4.2.3, observe that $\mathbb{SM}_{2k,i}(\vec{t})$ is invertible if and only if no two points $t_l, t_j \in S^1 \setminus \{t_i\}$ are equal or antipodal. Hence, $\mathbb{SM}_{2k}(\vec{t})$ contains $2k$ linearly independent columns and has nullity exactly one.

Next, we prove $\vec{\lambda}$ is contained in the nullspace of $\mathbb{SM}_{2k}(\vec{t})$. To ease notation, write

$$\mathbb{SM}_{2k}(\vec{t})\vec{\lambda} = \begin{pmatrix} C_1 & S_1 & C_3 & S_3 & \cdots & C_{2k-1} & S_{2k-1} \end{pmatrix}^\intercal.$$

We will prove that $C_j = S_j = 0$ for all $j = 1, 3, 5, \dots, 2k-1$.

Note $\lambda_i = (-1)^i \frac{1}{\kappa} \det\left(\mathbb{SM}_{2k,i}(\vec{t})\right)$, and hence for $n = 1, 3, 5, \dots, 2k-1$ we have

$$C_n = \sum_{i=0}^{2k} \cos(nt_i)\lambda_i = \frac{1}{\kappa} \sum_{i=0}^{2k} (-1)^i \cos(nt_i) \det\left(\mathbb{SM}_{2k,i}(\vec{t})\right).$$

Therefore, C_n is equal to $\frac{1}{\kappa}$ times the determinant of the matrix

$$\begin{pmatrix} \cos(nt_0) & \cos(nt_1) & \dots & \cos(nt_{2k}) \\ \cos(t_0) & \cos(t_1) & \dots & \cos(t_{2k}) \\ \sin(t_0) & \sin(t_1) & \dots & \sin(t_{2k}) \\ \cos(3t_0) & \cos(3t_1) & \dots & \cos(3t_{2k}) \\ \sin(3t_0) & \sin(3t_1) & \dots & \sin(3t_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ \cos((2k-1)t_0) & \cos((2k-1)t_1) & \dots & \cos((2k-1)t_{2k}) \\ \sin((2k-1)t_0) & \sin((2k-1)t_1) & \dots & \sin((2k-1)t_{2k}) \end{pmatrix}.$$

Since $n = 2j - 1$ for some $1 \le j \le k$, the first row of this matrix is equal to one of the other rows. Hence, the matrix is singular, giving that $C_n = 0$.

Similarly, it follows that S_n is equal to $\frac{1}{\kappa}$ times the determinant of the same matrix, except with the first row replaced by $(\sin(nt_0), \sin(nt_1), \ldots, \sin(nt_{2k}))$. For the same reasons as before, it follows that $S_n = 0$. $\square$

We have shown that the nullspace of $\mathbb{SM}_{2k}(\vec{t})$ is spanned by $\vec{\lambda}$ as defined in Lemma 4.2.4. For convenience, we rescale $\vec{\lambda}$ by $\gamma := \prod_{0 \le j < l \le 2k} \frac{1}{\sin(t_l - t_j)}$ (which is well-defined for $t_1, \ldots, t_{2k}$ distinct) to obtain

$$\gamma\vec{\lambda} = \left(\alpha_0(t_0,\ldots,t_{2k})^{-1}, \ldots, \alpha_{2k}(t_0,\ldots,t_{2k})^{-1}\right)^{\intercal}, \quad \text{where} \quad \alpha_i(t_0,\ldots,t_{2k}) = \prod_{\substack{0 \le j \le 2k \\ j \ne i}} \sin(t_j - t_i).$$

Recall that entries of $\vec{\lambda}$ are the coefficients in the linear combination $\vec{0} = \sum_{i=0}^{2k} \lambda_i \mathrm{SM}_{2k}(t_i)$. Hence, the origin may be contained in the convex hull of $\{\mathrm{SM}_{2k}(t_0), \ldots, \mathrm{SM}_{2k}(t_{2k})\}$ only in the case that the terms $\alpha_i(t_0,\ldots,t_{2k})$ share the same sign. We next relate the sign of each term $\alpha_i(t_0,\ldots,t_{2k})$ to the configuration of points $t_0,\ldots,t_{2k} \in S^1$. For what follows, χ is defined as in Definition 2.2.14, namely, $\chi(t_i) = \left|\left\{t_j \mid t_j \in (t_i, t_i + \pi)_{S^1}\right\}\right|$.

Lemma 4.2.5. *Let $t_0,\ldots,t_{2k} \in S^1$ be given with no two points equal or antipodal. Then, the numbers $\alpha_i(t_0,\ldots,t_{2k})$ have the same sign for all $0 \le i \le 2k$ if and only if $\chi(t_i) = k$ for all i.*

Proof. Throughout, we assume that the points $t_0,\ldots,t_{2k} \in S^1$ are distinct, with no two points antipodal, and furthermore that they are ordered by index with a counterclockwise orientation. Observe that $\operatorname{sign}(\alpha_i(t_0,\ldots,t_{2k})) = (-1)^{\chi(t_i)}$.

We first prove two preliminary properties.

(i) $\sum_{i=0}^{2k} \chi(t_i) = k(2k+1)$.

(ii) If $t_0,\ldots,t_{2k}$ are not all contained in a semicircle, then $1 \ge \chi(t_{i+1}) - \chi(t_i)$ for $0 \le i \le 2k$, where we set $t_{2k+1} = t_0$.

For (i), note that since no two points are equal or antipodal, we have that $t_j \in (t_i + \pi, t_i)_{S^1}$ if and only if $t_i \notin (t_j + \pi, t_j)_{S^1}$. Therefore $\sum_{i=0}^{2k} \chi(t_i) = \binom{2k+1}{2} = k(2k+1)$.

For (ii), observe that the open arc $(t_{i+1} + \pi, t_i)_{S^1}$ contains exactly $\chi(t_{i+1}) - 1$ points. Indeed, $(t_{i+1} + \pi, t_i)_{S^1}$ contains exactly $\chi(t_{i+1}) - 1$ points for all i if and only if $t_i \in (t_{i+1} + \pi, t_{i+1})_{S^1}$ for all i, which is true if and only if the points are not contained in a semicircle. Hence, $(t_i + \pi, t_{i+1} + \pi)_{S^1}$ must contain exactly $\chi(t_i) - (\chi(t_{i+1}) - 1)$ points. Because this number is non-negative, it follows that $1 \geq \chi(t_{i+1}) - \chi(t_i)$.

We now prove Lemma 4.2.5. In the case that $\chi(t_i) = k$ for all i, we see that the numbers $\alpha_i(t_0, \ldots, t_{2k})$ are all positive or are all negative.

Conversely, suppose the numbers $\alpha_i(t_0, \ldots, t_{2k})$ have the same sign. Then, the numbers $\chi(t_i)$ must have the same parity because $\operatorname{sign}(\alpha_i(t_0, \ldots, t_{2k})) = (-1)^{\chi(t_i)}$. Further, in the case k is odd (respectively, even), (i) implies each $\chi(t_i)$ is odd (respectively, even). Therefore, in either case, we may write $\chi(t_i) = k + 2n_i$ for some integer $n_i \in \mathbb{Z}$. Note that (i) implies

$$k(2k+1) = \sum_{i=0}^{2k} \chi(t_i) = \sum_{i=0}^{2k} (k + 2n_i) = k(2k+1) + 2\sum_{i=0}^{2k} n_i,$$

giving $\sum_{i=0}^{2k} n_i = 0$. Therefore, it is sufficient to prove that $n_i = n_j$ for all i, j. Toward that end, define $t_{2k+1} = t_0$ and $n_{2k+1} = n_0$, and observe

$$0 = \sum_{i=0}^{2k} n_{i+1} = \sum_{i=0}^{2k} (n_{i+1} - n_i + n_i) = \sum_{i=0}^{2k} (n_{i+1} - n_i) + \sum_{i=0}^{2k} n_i = \sum_{i=0}^{2k} (n_{i+1} - n_i).$$

It cannot be the case that all of the points t_i are contained in a semicircle, since then $\chi(t_i)$ would obtain all of the values $0, 1, \ldots, 2k$, contradicting the fact that these values have the same parity. Therefore, we may apply (ii) to obtain

$$1 \geq (k + 2n_{i+1}) - (k + 2n_i) = 2(n_{i+1} - n_i),$$

which implies $0 \geq n_{i+1} - n_i$ for all i. Since $\sum_{i=0}^{2k} (n_{i+1} - n_i) = 0$, this gives $n_{i+1} = n_i$ for all i. $\square$

We are now prepared to prove Theorem 4.2.1.

Proof of Theorem 4.2.1. We must show that the convex hull $\operatorname{conv}(\mathrm{SM}_{2k}(X))$ does not contain the origin $\vec{0} \in \mathbb{R}^{2k}$ whenever $\operatorname{diam}(X) < \frac{2\pi k}{2k+1}$. Let d denote the geodesic metric on S^1 and let distinct $t_0,\ldots,t_{2k} \in S^1$ be given in counterclockwise order. If $\chi(t_i) = k$ for all i, it follows that $\operatorname{diam}(\{t_0,\ldots,t_{2k}\}) \geq \frac{2\pi k}{2k+1}$ by Lemma 2.2.16. Thus, if $\operatorname{diam}(\{t_0,\ldots,t_{2k}\}) < \frac{2\pi k}{2k+1}$, then $\chi(t_i) \neq k$ for some $0 \leq i \leq 2k$. Hence Lemmas 4.2.4 and 4.2.5 imply that there do not exist positive scalars λ_i with $\vec{0} = \sum_{i=0}^{2k} \lambda_i \mathrm{SM}_{2k}(t_i)$.

To see that this bound is sharp, let $t_i \in S^1$ denote the vertices of a regular inscribed $(2k+1)$-gon and note that $\vec{0} = \sum_{i=0}^{2k} \frac{1}{2k+1} \mathrm{SM}_{2k}(t_i)$ in this case. □

Remark 4.2.6. In this subsection, the matrix $\mathbb{SM}_{2k}$, together with the determinants of its $2k \times 2k$-submatrices, were used to prove Theorem 4.2.1, which lower bounds the diameter of the preimages of Carathéodory subsets of SM_{2k}. We do not consider the analogous matrix $\mathbb{M}_{2k}$ here because Theorem 6.0.2 allows for a more straightforward proof of the analogous result, Theorem 4.1.1, regarding the preimages of Carathéodory subsets of the moment curve M_{2k}.

4.3 The Carathéodory coorbitope cone

In this section, we demonstrate how the matrices $\mathbb{M}^1_{2k}$, as defined in Subsection 2.2.8, may be used to better understand the facial structure of the Carathéodory orbitopes. In particular, we will use these matrices to explicitly construct the coorbitope cone to $\mathscr{C}_{2k}$. Recall (see Remark 2.2.8) that this cone consists of the set of coefficient vectors of trigonometric polynomials of degree at most k defining faces of $\mathscr{C}_{2k}$.

Proposition 4.3.1. *For an integer $k \geq 1$, let $\vec{t} = (t_0,\ldots,t_{2k})^\intercal \in \mathbb{R}^{2k+1}$. Then,*

$$\det(\mathbb{M}^1_{2k}(\vec{t})) = \kappa \prod_{0\leq j<l\leq 2k} \sin\left(\frac{t_l - t_j}{2}\right)$$

for some nonzero constant κ depending on k.

The proof of Proposition 4.3.1 is similar to the proof of Lemma 4.2.2 and is contained in Appendix A.

In what follows, given the set up of Proposition 4.3.1, we will think of t_0 as a variable and $t_1, \ldots, t_{2k}$ as fixed. Along those lines, fix $s_1, \ldots, s_{2k} \in \mathbb{R}$ and define $\vec{s} := (s_1, \ldots, s_{2k})^\intercal$. For $t \in \mathbb{R}$, let $\vec{t} = (t, s_1, s_2, \ldots, s_{2k})$ and write $\mathbb{M}^1_{2k}(t, \vec{s}) := \mathbb{M}^1_{2k}(\vec{t})$. By considering the cofactor expansion of the determinant of $\mathbb{M}^1_{2k}(t, \vec{s})$ along the first column, observe that $\det(\mathbb{M}^1_{2k}(t, \vec{s}))$ is a degree k trigonometric polynomial in t. Writing

$$f_{\vec{s}}(t) = \prod_{1 \le j \le 2k} \sin\left(\frac{s_j - t}{2}\right),$$

it follows that

$$\det(\mathbb{M}^1_{2k}(t, \vec{s})) = \left(\kappa \prod_{1 \le j < l \le 2k} \sin\left(\frac{s_l - s_j}{2}\right)\right) f_{\vec{s}}(t) = \tilde{\kappa} f_{\vec{s}}(t)$$

for some constant $\tilde{\kappa}$. This proves the following corollary of Proposition 4.3.1.

Corollary 4.3.2. *For any $\vec{s} = (s_1, \ldots, s_{2k})^\intercal \in \mathbb{R}^{2k}$,*

$$f_{\vec{s}}(t) := \prod_{1 \le j \le 2k} \sin\left(\frac{s_j - t}{2}\right)$$

is a degree k trigonometric polynomial in t.

An important feature of the polynomial $f_{\vec{s}}$ defined in Corollary 4.3.2 is that its zero set is precisely $\{s_1, \ldots, s_{2k}\}$. Because a degree k trigonometric polynomial with $2k$ roots is uniquely determined up to a non-zero scalar, the set $\{\alpha f_{\vec{s}} \mid \alpha \in \mathbb{R} \setminus \{0\}, \vec{s} = (s_1, \ldots, s_{2k})^\intercal \in \mathbb{R}^{2k}\}$ is precisely the collection of all such polynomials. The following lemma makes this precise.

Lemma 4.3.3. *Let*

$$f(t) = c + \sum_{j=1}^{k} \left(a_j \cos(jt) + b_j \sin(jt)\right)$$

with $a_k \neq 0$ or $b_k \neq 0$ denote a degree k trigonometric polynomial having $2k$ roots in S^1 counted with multiplicity. Then, f is uniquely determined up to a non-zero constant multiple.

Proof. Suppose the multi-set of the roots of f is $\{t_1, \dots, t_{2k}\} \subseteq S^1$. Let us make the substitution $z = e^{it}$ in the expression for $f(t)$. Noting that

$$\cos(jt) = \frac{z^j + z^{-j}}{2} \qquad \text{and} \qquad \sin(jt) = \frac{z^j - z^{-j}}{2i},$$

we can write $f(t) = z^{-k} D(z)$, where

$$D(z) := cz^k + \sum_{j=1}^{k} \frac{(a_j - ib_j)}{2} z^{k+j} + \sum_{j=1}^{k} \frac{(a_j + ib_j)}{2} z^{k-j}.$$

Because D is a complex polynomial of degree $2k$, and because it has $2k$ roots $\{e^{it_1}, \dots, e^{it_{2k}}\}$ by hypothesis, the fundamental theorem of algebra implies that D, and consequently f, is uniquely determined up to a non-zero scalar. $\square$

Hence, by choosing vectors $\vec{s} = (s_1, \dots, s_{2k})$ such that each s_i appears with even multiplicity, we obtain all non-negative trigonometric polynomials of degree k with prescribed roots at the s_i, and each such polynomial defines a proper face of the Carathéodory orbitope $\mathscr{C}_{2k}$. Therefore, we have the following explicit description of its coorbitope cone.

Theorem 4.3.4. *Let the trigonometric polynomial f_s be defined as in Corollary 4.3.2. The coorbitope cone $\widehat{\mathscr{C}_{2k}^{\circ}}$ is generated by the following set of vectors:*

- *the zero vector $(0, 0, \dots, 0) \in \mathbb{R}^{2k+1}$, corresponding to the face $\mathscr{C}_{2k}$ of $\mathscr{C}_{2k}$,*
- *the vector $(1, 0, \dots, 0) \in \mathbb{R}^{2k+1}$, corresponding to the face $\varnothing$ of $\mathscr{C}_{2k}$,*
- *the coefficient vectors of polynomials $f_{\vec{s}}$ defined by $\vec{s} = (s_1, \dots, s_{2j}) \in \mathbb{R}^{2j}$, for $1 \le j \le k$, such that each s_i appears with even multiplicity, corresponding to the proper faces of $\mathscr{C}_{2k}$.*

Example 4.3.5. By considering the cofactor expansion of the determinant of $\mathbb{M}^1_{2k}(t, \vec{s})$ along the first column, we may explicitly compute the coefficient vectors for the polynomials

$$f_{\vec{s}}(t) = c + \sum_{j=1}^{k} \big(a_j \cos(jt) + b_j \sin(jt)\big)$$

as defined in Corollary 4.3.2. Here, we list the coefficients of f_s, up to a nonzero scalar multiple, for degrees $k = 1$ and $k = 2$ in terms of $\vec{s} = (s_1, s_2, \ldots, s_{2k})$.

$k = 1$

c	$\cos\left(\frac{s_1 - s_2}{2}\right)$
a_1	$-\cos\left(\frac{s_1 + s_2}{2}\right)$
b_1	$-\sin\left(\frac{s_1 + s_2}{2}\right)$

$k = 2$

c	$\cos\left(\frac{s_1+s_2-s_3-s_4}{2}\right) + \cos\left(\frac{s_1-s_2+s_3-s_4}{2}\right) + \cos\left(\frac{s_1-s_2-s_3+s_4}{2}\right)$
a_1	$-\left(\cos\left(\frac{-s_1+s_2+s_3+s_4}{2}\right) + \cos\left(\frac{s_1-s_2+s_3+s_4}{2}\right) + \cos\left(\frac{s_1+s_2-s_3+s_4}{2}\right) + \cos\left(\frac{s_1+s_2+s_3-s_4}{2}\right)\right)$
b_1	$-\left(\sin\left(\frac{-s_1+s_2+s_3+s_4}{2}\right) + \sin\left(\frac{s_1-s_2+s_3+s_4}{2}\right) + \sin\left(\frac{s_1+s_2-s_3+s_4}{2}\right) + \sin\left(\frac{s_1+s_2+s_3-s_4}{2}\right)\right)$
a_2	$\cos\left(\frac{s_1+s_2+s_3+s_4}{2}\right)$
b_2	$\sin\left(\frac{s_1+s_2+s_3+s_4}{2}\right)$

Lemma 4.3.6. *Fix an integer $k \geq 1$. Let $\mathbf{e}_j(e^{it_1}, e^{it_2}, \ldots, e^{it_{2k}})$ denote the j-th elementary symmetric polynomial in the variables $e^{it_1} \ldots, e^{it_{2k}}$ and let $\exp(z) = e^z$. Define*

$$\zeta_{k,j} := (-1)^{k+j} \exp\left(-\frac{1}{2} i \sum_{l=1}^{2k} t_l\right) \mathbf{e}_j\left(e^{it_1}, e^{it_2}, \ldots, e^{it_{2k}}\right),$$

Then, up to a nonzero scalar, the coefficient vector $(c, a_1, b_1, \ldots, a_k, b_k)$ of the trigonometric polynomial defined in Corollary 4.3.2 is given by $c = \mathrm{Re}(\zeta_{k,k})$, $a_i = \mathrm{Re}(\zeta_{k,k+i})$, and $b_i = \mathrm{Im}(\zeta_{k,k+i})$ for all $1 \leq i \leq k$.

The proof of Lemma 4.3.6 is analogous to the proofs of Lemma 4.2.2 and Proposition 4.3.1 and follows by first converting trigonometric polynomials to exponential functions and then computing subdeterminants of $\mathbb{M}^1_{2k}(t, \vec{s})$. The details are omitted.

Example 4.3.7. Let $k = 2$ and choose distinct points $s_1, s_2 \in S^1$. Then, for $\vec{s} = (s_1, s_1, s_2, s_2)^\intercal \in \mathbb{R}^4$, the function

$$f_{\vec{s}}(t) = \sin\left(\frac{s_1 - t}{2}\right)^2 \sin\left(\frac{s_2 - t}{2}\right)^2$$

is a non-negative trigonometric polynomial of degree 2 with roots s_1 and s_2. Hence, the coefficient vector of $f_{\vec{s}}$ belongs to the coorbitope cone $\widehat{\mathscr{C}_4^\circ}$, and these trigonometric polynomials correspond to the faces of $\mathscr{C}_4$ of the form $\operatorname{conv}(\{\mathrm{M}_4(s_1), \mathrm{M}_4(s_2)\})$ for any choice of $s_1, s_2 \in S^1$. Up to a nonzero scalar, the coefficient vector of $f_{\vec{s}}$ is

$$\big(2 + \cos(s_1 - s_2), -2(\cos(s_1) + \cos(s_2)), -2(\sin(s_1) + \sin(s_2)), \cos(s_1 + s_2), \sin(s_1 + s_2)\big),$$

in light of Lemma 4.3.6. A collection of these polynomials, for various vectors $\vec{s} = (0, 0, s_2, s_2)$, is shown in Figure 4.3.1.

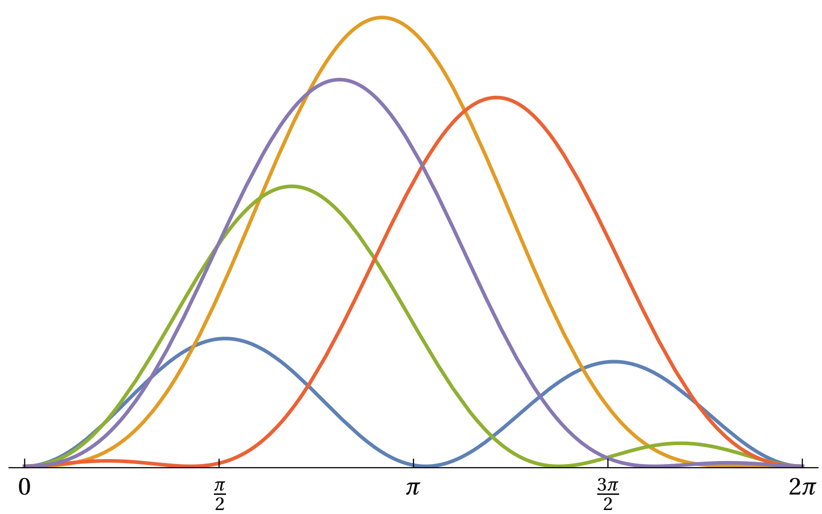


Figure 4.3.1: A set of non-negative trigonometric polynomials $f_{\vec{s}}$, as defined in Example 4.3.7, each of which defines a 1-dimensional face on the boundary of the Carathéodory orbitope $\mathscr{C}_4$. The non-zero root of each polynomial has been chosen at random.

Similarly, by choosing $\vec{s} = (s_1, s_1, s_1, s_1)^\intercal$, we obtain a non-negative trigonometric polynomial of degree 2 with a single root s_1. Hence, these trigonometric polynomials correspond to the faces $\{\mathrm{M}_4(s_1)\}$ of $\mathscr{C}_4$ for any $s_1 \in S$. Up to a nonzero scalar, the coefficient vector of $f_{\vec{s}}$ is

$$\big(3, -4\cos(s_1), -4\sin(s_1), \cos(2s_1), \sin(2s_1)\big).$$

4.4 The Barvinok–Novik coorbitope cone

The results of Section 4.3 suggest that one may obtain an explicit description of the coorbitope cone $\widehat{\mathscr{B}^{\circ}_{2k}}$ of the Barvinok–Novik orbitopes by computing the determinants of the analogous matrices $\mathbb{SM}^1_{2k}(t, \vec{s})$ for $\vec{s} = (s_1, \dots, s_{2k})^\intercal \in \mathbb{R}^{2k}$. However, the resulting trigonometric polynomials are more complicated. In the case $k = 2$, for $\vec{s} = (s_1, \dots, s_4)^\intercal$, Mathematica computes

$$\det(\mathbb{SM}^1_4(t, \vec{s})) =$$
$$\kappa \left(\prod_{1 \le l \le 4} \sin\left(\frac{s_l - t}{2}\right) \prod_{1 \le j < l \le 4} \sin\left(\frac{s_l - s_j}{2}\right) \right) \left(2 + \sum_{1 \le j \le 4} \cos(s_l - t) + \sum_{1 \le j < l \le 4} \cos\left(s_l - s_j\right) \right).$$

As before (cf. Proposition 4.3.1 and Corollary 4.3.2), factoring the constant $\kappa \prod_{1 \le j < l \le 4} \sin\left(\frac{s_l - s_j}{2}\right)$ from this expression proves that

$$g_{\vec{s}}(t) := \left(\prod_{1 \le l \le 4} \sin\left(\frac{s_l - t}{2}\right) \right) \left(2 + \sum_{1 \le j \le 4} \cos(s_l - t) + \sum_{1 \le j < l \le 4} \cos\left(s_l - s_j\right) \right) \tag{4.2}$$

is a raked trigonometric polynomial of degree $2k - 1 = 3$ in t. We note that $g_{\vec{s}}$ has a root at each s_i. However, the sum of cosines makes it more difficult to determine vectors $\vec{s}$ such that $g_{\vec{s}}$ is non-negative on S^1.

Remark 4.4.1. A straightforward computation shows

$$\det(\mathbb{SM}_{2k}(t,\vec{s})) = e_k \frac{i^k}{2^k} \omega \det \begin{pmatrix} 1 & e^{2it} & e^{4it} & \cdots & e^{(2k-2)it} & e^{(2k-1)it} & e^{(2k)it} & \cdots & e^{(2(2k-1))it} \\ 1 & e^{2is_1} & e^{4is_1} & \cdots & e^{(2k-2)is_1} & e^{(2k-1)is_1} & e^{(2k)is_1} & \cdots & e^{(2(2k-1))is_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{2is_{2k}} & e^{4is_{2k}} & \cdots & e^{(2k-2)is_{2k}} & e^{(2k-1)is_{2k}} & e^{(2k)is_{2k}} & \cdots & e^{(2(2k-1))is_{2k}} \end{pmatrix},$$

where $e_k \in \{-1,+1\}$ and $\omega = e^{-(2k-1)i(t+s_1+s_2+\cdots+s_{2k})}$. The matrix in this expression is almost a Vandermonde matrix: it has an additional column, and is referred to as a **generalized Vandermonde matrix**. It is well known that the determinant of a generalized Vandermonde matrix factors as the product of an ordinary Vandermonde determinant (Definition 2.2.18) and a Schur polynomial; in fact, this is often how a Schur polynomial is defined (see [39, 58, 75, 96], for example). Hence, $\det(\mathbb{SM}_{2k}(t,\vec{s}))$ will always contain the factor

$$\left(\prod_{1\le l\le 2k} \sin\left(\frac{s_l - t}{2}\right) \prod_{1\le j<l\le 2k} \sin\left(\frac{s_l - s_j}{2}\right)\right)$$

arising from the Vandermonde determinant. Consequently,

$$g_{\vec{s}}(t) := \frac{\det(\mathbb{SM}^1_{2k}(t,\vec{s}))}{\prod_{1\le j<l\le 2k} \sin\left(\frac{s_l - s_j}{2}\right)}$$

is a well-defined raked trigonometric polynomial of degree $2k-1$.

Remark 4.4.2. For $\vec{s} = (s_1,\ldots,s_{2k})^\intercal$ and $g_{\vec{s}}$ as defined in Remark 4.4.1, let us write

$$g_{\vec{s}}(t) = \left(\prod_{1\le l\le 2k} \sin\left(\frac{s_l - t}{2}\right)\right) p_{\vec{s}}(t),$$

where $p_{\vec{s}}$ denotes the factor of $g_{\vec{s}}$ arising from the Schur polynomial part of the determinant of $\mathbb{SM}_{2k}(t,\vec{s})$. Let $t_1,\ldots,t_n \in S^1$ denote the distinct roots of $p_{\vec{s}}$, and let $m_1,\ldots,m_n$ denote their multiplicities. Because $g_{\vec{s}}$ is a raked trigonometric polynomial of degree $2k-1$, Theorem 3.1.1 of [22] implies that the sum of the multiplicities of the distinct roots of $g_{\vec{s}}$ can be at most $4k-2$. Hence, because $g_{\vec{s}}$ has a root at each s_i, it follows that $\sum_{i=1}^{n} m_i \leq 2k-2$. In particular, $p_{\vec{s}}$ can have at most $2k-2$ roots counted with multiplicity. Furthermore, if $\{s_1,\ldots,s_{2k}\}$ belong to an arc $\Gamma \subseteq S^1$ of length less than π, then Theorem 2.2.13 implies that any root of $p_{\vec{s}}$ must belong to the arc $\Gamma+\pi$.

In analogy with Theorem 4.3.4, we conjecture that the polynomials $g_{\vec{s}}$ determine the proper faces of the Barvinok–Novik orbitopes.

Conjecture 4.4.3. *Let the trigonometric polynomial $g_{\vec{s}}$ be defined as in Remark 4.4.1. The coorbitope cone $\widehat{\mathscr{B}_{2k}^{\circ}}$ is generated by the following set of vectors:*

- *the zero vector $(0,0,\ldots,0) \in \mathbb{R}^{2k+1}$, corresponding to the face $\mathscr{B}_{2k}$ of $\mathscr{B}_{2k}$,*
- *the vector $(1,0,\ldots,0) \in \mathbb{R}^{2k+1}$, corresponding to the face $\varnothing$ of $\mathscr{B}_{2k}$,*
- *the coefficient vectors of polynomials $g_{\vec{s}}$ defined by $s=(s_1,\ldots,s_{2j})^\intercal$, for $1 \leq j \leq k$, such that $g_{\vec{s}}$ is non-negative on S^1, corresponding to the proper faces of $\mathscr{B}_{2k}$.*

Figure 4.4.1 shows a collection of polynomials $g_{\vec{s}}$, as defined in Equation 4.2 (up to a non-zero constant), for vectors of the form $s=(0,0,r,r)^\intercal$ with $r \leq \frac{2\pi}{3}$. Because each polynomial is a non-negative raked trigonometric polynomial of degree 3, each defines a face of the Barvinok–Novik orbitope $\mathscr{B}_4$. In particular, five polynomials in the figure correspond to edges of length less than $\frac{2\pi}{3}$, and one corresponds to the dimension-maximal proper face of diameter exactly $\frac{2\pi}{3}$.

Experimentally, and as evidence toward Conjecture 4.4.3, every non-negative polynomial $g_{\vec{s}}$, as defined in Equation 4.2, defines a face of $\mathscr{B}_4$, and, conversely, all faces of $\mathscr{B}_4$ arise in this way.

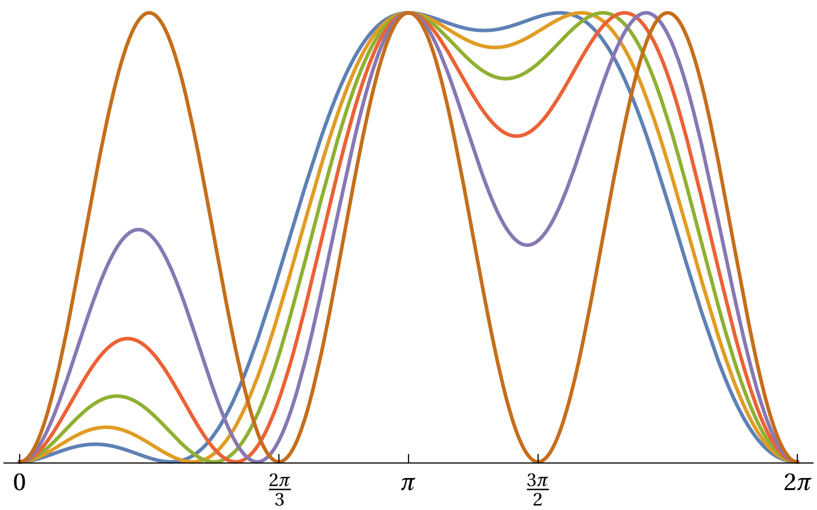


Figure 4.4.1: Non-negative trigonometric polynomials $g_{\vec{s}}$, as defined in Equation 4.2, each of which defines a face of the Barvinok–Novik orbitope $\mathscr{B}_4$. For clarity, each polynomial has been multiplied by a non-zero constant to achieve the same maximum value.

Chapter 5

Generalizations of the Borsuk–Ulam theorem

5.1 Maps $S^1 \to \mathbb{R}^k$ with $k > 1$

In this chapter, we give generalizations of the Borsuk–Ulam theorem in which the dimension of the codomain of an odd map may surpass the dimension of the domain.

Definition 5.1.1. Let X be a topological space equipped with a $\mathbb{Z}/2\mathbb{Z}$-action. Define the $\mathbb{Z}/2\mathbb{Z}$-**index of** X by

$$\mathrm{ind}_{\mathbb{Z}/2\mathbb{Z}}(X) := \min\{m \in \{0,1,2,\dots\} \mid \text{there exists an odd map } X \to S^m\}.$$

Analogously, define the $\mathbb{Z}/2\mathbb{Z}$-**coindex of** X by

$$\mathrm{coind}_{\mathbb{Z}/2\mathbb{Z}}(X) := \max\{m \in \{0,1,2,\dots\} \mid \text{there exists an odd map } S^m \to X\}.$$

Observe that $\mathrm{coind}_{\mathbb{Z}/2\mathbb{Z}}(X) \leq \mathrm{ind}_{\mathbb{Z}/2\mathbb{Z}}(X)$ for any $\mathbb{Z}/2\mathbb{Z}$-space X by the Borsuk–Ulam theorem.

Properties of the symmetric moment curve allow us to determine the (co)index of Vietoris–Rips metric thickenings of the circle. Because SM_{2k} is an odd map, note that the boundary $\partial\mathscr{B}_{2k} \cong S^{2k-1}$ is naturally equipped with a $\mathbb{Z}/2\mathbb{Z}$-action specified by extending the antipodal action on the domain of SM_{2k}, that is,

$$\sum_i \lambda_i \mathrm{SM}_{2k}(t_i) \mapsto \sum_i \lambda_i \mathrm{SM}_{2k}(-t_i) = -\sum_i \lambda_i \mathrm{SM}_{2k}(t_i),$$

where $\{\lambda_i\}_i$ denote convex coefficients.

Lemma 5.1.2. *For all positive integers $k \geq 1$,*

$$\mathrm{ind}_{\mathbb{Z}/2\mathbb{Z}}\left(\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1;r)\right) = \mathrm{coind}_{\mathbb{Z}/2\mathbb{Z}}\left(\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1;r)\right) = 2k-1 \qquad \textit{whenever} \qquad \tfrac{2\pi(k-1)}{2k-1} \leq r < \tfrac{2\pi k}{2k+1}.$$

Proof. Observe that $p \circ \mathrm{SM}_{2k} \colon \mathrm{VR}^{\mathrm{m}}_{\le}(S^1;r) \to \partial\mathscr{B}_{2k}$ is an odd map that is well-defined at these scales by Theorem 4.2.1 and continuous by [3, Lemma 5.2]. Further, the linear projection from $\partial\mathscr{B}_{2k}$ to the $(2k-1)$-sphere in $\mathbb{R}^{2k}$ centered at the origin is odd and continuous. It follows that $\mathrm{ind}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{VR}^{\mathrm{m}}_{\le}(S^1;r)) \le 2k-1$.

On the other hand, the linear projection from the the $(2k-1)$-sphere in $\mathbb{R}^{2k}$ centered at the origin to $\partial\mathscr{B}_{2k}$ is odd and continuous, and the inclusion $\iota \colon \partial\mathscr{B}_{2k} \to \mathrm{VR}^{\mathrm{m}}_{\le}(S^1;r)$ is an odd map that is well-defined by Theorem 2.2.12 at these scales and continuous by Lemma 3.1.4. Hence, $\mathrm{coind}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{VR}^{\mathrm{m}}_{\le}(S^1;r)) \ge 2k-1$. □

Knowledge of the $\mathbb{Z}/2\mathbb{Z}$-index of these Vietoris–Rips metric thickenings yields the following generalization of the Borsuk–Ulam theorem.

Theorem 5.1.3. *If $f \colon S^1 \to \mathbb{R}^{2k+1}$ is odd and continuous, then there is a subset $X \subseteq S^1$ of diameter at most $\frac{2\pi k}{2k+1}$ such that* $\mathrm{conv}(f(X))$ *contains the origin, and this diameter bound is sharp.*

Proof. Let $f \colon S^1 \to \mathbb{R}^{2k+1}$ be a continuous odd map. Because f is bounded, the induced map $F \colon \mathrm{VR}^{\mathrm{m}}_{\le}(S^1;\frac{2\pi k}{2k+1}) \to \mathbb{R}^{2k+1}$, defined by $F(\sum_i \lambda_i x_i) = \sum_i \lambda_i f(x_i)$ is continuous by [3, Lemma 5.2]. Notice that F commutes with the antipodal action on $\mathrm{VR}^{\mathrm{m}}_{\le}(S^1;r)$ and S^1, that is,

$$F\left(\sum_{i=1}^{k} \lambda_i \delta_{-x_i}\right) = \sum_{i=1}^{k} \lambda_i f(-x_i) = -\sum_{i=1}^{k} \lambda_i f(x_i) = -F\left(\sum_{i=1}^{k} \lambda_i \delta_{x_i}\right).$$

Because the domain of this odd map has $\mathbb{Z}/2\mathbb{Z}$-index $2k+1$ by Lemma 5.1.2, it must have a zero. In other words, there must be a subset $X \subseteq S^1$ of diameter at most $\frac{2\pi k}{2k+1}$ such that $\mathrm{conv}(f(X))$ contains the origin. Furthermore, by Carathéodory's theorem, we can take the size of X to be at most $2k+2$. Theorem 4.2.1 shows that this diameter bound is sharp. □

Equivalently, if $f \colon S^1 \to \mathbb{R}^{2k+1}$ is continuous, then there exists a subset $\{x_1,\dots,x_m\} \subseteq S^1$ of diameter at most $\frac{2\pi k}{2k+1}$ such that $\sum_{i=1}^m \lambda_i f(x_i) = \sum_{i=1}^m \lambda_i f(-x_i)$, for some choice of convex coefficients $\lambda_1,\dots,\lambda_m$.

We will call theorems of this type, for maps $S^n \to \mathbb{R}^k$ with $k > n$, **convex Borsuk–Ulam approximations**.

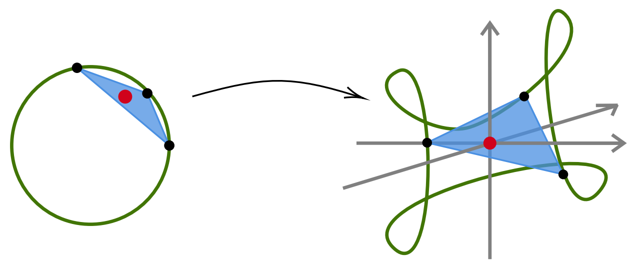


Figure 5.1.1: A subset of S^1 of small diameter whose image is a Carathéodory subset under an odd map.

For example, if $f = \mathrm{SM}_{2k}\colon S^1 \to \mathbb{R}^{2k} \subseteq \mathbb{R}^{2k+1}$, then this set X is easy to find: we can let X be $2k+1$ evenly-spaced points on the circle. Theorem 4.2.1 shows that the diameter bound $\frac{2\pi k}{2k+1}$ is sharp, both for maps $S^1 \to \mathbb{R}^{2k+1}$ and for maps $S^1 \to \mathbb{R}^{2k}$. Indeed, $\mathrm{SM}_{2k}\colon S^1 \to \mathbb{R}^{2k} \subseteq \mathbb{R}^{2k+1}$ is an odd map in which the convex hull of the image of every set of diameter strictly less than $\frac{2\pi k}{2k+1}$ misses the origin.

Corollary 5.1.4. *Fix a list of odd maps $f_i\colon S^1 \to \mathbb{R}^1$ for $1 \le i \le 2k+2$. Then, there exists a subset $\{x_1, \ldots, x_m\} \subseteq S^1$ of diameter at most $\frac{2\pi k}{2k+1}$ and a set of convex coefficients $\{\lambda_1, \ldots, \lambda_m\}$ such that*

$$\sum_{i=1}^{m} \lambda_i f_1(x_i) = \sum_{i=1}^{m} \lambda_i f_2(x_i) = \cdots = \sum_{i=1}^{m} \lambda_i f_{2k+2}(x_i).$$

Furthermore, this diameter bound is sharp.

Proof. Apply Theorem 5.1.3 to the odd map $g\colon S^1 \to \mathbb{R}^{2k+1}$ with components $g_j := f_j - f_{2k+2}$. $\square$

Recall that the Barvinok–Novik metric thickenings $\mathrm{BN}^{\mathrm{m}}(k)$ defined in Section 3.1 are homeomorphic to the boundaries of the Barvinok–Novik orbitopes, that is, $\mathrm{BN}^{\mathrm{m}}(k) \cong \partial\mathscr{B}_{2k} \cong S^{2k-1}$. Because this homeomorphism respects the $\mathbb{Z}/2\mathbb{Z}$-action on both spaces, it follows that the (co)index of $\mathrm{BN}^{\mathrm{m}}(k)$ is $2k-1$. Hence, we may use facts about Barvinok–Novik orbitopes outlined in Subsection 2.2.7 (see Theorem 2.2.12 and Lemma 2.2.17) to obtain the following stronger version of Theorem 5.1.3.

Theorem 5.1.5. *Let $f\colon S^1 \to \mathbb{R}^{2k+1}$ be an odd map for any $k \ge 0$. Then, there exists a finite subset $Y \subseteq S^1$ such that $\vec{0} \in \mathrm{conv}(f(Y))$ and the points of Y define a face of the Barvinok–Novik orbitope*

$\mathscr{B}_{2k+2}$. *In particular,* $|Y| \leq 2k+1$*, the diameter of* Y *is at most* $\frac{2\pi k}{2k+1}$*, and if* $|Y| = 2k+1$*, then the points of* Y *form the vertices of a regular inscribed* $(2k+1)$*-gon in* S^1.

In the conclusion of this theorem, note that $|Y| \leq 2k+1$, whereas the cardinality of Y is a priori only bounded above by $2k+2$ by Carathéodory's theorem. The proof of this theorem is analogous to the proof of Theorem 5.1.3 and is omitted.

5.2 Maps $S^n \to \mathbb{R}^k$ with $k > n$.

Next, we consider convex Borsuk–Ulam approximations for maps of higher-spheres.

Theorem 5.2.1. *If* $f\colon S^{2n-1} \to \mathbb{R}^{2kn+2n-1}$ *is odd and continuous, then there is a subset* $X \subseteq S^{2n-1}$ *of diameter at most* $\frac{2\pi k}{2k+1}$ *such that* $\mathrm{conv}(f(X))$ *contains the origin.*

Proof. The case of $k=0$ follows from the standard Borsuk–Ulam theorem.

For $k \geq 1$, we will think of S^{2n-1} as a join of n circles $(S^1)^{*n}$. Explicitly, if S^{2n-1} is viewed as the unit sphere in $\mathbb{R}^{2n}$, then the subset of S^{2n-1} with all coordinates zero, with the (possible) exception of coordinates $2j-1$ and $2j$, is a circle. The distance between any two points in distinct such circles is $\frac{\pi}{2}$ in the geodesic metric. Let $r = \frac{2\pi k}{2k+1}$. Since $r > \frac{\pi}{2}$, this will allow us to construct a $\mathbb{Z}/2\mathbb{Z}$-equivariant embedding of $(\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1;r))^{*n}$ into $\mathrm{VR}^{\mathrm{m}}_{\leq}(S^{2n-1};r)$. A point in $\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1;r)$ can be written as $\sum_{x\in X}\lambda_x\delta_x$, where the vertex set X of the simplex containing this point has diameter at most r and where $\{\lambda_x\}_{x\in X}$ is a set of convex coefficients. Hence, a point in $(\mathrm{VR}(S^1;r))^{*n}$ consists of n collections of such points $\sum_{x\in X_i}\lambda_x\delta_x$ for $1 \leq i \leq n$, along with non-negative numbers $\kappa_1,\ldots,\kappa_n$ that sum to one. We map the points in X_i to the i-th copy of S^1 in $S^{2n-1} = (S^1)^{*n}$, and we multiply their weights by κ_i. This gives the barycentric coordinates of a well-defined point in $\mathrm{VR}^{\mathrm{m}}_{\leq}(S^{2n-1};r)$; the diameter of the supporting simplex is at most r since $r > \frac{\pi}{2}$. Furthermore, this map respects the antipodal $\mathbb{Z}/2\mathbb{Z}$-actions on $(\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1;r))^{*n}$ and $\mathrm{VR}^{\mathrm{m}}_{\leq}(S^{2n-1};r)$. By [79, Proposition 5.3.2], $\mathrm{ind}_{\mathbb{Z}/2\mathbb{Z}}(Y * Y') \leq \mathrm{ind}_{\mathbb{Z}/2\mathbb{Z}}(Y) + \mathrm{ind}_{\mathbb{Z}/2\mathbb{Z}}(Y') + 1$ for any $\mathbb{Z}/2\mathbb{Z}$-spaces Y and Y'. Hence, Lemma 5.1.2 implies

$$\mathrm{ind}_{\mathbb{Z}/2\mathbb{Z}}\left((\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1;r))^{*n}\right) \leq (2k+1)n + n - 1 = 2kn + 2n - 1.$$

Thus, any odd map from $(\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1;r))^{*n}$, and hence also from $\mathrm{VR}^{\mathrm{m}}_{\leq}(S^{2n-1};r)$, into $\mathbb{R}^{2kn+2n-1}$ must hit the origin. This gives a subset $X \subseteq S^{2n-1}$ of diameter at most $r = \frac{2\pi k}{2k+1}$ such that $\mathrm{conv}(f(X))$ contains the origin. □

Recall (see Definition 2.1.33) that Δ_n denotes the diameter of an inscribed regular $(n+1)$-simplex.

Theorem 5.2.2. *If $f\colon S^n \to \mathbb{R}^{n+2}$ is odd and continuous, then there is a subset $X \subseteq S^n$ of diameter at most Δ_n such that $\mathrm{conv}(f(X))$ contains the origin, and this diameter bound is sharp.*

The sharpness of the bound Δ_n follows from the fact that the standard inclusion $f\colon S^n \hookrightarrow \mathbb{R}^{n+1} \subseteq \mathbb{R}^{n+2}$ is an odd map that satisfies $\vec{0} \notin \mathrm{conv}(f(X))$ for all $X \subseteq S^n$ of diameter less than Δ_n [71, Proof of Lemma 3].

Our proof of Theorem 5.2.2 requires the following intermediate result, in which the $\mathbb{Z}/2\mathbb{Z}$-index of a space is bounded below by its connectivity. We say a space is **n-connected** if it is nonempty, path connected, and its homotopy groups vanish up to and including dimension n.

Theorem 5.2.3. *Let X be a $(k-1)$-connected space equipped with a $\mathbb{Z}/2\mathbb{Z}$-action. Then,*

$$k \leq \mathrm{ind}_{\mathbb{Z}/2\mathbb{Z}}(X).$$

Proof. We follow the proof of [79, Proposition 5.3.2(iv)]. Before proceeding, we fix the following notation for all $n \geq 1$. First, let $\pi_n\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection map that deletes the n^{th} coordinate. Next, let $S^n_+ := \{(x_1,\ldots,x_{n+1}) \in S^n \mid x_{n+1} \geq 0\}$ denote the closed upper hemisphere of S^n (which is homeomorphic to the ball B^n). Last, we consider S^{n-1} to be the equator of S^n defined by $S^{n-1} := \{(x_1,\ldots,x_{n+1}) \in S^n \mid x_{n+1} = 0\}$.

Now, by the Borsuk–Ulam theorem, observe that it is sufficient to exhibit an odd map $S^k \to X$. Further, it is clear that an odd map $S^0 \to X$ exists. We will proceed by induction on n to construct an odd map $S^k \to X$. For the induction step, suppose $g_{n-1}\colon S^{n-1} \to X$ is an odd map for some $n \leq k$. Since X is $(n-1)$-connected, we can extend g_{n-1} to a map $\overline{g}_{n-1}\colon B^n \to X$, and we may then compose with π_{n+1} to obtain a map $\overline{g}_{n-1} \circ \pi_{n+1} \colon S^n_+ \to X$. Finally, for $x \in S^n_+$, define $g_n(x) := \overline{g}_{n-1}(\pi_{n+1}(x))$ and $g_n(-x) := \nu\overline{g}_{n-1}(\pi_{n+1}(x))$, where ν denotes the $\mathbb{Z}/2\mathbb{Z}$-action on X. One easily checks that g_n is well-defined, continuous, and odd. □

This theorem immediately implies the following.

Corollary 5.2.4. *Let X be a $(k-1)$-connected topological space equipped with a $\mathbb{Z}/2\mathbb{Z}$-action. Given a continuous odd map $f \colon X \to \mathbb{R}^k$, there exists $x_0 \in X$ such that $f(x_0) = \vec{0}$.*

Proof. Theorem 5.2.3 implies that there is no $\mathbb{Z}/2\mathbb{Z}$-equivariant map from X into S^{k-1}. Hence, any odd map $f \colon X \to \mathbb{R}^k$ must hit the origin, because otherwise we would obtain a continuous odd map $\frac{f}{|f|} \colon X \to S^{k-1}$. □

Remark 5.2.5. In fact, Theorem 5.2.3 and Corollary 5.2.4 hold for more general spaces: it is sufficient to assume only that the $\mathbb{Z}/2\mathbb{Z}$-homology of X vanishes up to and including dimension $(n-1)$ [107].

Proof of Theorem 5.2.2. The space $\mathrm{VR}^{\mathrm{m}}(S^n;\Delta_n)$ has a $\mathbb{Z}/2\mathbb{Z}$-action that maps a convex combination $\sum_{i=1}^k \lambda_i \delta_{x_i}$ of Dirac measures for points $x_1, \ldots, x_k$ on S^n to $\sum_{i=1}^k \lambda_i \delta_{-x_i}$, that is, to the measure that is supported on the antipodal point set with the same weights λ_i. This action is free since antipodal points on S^n are farther than Δ_n apart.

Let $f \colon S^n \to \mathbb{R}^{n+2}$ be odd and continuous. Because f is bounded, [3, Lemma 5.2] implies that f induces a continuous map $F \colon \mathrm{VR}^{\mathrm{m}}(S^n; r_n) \to \mathbb{R}^{n+2}$ defined by $F(\sum_{i=1}^k \lambda_i \delta_{x_i}) = \sum_{i=1}^k \lambda_i f(x_i)$. Notice that F commutes with the antipodal action on $\mathrm{VR}^{\mathrm{m}}(S^n;\Delta_n)$ and S^n:

$$F\left(\sum_{i=1}^k \lambda_i \delta_{-x_i}\right) = \sum_{i=1}^k \lambda_i f(-x_i) = -\sum_{i=1}^k \lambda_i f(x_i) = -F\left(\sum_{i=1}^k \lambda_i \delta_{x_i}\right).$$

Next, fix a regular $(n+1)$-simplex Δ inscribed in S^n, and let A_{n+2} denote the group of rotational symmetries of Δ, that is, the alternating group on $n+2$ elements. In a noncanonical fashion, we may identify A_{n+2} as a subgroup of $\mathrm{SO}(n+1)$ by associating to each $g \in A_{n+2}$ the matrix $M_g \in \mathrm{SO}(n+1)$ such that $M_g \cdot v = g \cdot v$ for each vertex v of Δ. In this way, we obtain the orbit space $\frac{\mathrm{SO}(n+1)}{A_{n+2}}$ of $\mathrm{SO}(n+1)$ under the action of A_{n+2} by left multiplication. By Theorem 2.1.41 the homotopy type of $\mathrm{VR^m}(S^n; r_n)$ is $S^n * \frac{\mathrm{SO}(n+1)}{A_{n+2}}$, and because $\frac{\mathrm{SO}(n+1)}{A_{n+2}}$ is connected, its join with S^n is $(n+1)$-connected. Thus, the map F has a zero by Corollary 5.2.4. That is, there are points $x_1, \ldots, x_m \in S^n$ that are pairwise at distance at most Δ_n and such that $\sum_{i=1}^m \lambda_i f(x_i) = \vec{0}$ for some $\lambda_1, \ldots, \lambda_m \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$. □

In what follows, we consider generalizations of the Borsuk–Ulam theorem in which the dimension of the codomain may be arbitrarily large with respect to the dimension of the domain. For notational convenience, we make the following definition.

Definition 5.2.6. Fix integers $k \geq n \geq 1$. We say $t \in [0, \pi]$ is (n, k)-**filling** if, given any odd and continuous map $f\colon S^n \to \mathbb{R}^k$, there exists a subset $X \subseteq S^n$ with $\mathrm{diam}(X) \leq t$ such that $\vec{0} \in \mathrm{conv}(f(X))$.

We observe that $t \in [0, \pi]$ is not (n, k)-filling when there exists an odd continuous map $g\colon S^n \to \mathbb{R}^k$ such that if $X \subseteq S^n$ with $\mathrm{diam}(X) \leq t$, then $\vec{0} \notin \mathrm{conv}(g(X))$.

Definition 5.2.7. We define

$$s_{n,k} \coloneqq \inf\{t \in [0, \pi) \mid t \text{ is } (n,k)\text{-filling}\}.$$

We prove in the following proposition that this infimum is taken over a nonempty set and that the infimum is attained. Because the number $s_{n,k}$ bounds the diameter of the preimage of any Carathéodory subset (Definition 2.2.2) of $f(S^n)$ for any odd and continuous map f, we call $s_{n,k}$ the **spherical Carathéodory diameter** for this choice of n and k.

Proposition 5.2.8. *For all integers $k \geq n \geq 1$, the spherical Carathéodory diameter $s_{n,k}$ is a real number less than π and the infimum is attained.*

Proof. As an intermediate step, define

$$\tilde{s}_{n,k} \coloneqq \inf\{t \in [0, \pi] \mid t \text{ is } (n,k)\text{-filling}\},$$

where $[0,\pi)$ has been replaced with $[0,\pi]$. Note that Theorem 5.2.1 implies that, given any positive integers n and k_0, there exists some $K \geq k_0$ and diameter $D < \pi$ so that any odd map $S^{2n-1} \to \mathbb{R}^K$ has some set $X \subseteq S^n$ of diameter at most D whose image is a Carathéodory subset of $\mathbb{R}^K$. Observe the same diameter $D < \pi$ works for any odd map $S^{2n-1} \to \mathbb{R}^k$ with $k < K$, since the embedding $\mathbb{R}^k \hookrightarrow \mathbb{R}^K$ obtained by appending zeros is odd. Hence, for all positive integers n and k, there exists a number $t \in [0,\pi)$ that is $(2n-1,k)$-filling. Further, observe that any odd map $S^{2n} \to \mathbb{R}^k$ restricted to the equator of S^{2n} is again an odd map $S^{2n-1} \to \mathbb{R}^k$; hence, t is also $(2n,k)$-filling. In summary, $0 \leq \tilde{s}_{n,k} < \pi$ for all integers $k \geq n \geq 1$.

It remains to prove that the infimum in the definition of $\tilde{s}_{n,k}$ is attained. Toward that end, let $f\colon S^n \to \mathbb{R}^k$ be an odd and continuous map, and let $\varepsilon > 0$. Then, for each integer $m \geq 1$, there exists a subset $X_m \subseteq S^n$ of diameter at most $s_{n,k} + \frac{\varepsilon}{m}$ such that $\vec{0} \in f(X_m)$. Further, by Carathéodory's Theorem, we may assume $|X_m| \leq k+1$. If $|X_m| < k+1$, duplicate an arbitrary point in X_m to obtain a multi-set of size exactly $k+1$. Arbitrarily order these points so that X_m can be thought of as a point in $(S^n)^{k+1}$. By compactness of this product of spheres, the sequence $\{X_m\}$ has a subsequence converging to a limit configuration $X \in (S^n)^{k+1}$ of diameter at most $s_{n,k}$ and with $\vec{0} \in \mathrm{conv}(f(X))$. Removing duplicate points (and ignoring the ordering) gives us the desired subset $X \subseteq S^n$. □

It follows from Proposition 5.2.8 that the set of numbers $t \in [0,\pi]$ that are (n,k)-filling is precisely $[s_{n,k},\pi]$, and the set of numbers that are not (n,k)-filling is $[0,s_{n,k})$.

In particular, we have the following generalization of the Borsuk–Ulam theorem in which the dimension of the codomain may be arbitrarily large. The proof of this theorem is simply by definition of the spherical Carathéodory diameter $s_{n,k}$; what is more interesting is that we give the exact values of $s_{n,k}$ for $n = 1$ or $k \leq n+2$ in Theorems 5.1.3 and 5.2.2, and we give nontrivial upper bounds for all $s_{n,k}$ in Theorem 5.2.1.

Theorem 5.2.9. *Given integers $k \geq n \geq 1$, let $f\colon S^n \to \mathbb{R}^k$ be any continuous odd map. Then, there exists a finite subset $X \subseteq S^n$ of diameter at most the spherical Carathéodory diameter $s_{n,k} < \pi$ such that $f(X)$ is a Carathéodory subset of $\mathbb{R}^k$.*

Known values of $s_{n,k}$ for $n \le 4$ and $k \le 6$							
	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$n=1$	0	$\Delta_1 = \frac{2\pi}{3}$	$\Delta_1 = \frac{2\pi}{3}$	$\frac{4\pi}{5}$	$\frac{4\pi}{5}$	$\frac{6\pi}{7}$	$\frac{6\pi}{7}$
$n=2$	0	0	Δ_2	Δ_2			
$n=3$	0	0	0	Δ_3	Δ_3		
$n=4$	0	0	0	0	Δ_4	Δ_4	

The table above lists some known values of the spherical Carathéodory diameters $s_{n,k}$. Because any odd map $S^n \to \mathbb{R}^k$ restricts to an odd map of an equatorial sphere $S^{n-1} \hookrightarrow S^n \to \mathbb{R}^k$, it follows that $s_{n,k}$ is non-increasing in n, that is, $s_{n,k} \le s_{n',k}$ for any $n \ge n'$. Furthermore, any odd map $S^n \to \mathbb{R}^k$ restricts to an odd map $S^n \to \mathbb{R}^k \xrightarrow{\pi_k} \mathbb{R}^{k-1}$, where π_k is the projection deleting the k^{th} coordinate, and it follows that $s_{n,k}$ is non-decreasing in k, that is, $s_{n,k} \le s_{n,k'}$ for any $k' \ge k$. This implies, for example,

$$1.910633 < \arccos\left(-\tfrac{1}{3}\right) = \Delta_2 = s_{2,4} \le s_{2,5}$$

and

$$s_{2,5} \le s_{1,5} = \tfrac{4\pi}{5} < 2.513275.$$

Hence, $1.910633 < s_{2,5} < 2.513275$. On the other hand, Theorem 5.2.1 often gives an upper bound for $s_{n,k}$ that is tighter than the bound $s_{n,k} \le s_{n',k}$ for $n \ge n'$.

Note that all convex Borsuk–Ulam approximations in this chapter have analogous versions for continuous maps that are not necessarily odd. In particular, given any continuous map $S^n \to \mathbb{R}^k$, there exists a finite subset $X = \{x_1, \ldots, x_m\} \subseteq S^n$ of diameter at most $s_{n,k}$ such that $\sum_{i=1}^m \lambda_i f(x_i) = \sum_{i=1}^m \lambda_i f(-x_i)$ for some convex coefficients $\{\lambda_i\}_i$.

5.3 Corollaries

In this section, we give generalizations of corollaries of the Borsuk–Ulam theorem in terms of the spherical Carathéodory diameter $s_{n,k}$ (Definition 5.2.7).

5.3.1 Generalization of the Stone–Tukey theorem

Convex Borsuk–Ulam approximations allow us to extend the Stone–Tukey theorem (Theorem 2.3.5) to the setting in which the number of measures exceeds the dimension of the ambient space. For the purpose of illustration, we refer to the following as the "log bundle" theorem.

Informally, in the case of three measurable subsets of the disk, Theorem 5.3.1 says the following. Suppose a bundle of three logs needs to be divided up using only planar cuts to obtain an equipartition of each log. Furthermore, our equipment permits only two kinds of cuts: first, we can perform at most three horizontal cuts perpendicular to the bundle (Figure 5.3.1 (left)); then, through the centers of each resulting shorter bundle, we can perform a single vertical cut to produce two hemi-bundles (Figure 5.3.1 (right)). Furthermore, suppose the blade used to perform each "vertical cut" is on a fixed pivot that can swivel by an angle of at most $\frac{2\pi}{3}$. Then, it is always possible to obtain an equipartition of each log using only the cuts described above by selecting exactly one of each of the resulting hemi-bundles.

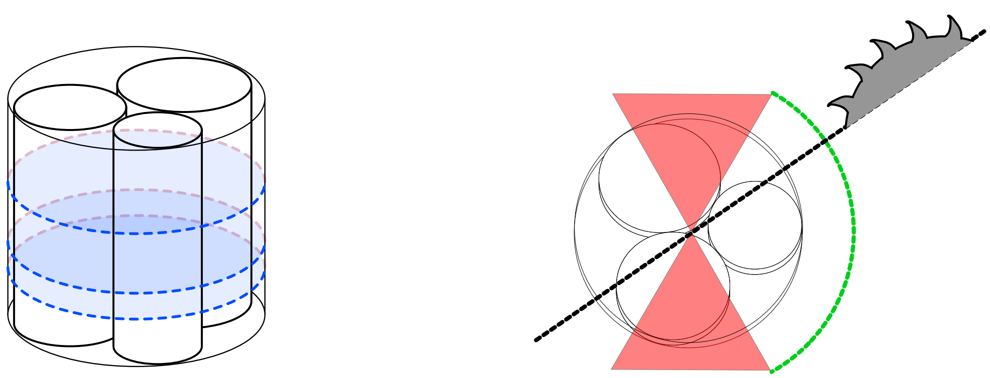


Figure 5.3.1: (Left) A bundle of three logs. Dashed blue lines indicate horizontal cuts. (Right) A vertical cut through the center of one slice of the log bundle. In this case, the saw blade is on a fixed pivot that can not swivel by an angle of more than $\frac{2\pi}{3}$.

Theorem 5.3.1 (Generalization of Theorem 2.3.5)**.** *Fix integers $n, k \geq 1$. Let*

$$D^{n+1} = \{(x_1, x_2, \ldots, x_{n+1}, 0) \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 \leq 1\} \subseteq \mathbb{R}^{n+2}$$

and suppose $A_1, \ldots, A_k \subseteq D^{n+1}$ are support sets of finite Borel measures such that every hyperplane in $\mathbb{R}^{n+1}$ has measure 0. For $p \in \partial D^{n+1}$, let $H_p = \{x \in \mathbb{R}^{n+2} \mid \langle x, p\rangle = 0\}$ denote the hyperplane passing through the origin normal to p. Furthermore, let H_p^+ and H_p^- denote the (closed) half-spaces of $\mathbb{R}^{n+2}$ determined by the inequalities $\langle x, p\rangle \geq 0$ and $\langle x, p\rangle \leq 0$, respectively. Then, there exist numbers $0 = t_0 \leq t_1 \leq \cdots \leq t_k \leq t_{k+1} = 1$ and vectors $p_1, \ldots, p_{k+1} \in \partial D^{n+1}$ such that

1. *the vectors p_i are close in the sense that $\arccos(\langle p_i, p_j\rangle) \leq s_{n,k} < \pi$ for all i and j (in particular, no two vectors are antipodal), and*

2. *there exists an equipartition of the k masses $A_j \times I$ given by taking $(A_j \times [t_{i-1}, t_i]) \cap H_{p_i}^+$ for each $1 \leq i \leq k+1$.*

Furthermore, the bound $\arccos(\langle p_i, p_j\rangle) \leq s_{n,k}$ is sharp.

In the case $n = 1$, we have $s_{n,2k} = s_{n,2k+1} = \frac{2\pi k}{2k+1}$.

Proof. For $p \in S^n = \partial D^{n+1}$, define $h_p^+ := H_p^+ \cap D^{n+1}$ and $h_p^- := H_p^- \cap D^{n+1}$. Define a function $f : S^n \to \mathbb{R}^k$ by

$$p \mapsto \Big(\text{the measure of } A_1 \cap h_p^+, \ldots, \text{the measure of } A_k \cap h_p^+\Big),$$

which is continuous by an application of Lebesgue's dominated convergence theorem. By the convex Borsuk–Ulam approximation of Theorem 5.2.9, there exist vectors $p_1, \ldots, p_{k+1} \in S^n$ and convex coefficients $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{k+1} \leq 1$ such that $\arccos(\langle p_i, p_j\rangle) \leq s_{n,k}$ for all i and j and

$$\sum_{i=1}^{k+1} \lambda_i f(p_i) = \sum_{i=1}^{k+1} \lambda_i f(-p_i).$$

Note that this diameter bound is optimal by Definition 5.2.7. Last, observe that

$$f(-p) = (\text{the measure of } A_1 \cap h_p^-,\ \ldots,\ \text{the measure of } A_k \cap h_p^-).$$

Setting $t_i = \sum_{j=1}^{i} \lambda_j$ for each $0 \leq i \leq k+1$ completes the proof. $\square$

We may obtain a stronger version of Theorem 5.3.1, allowing one more set to be equipartitioned, by dropping the requirement that each hyperplane H_p must pass through the origin. In this setting, $k+1$ sets may be equipartitioned by $k+1$ hyperplanes determined by vectors $p_1, \ldots, p_{k+1}$ which again satisfy $\arccos(\langle p_i, p_j \rangle) \le s_{n,k} < \pi$ for all i and j.

5.3.2 Generalization of the Lyusternik–Shnirel'man–Borsuk theorem

Convex Borsuk–Ulam approximations allow us to generalize the Lyusternik–Shnirel'man–Borsuk covering theorem (Theorem 2.3.6) to the setting in which the number of sets in the cover may be arbitrarily large with respect to the dimension of the sphere.

Theorem 5.3.2 (Generalization of Theorem 2.3.6)**.** *For integers $k \ge n \ge 1$, suppose $A_1, \ldots, A_{k+1}$ is a cover of the sphere S^n by $k+1$ sets such that the first k sets $A_1, \ldots, A_k$ are each open or closed. Furthermore, suppose that any subset of the sphere of diameter at most $s_{n,k}$ is contained in some subset A_i. Then, there is at least one set A_i containing a pair of antipodal points.*

The above theorem generalizes Theorem 2.3.6 because if $n = k$, then the condition that any subset of the sphere of diameter at most $s_{n,k} = 0$ is in some subset A_i simply implies that the sets A_i cover the sphere.

Proof of Theorem 5.3.2. Assume, for the sake of contradiction, that no set in the cover contains antipodal points. Define a continuous map $f\colon S^n \to \mathbb{R}^k_{\ge 0}$ by $f(x) = (d(x, A_1), \ldots, d(x, A_k))$. By the convex Borsuk–Ulam approximation of Theorem 5.2.9, there exists a subset $\{x_1, \ldots, x_m\} \subset S^n$ of diameter at most $s_{n,k}$ and convex coefficients $\{\lambda_i\}_i$ such that

$$y := \sum_{i=1}^{m} \lambda_i f(x_i) = \sum_{i=1}^{m} \lambda_i f(-x_i). \tag{5.1}$$

Since A_{k+1} does not contain antipodal points by assumption, at least one of x_i and $-x_i$ must be contained in some element of $\{A_1, \ldots, A_k\}$ for each $1 \le i \le m$. In fact, we claim that there must exist a single $A_j \in \{A_1, \ldots, A_k\}$ containing all of $\{x_1, \ldots, x_m\}$ or all of $\{-x_1, \ldots, -x_m\}$. Toward proving the claim, note that the points $x_1, \ldots, x_m$ are all contained in some element of the cover because $\operatorname{diam}(\{x_1,\dots,x_m\}) \le s_{n,k}$. Hence, either $\{x_1,\dots,x_m\} \subseteq A_j$ for some $1 \le j \le k$, or $\{x_1,\dots,x_m\} \subseteq A_{k+1}$. In the latter case, because A_{k+1} does not contain antipodal points, we have $\{-x_1,\dots,-x_m\} \subseteq S^n \setminus A_{k+1}$. Then, because $\operatorname{diam}(\{x_1,\dots,x_m\}) = \operatorname{diam}(\{-x_1,\dots,-x_m\})$, it follows that $\{-x_1,\dots,-x_m\} \subseteq A_j$ for some $1 \le j \le k$. This proves the claim.

Now, observe that $d(x_i, A_j) = 0$ for all i or $d(-x_i, A_j) = 0$ for all i. Furthermore, by considering the j^{th} coordinate of y in Equation 5.1 above, it follows that both $d(x_i, A_j) = 0$ and $d(-x_i, A_j) = 0$ for all i. There are two cases:

1. Suppose A_j is closed. In this case, $d(x_i, A_j) = 0$ and $d(-x_i, A_j) = 0$ imply that $\{x_i, -x_i\} \subseteq A_j$ for all i, contradicting the assumption that no set in the cover contains antipodal points.

2. Suppose A_j is open. Note that $d(-x_i, A_j) = 0$ implies $-x_i \in \overline{A_j}$ for all i. In turn, $\overline{A_j}$ is contained in the closed set $S^n \setminus (-A_j) \supseteq A_j$. Hence, each $-x_i$ belongs to $S^n \setminus (-A_j)$, which implies that $x_i \notin A_j$ for all i. Swapping the roles of x_i and $-x_i$, a similar argument shows that $-x_i \notin A_j$ for all i. Finally, this contradicts the fact that A_j contains all of $\{x_1,\dots,x_m\}$ or all of $\{-x_1,\dots,-x_m\}$.

□

5.3.3 Traversals of bounded diameter

Definition 5.3.3. Given a set X and a collection of nonempty subsets $\mathscr{U} = \{U_\alpha \subseteq X \mid \alpha \in A\}$ for some index set A, we say $T \subseteq X$ is a **traversal** of $\mathscr{U}$ if $T \cap U_\alpha \neq \emptyset$ for all $\alpha \in A$.

Observe that any set X is itself a traversal of any collection of nonempty subsets of X. On the other hand, in the case that X is a metric space, we are interested in finding traversals of minimal diameter. In particular, our generalization of the Lyusternik–Schnirel'man–Borsuk covering theorem (Theorem 5.3.2) implies the existence of traversals of bounded diameter for certain subsets of a sphere.

Theorem 5.3.4. *Fix an integer $k \geq 1$. Let $\mathscr{U} = \{U_i \subseteq S^n \mid 1 \leq i \leq k+1\}$ denote a collection of $k+1$ subsets of S^n such that the first k sets are open or closed and such that $S^n \setminus U_i$ does not contain antipodal points for all $1 \leq i \leq k+1$. Then, there is a traversal of $\mathscr{U}$ of diameter at most $s_{n,k}$.*

Proof. In the case that $U_1 \cap \cdots \cap U_{k+1}$ is nonempty, any point of common intersection is a traversal of $\mathscr{U}$ of diameter 0.

Otherwise, suppose $U_1 \cap \cdots \cap U_{k+1}$ is empty. For $1 \leq i \leq k+1$, define $K_i := S^n \setminus U_i$. Observe,

$$K_1 \cup \cdots \cup K_{k+1} = \left(S^n \setminus U_1\right) \cup \cdots \cup \left(S^n \setminus U_{k+1}\right) = S^n \setminus (U_1 \cap \cdots \cap U_{k+1}) = S^n.$$

Hence, $\{K_1, \ldots, K_{k+1}\}$ is a cover of S^n such that no element in the cover contains antipodal points. Furthermore, each K_i is either open or closed. Hence, Theorem 5.3.2 implies that there is a set $X \subseteq S^n$ of diameter at most $s_{n,k}$ that is not contained in any K_i. It follows that for each $1 \leq i \leq k+1$ there must be some $x_i \in X$ such that $x_i \in S^n \setminus K_i = U_i$. Thus, $\{x_1, \ldots, x_{k+1}\}$ is a traversal of $\mathscr{U}$ of diameter at most $s_{n,k}$. □

Theorem 5.3.4 applies, for example, to any collection of closed hemispheres.

Corollary 5.3.5. *Given a collection $\mathscr{H} = \{H_1, \ldots, H_{k+1}\}$ of $k+1$ closed hemispheres of S^n, there is a traversal of $\mathscr{H}$ of diameter at most $s_{n,k}$.*

The following corollary is immediate by considering closed hemispheres centered at the points of X.

Corollary 5.3.6. *Given a set of $k+1$ points $X = \{x_1, \ldots, x_{k+1}\} \subseteq S^n$, there exists a function $\varphi \colon X \to S^n$ such that $\operatorname{diam}(\varphi(X)) \leq s_{n,k}$, and $d(x_i, \varphi(x_i)) \leq \frac{\pi}{2}$ for all $1 \leq i \leq k$.*

Chapter 6

Zeros of trigonometric polynomials

In this chapter, we compile results about the zeros of trigonometric polynomials.

In [50], Gilbert and Smyth establish a sharp upper bound on the length of gaps between roots of a homogeneous trigonometric polynomial. We state this result in Theorem 6.0.2, which is a corollary of [50, Corollary 1].

Theorem 6.0.1 (Corollary 1 of [50])**.** *Let $\mathscr{F}_k$ denote the collection of homogeneous trigonometric polynomials of degree at most k containing only cosine terms. Then, for each $n \geq 1$ and for any $0 < \theta < \frac{\pi k}{k+1}$, there exists a polynomial $p \in \mathscr{F}_k$ of degree k that is positive on the interval $[0,\theta]_{S^1}$. Furthermore, there are no polynomials in $\mathscr{F}_k$ that are positive on $[0,\theta]_{S^1}$ if $\theta \geq \frac{\pi k}{k+1}$.*

Theorem 6.0.2. *Let $[a,b]_{S^1} \subseteq S^1$ denote a closed circular arc of length less than $\frac{2\pi k}{k+1}$. Then, there is a homogeneous trigonometric polynomial of degree k that is positive on $[a,b]_{S^1}$. Moreover, no homogeneous trigonometric polynomial of degree at most k is positive on any subset that contains a closed circular arc of length $\frac{2\pi k}{k+1}$.*

Proof. First, assume $\theta = b-a < \frac{2\pi k}{k+1}$. By Theorem 6.0.1, there exists a homogeneous cosine polynomial p of degree k that is positive on the interval $[0,\theta/2]_{S^1}$. Because p is symmetric about the origin, it is also positive over $[-\theta/2,\theta/2]_{S^1}$. Finally, because the space of homogeneous trigonometric polynomials is invariant under composition with translations, this implies the first part of the theorem.

Next, suppose for the sake of contradiction that a homogeneous trigonometric polynomial g of degree at most k is positive on an arc $[a,b]_{S^1}$ with $\theta = b-a \geq \frac{2\pi k}{k+1}$. By translation invariance, there exists a trigonometric polynomial h of degree at most k that is positive on $[-\theta/2,\theta/2]_{S^1}$. Furthermore, note that $h(-t) > 0$ for $t \in [-\theta/2,\theta/2]_{S^1}$ because this interval is symmetric about 0. In particular, this implies $h(t) + h(-t) > 0$ for $t \in [0,\theta/2]_{S^1}$. Finally, note that $\tilde{h}$ defined by

$\tilde{h}(t) := h(t) + h(-t)$ is a homogeneous cosine polynomial positive on $[0, \theta/2]_{S^1} = [0, \frac{\pi k}{k+1}]_{S^1}$, contradicting the second part of Theorem 6.0.1. $\square$

We establish the following analogous result about the roots of raked homogeneous trigonometric polynomials in [11].

Theorem 6.0.3. *Let $X \subseteq S^1$ be such that* $\operatorname{diam}(X) < \frac{2\pi k}{2k+1}$. *Then there is a raked homogeneous trigonometric polynomial of degree $2k-1$ that is positive on X. Moreover, no raked homogeneous trigonometric polynomial of degree at most $2k-1$ is positive on any subset that contains the vertices of a regular inscribed $(2k+1)$-gon.*

Proof. The first part of this theorem follows from Theorem 4.2.1, which says that $\operatorname{conv}(\mathrm{SM}_{2k}(X))$ does not contain the origin. Hence, there is a separating hyperplane H_z with orthogonal vector $z \in \mathbb{R}^{2k}$ and closed half-space $H_z^+ = \{x \in \mathbb{R}^{2k} \mid z^\intercal x > 0\}$ such that $\mathrm{SM}_{2k}(X) \subseteq H_z^+$. Therefore, the raked homogeneous trigonometric polynomial of degree $2k-1$ given by $p_z(x) := z^\intercal \mathrm{SM}_{2k}(x)$ is positive on all of the points of X.

Next, let $\{t_0, \ldots, t_{2k}\} \subseteq S^1$ denote the vertices of any inscribed regular $(2k+1)$-gon. Given any raked homogeneous trigonometric polynomial p of degree at most $2k-1$, note that we can write $p(t) = z^\intercal \mathrm{SM}_{2k}(t)$ for some $z \in \mathbb{R}^{2k}$ and for all $t \in S^1$. As remarked in the proof of Theorem 4.2.1, we have $\vec{0} = \sum_{i=0}^{2k} \frac{1}{2k+1} \mathrm{SM}_{2k}(t_i)$, and it follows that

$$\sum_{i=0}^{2k} \frac{1}{2k+1} p(t_i) = \sum_{i=0}^{2k} \frac{1}{2k+1} z^\intercal \mathrm{SM}_{2k}(t_i) = z^\intercal \sum_{i=0}^{2k} \frac{1}{2k+1} \mathrm{SM}_{2k}(t_i) = z^\intercal \vec{0} = \vec{0}.$$

Hence, $p(t_i)$ must be non-positive for some $0 \le i \le 2k$. $\square$

Lemma 6.0.4. *Fix a list of odd continuous functions $f_i(t)\colon S^1 \to \mathbb{R}$ for $1 \le i \le 2k+1$. Let P be the set of functions of the form $p\colon S^1 \to \mathbb{R}$ defined by $p(t) = \sum_{j=1}^{2k+1} z_j f_j(t)$ with $z_j \in \mathbb{R}$. Then there is a subset $X \subseteq S^1$ of diameter at most $\frac{2\pi k}{2k+1}$ such that no function in P is strictly positive on X.*

Proof. Consider the odd map $f\colon S^1 \to \mathbb{R}^{2k+1}$ given by $f(t) = (f_1(t), \ldots, f_{2k+1}(t))$. Note that each function $p \in P$ is specified by a coefficient vector $z \in R^{2k+1}$, in the sense that $p(t) = z^\intercal f(t)$

for all $t \in S^1$. By Theorem 5.1.3, there exists a subset $X \subseteq S^1$ of diameter at most $\frac{2\pi k}{2k+1}$ such that $\operatorname{conv}(f(X))$ contains the origin. Hence, if we write $X = \{x_1, \ldots, x_m\}$ with $\sum_{i=1}^m \lambda_i f(x_i) = \vec{0}$ for some convex coefficients $\lambda_i \geq 0$, then $\sum_{i=1}^m \lambda_i p(x_i) = \sum_{i=1}^m \lambda_i z^\intercal f(x_i) = z^\intercal \sum_{i=1}^m \lambda_i f(x_i) = z^\intercal \vec{0} = 0$. In particular, $p(x_i)$ must be non-positive for at least some i. □

The next corollary follows immediately from Corollary 6.0.4.

Corollary 6.0.5. *Fix a list of odd degrees d_i for $1 \leq i \leq 2k+1$, and fix a list of trigonometric functions $f_i(t) = \sin(t)$ or $f_i(t) = \cos(t)$. Let P be the set of all polynomials of the form $p(t) = \sum_{j=1}^{2k+1} z_j f_j(d_j t)$ with $z_j \in \mathbb{R}$. Then there is a subset $X \subseteq S^1$ of diameter at most $\frac{2\pi k}{2k+1}$ such that no polynomial in P is positive on X.*

For example, the above corollary applies if P is the set of all raked homogeneous trigonometric polynomials of degree at most $2k-1$, namely

$$p(t) = \sum_{j=1}^{k} a_j \cos(2j-1)t + \sum_{j=1}^{k} b_j \sin(2j-1)t,$$

after noting that we are considering the special case in which one of the constants z_j defining $p(t) = \sum_{j=1}^{2k+1} z_j f_j(d_j t)$ is zero.

In Subsection 3.2.4, Lemma 3.2.10 was used to separate cones over points along the symmetric moment curve. For convenience, we restate the lemma here before giving the proof.

Lemma 3.2.10. Fix an integer $k > 0$ and distinct $v_1, \ldots, v_{2k-1} \in S^1$ with no two points antipodal. Let $u_1, \ldots, u_{4k-2}$ denote the set of points $\{v_1, \ldots, v_{2k-1}\} \cup \{v_1 + \pi, \ldots, v_{2k-1} + \pi\}$ labeled in counterclockwise order such that $u_1 = v_1$. Then, there exists a raked homogeneous trigonometric polynomial f of degree $2k-1$ such that $f(u_i) = 0$ for $1 \leq i \leq 4k-2$. Further, $\operatorname{sign}(f(t)) = (-1)^i$ for $t \in (u_i, u_{i+1})_{S^1}$, where we define $u_{4k-1} = u_1$.

Proof. Let $H_z = \{x \in \mathbb{R}^{2k} \mid z^\intercal x = 0\}$ denote a hyperplane passing through the origin and each of $\mathrm{SM}_{2k}(v_i)$ for $1 \leq i \leq 2k-1$. Note that $f(t) := z^\intercal \mathrm{SM}_{2k}(t)$ is a raked homogeneous trigonometric polynomial of degree at most $2k-1$ with a root at each v_i. Let m_i denote the multiplicity of the root v_i. On the other hand, because SM_{2k} is centrally symmetric, f also has a root at each $v_i+\pi$. Let n_i denote the multiplicity of the root $v_i+\pi$. By [22, Theorem 3.1.2], $\sum_{i=1}^{2k-1} m_i + \sum_{i=1}^{2k-1} n_i \leq 4k-2$. Hence, because $\{v_1,\dots,v_{2k-1}\}\cup\{v_1+\pi,\dots,v_{2k-1}+\pi\}$ contains $4k-2$ distinct points, each root v_i and $v_i+\pi$ must have multiplicity 1 and these must be the only roots of f. □

In analogy with the construction of the polynomials generating the Carathéodory coorbitope cone in Section 4.3, we explicitly construct the polynomials satisfying the conditions of Lemma 3.2.10.

Lemma 6.0.6. *Fix an integer $k \geq 0$ and distinct $v_1,\dots,v_{2k-1}$ with no two points antipodal. Then, the expression*

$$f(t) = \prod_{1\leq l\leq 2k-1} \sin(v_l - t)$$

is a degree $2k-1$ raked homogeneous trigonometric polynomial that satisfies the conclusion of Lemma 3.2.10.

Proof. For $t\in S^1$, consider points $\mathrm{SM}_{2k}(t)\in\mathbb{R}^{2k}$ to be written as column vectors and define the $2k\times 2k$ matrix

$$N(t) := \begin{pmatrix} \mathrm{SM}_{2k}(t) & \mathrm{SM}_{2k}(v_1) & \mathrm{SM}_{2k}(v_2) & \cdots & \mathrm{SM}_{2k}(v_{2k-2}) & \mathrm{SM}_{2k}(v_{2k-1}) \end{pmatrix}.$$

By Lemma 4.2.2,

$$\det(N(t)) = \kappa\left(\prod_{1\leq j<l\leq 2k-1} \sin(v_l - v_j)\right)\left(\prod_{1\leq l\leq 2k-1} \sin(v_l - t)\right),$$

where κ is a nonzero constant that depends only on k. Further, by considering the cofactor expansion of this determinant along the first column of $N(t)$, observe that $\det(N(t))$ is a raked homogeneous trigonometric polynomial of degree $2k-1$. Further, since $\kappa\prod_{1\leq j<l\leq 2k-1}\sin(v_l - v_j)$ is a constant, observe that

$$f(t) := \prod_{1\leq l\leq 2k-1} \sin(v_l - t)$$

is itself a raked homogeneous trigonometric polynomial of degree $2k-1$. Note that f has distinct real roots $\{v_1, \ldots, v_{2k-1}\} \cup \{v_1 + \pi, \ldots, v_{2k-1} + \pi\}$ because no two elements of $\{v_1, \ldots, v_{2k-1}\}$ are equal or antipodal. Finally, observe for $t \notin \{v_1, \ldots, v_{2k-1}\} \cup \{v_1 + \pi, \ldots, v_{2k-1} + \pi\}$ that

$$\operatorname{sign}\big(f(t)\big) = \operatorname{sign}\left(\prod_{1 \le l \le 2k-1} \sin(v_l - t)\right) = (-1)^{\rho(t)},$$

where we define $\rho(t) := |\{v_l \mid v_l \in (t + \pi, t)_{S^1}, 1 \le l \le 2k-1\}|$. □

Chapter 7

Conclusion

In this thesis, we study simplicial metric thickenings from the perspectives of topology, geometry, and combinatorics. We give a geometric proof of the homotopy types of certain Čech and Vietoris–Rips metric thickenings of the circle by constructing deformation retractions of these spaces onto the boundaries of Carathéodory and Barvinok–Novik orbitopes, respectively. We use algebraic and combinatorial arguments to establish a sharp lower bound on the diameter of Carathéodory subsets of the centrally-symmetric version of the trigonometric moment curve and we give an explicit description of the vectors generating the Carathéodory coorbitope cone. We use topological information about metric thickenings to generalize the Borsuk–Ulam theorem and a selection of its corollaries. Finally, inspired by results about the zeros of trigonometric polynomials, we prove a centrally-symmetric analog of a result of Gilbert and Smyth [50] about the possible size of gaps between zeros of homogeneous trigonometric polynomials.

Many interesting questions about metric thickenings, convex bodies, and convex Borsuk–Ulam approximations remain unanswered. For example, in all known cases for which the homotopy types of both the geometric realization and the metric thickening of an open Čech or Vietoris–Rips complex are known, they are the same. In the case of the circle, an affirmative answer to Question 3.2.5 would prove that these geometric realizations and metric thickenings are homotopy equivalent at all scales. On the other hand, a negative answer to Question 3.2.5 would not be sufficient to disprove that these spaces are homotopy equivalent; it would only imply that a more complicated (that is, non-linear) homotopy would be required to establish $\iota \circ (p \circ \mathrm{M}_{2k}) \simeq \mathrm{id}_{\check{\mathrm{C}}^{\mathrm{m}}(S^1;r)}$ and $\iota \circ (p \circ \mathrm{SM}_{2k}) \simeq \mathrm{id}_{\mathrm{VR}^{\mathrm{m}}(S^1;r)}$ at higher scales.

The interesting relationships between (symmetric) moment curves and Čech and Vietoris–Rips thickenings of the circle suggest that similar maps from more general manifolds into Euclidean space may help reveal the topology of the analogous metric thickenings defined on these manifolds. For example, the homotopy type of $\mathrm{VR}^{\mathrm{m}}_{\le}(S^2;r)$ changes precisely at the scale

$r = \Delta_2$, and the inclusion $S^2 \hookrightarrow \mathbb{R}^3$ is an odd map satisfying the property that all Carathéodory subsets in the image of this map have diameter at least Δ_2 (as measured in the domain S^2). Supposing the homotopy type of $\mathrm{VR}^{\mathrm{m}}_{\leq}(S^2; r)$ changes again at some scale $r' > r$, does there exist an odd map $S^2 \to \mathbb{R}^k$, possibly generalizing the symmetric moment curve in some way, such that all Carathéodory subsets in the image of this map have diameter at least r'?

Along these lines, we also note that the spherical Carathéodory diameters $s_{1,2k} = s_{1,2k+1}$ correspond precisely to the diameters at which the homotopy type of $\mathrm{VR}^{\mathrm{m}}_{\leq}(S^1; r)$ changes. Is this true in general for higher spheres? If so, this fact would establish a strong relationship between metric thickenings of spheres and convex Borsuk–Ulam approximations.

Our explicit description of the trigonometric polynomials defining faces of Carathéodory orbitopes arising as factors of the determinants $\det\left(\mathbb{M}^1_{2k}\right)$ suggests that a better understanding of the determinants $\det\left(\mathbb{SM}^1_{2k}\right)$ may yield more information about the faces of Barvinok–Novik orbitopes. For example, an affirmative answer to Conjecture 4.4.3 would imply that a set $\{s_0, \ldots, s_{2k}\}$ defines a face of a Barvinok–Novik orbitope if and only if the corresponding trigonometric polynomial g_s for $s = (s_0, \ldots, s_{2k})$, as defined in Section 4.4, is non-negative over S^1.

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Appendix A

Proof of Proposition 4.3.1

Proof of Proposition 4.3.1. For an integer $k \geq 1$, let $\vec{t} = (t_0, \ldots, t_{2k}) \in \mathbb{R}^{2k+1}$. We must show that

$$\det(\mathbb{M}^1_{2k}(\vec{t})) = \kappa \prod_{0 \leq j < l \leq 2k} \sin\left(\frac{t_l - t_j}{2}\right)$$

for some nonzero constant κ depending on k. To ease notation, define $N := \mathbb{M}^1_{2k}(\vec{t})$. We will perform elementary row and column operations on N to obtain a Vandermonde matrix. Given a function $f : \mathbb{R} \to \mathbb{C}$ and $\underline{t} = (t_0, \ldots, t_{2k}) \in \mathbb{R}^{2k+1}$, let us write

$$f(\underline{t}) := (f(t_0), f(t_1), f(t_2), \ldots, f(t_{2k}))^\intercal.$$

Let $\underline{1} \in \mathbb{R}^{2k+1}$ denotes the vector of all 1's. Since

$$N = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \cos(t_0) & \cos(t_1) & \ldots & \cos(t_{2k}) \\ \sin(t_0) & \sin(t_1) & \ldots & \sin(t_{2k}) \\ \cos(2t_0) & \cos(2t_1) & \ldots & \cos(2t_{2k}) \\ \sin(2t_0) & \sin(2t_1) & \ldots & \sin(2t_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(kt_0) & \cos(kt_1) & \ldots & \cos(kt_{2k}) \\ \sin(kt_0) & \sin(kt_1) & \ldots & \sin(kt_{2k}) \end{pmatrix},$$

we have

$$\begin{aligned}
\det(N) = \det\left(N^{\mathsf{T}}\right) &= \det\begin{pmatrix} \underline{1} & \cos(\underline{t}) & \sin(\underline{t}) & \cos(2\underline{t}) & \sin(2\underline{t}) & \cdots & \cos(k\underline{t}) & \sin(k\underline{t}) \end{pmatrix} \\
&= \det\begin{pmatrix} \underline{1} & \frac{e^{i\underline{t}}+e^{-i\underline{t}}}{2} & \frac{e^{i\underline{t}}-e^{-i\underline{t}}}{2i} & \frac{e^{2i\underline{t}}+e^{-2i\underline{t}}}{2} & \frac{e^{2i\underline{t}}-e^{-2i\underline{t}}}{2i} & \dots & \frac{e^{ki\underline{t}}+e^{-ki\underline{t}}}{2} & \frac{e^{ki\underline{t}}-e^{-ki\underline{t}}}{2i} \end{pmatrix} \\
&= \frac{1}{2^{2k}}(-i)^k \det\begin{pmatrix} \underline{1} & e^{i\underline{t}}+e^{-i\underline{t}} & e^{i\underline{t}}-e^{-i\underline{t}} & \cdots & e^{ki\underline{t}}+e^{-ki\underline{t}} & e^{ki\underline{t}}-e^{-ki\underline{t}} \end{pmatrix}.
\end{aligned}$$

Next, let C_j denote the j-th column of the above matrix. For $j = 2, 4, \dots, 2k$, perform the column operations $C_j \mapsto C_j + C_{j+1}$, and then after each C_j has been updated, perform the column operations $C_{j+1} \mapsto C_{j+1} - \frac{1}{2}C_j$. It follows that

$$\begin{aligned}
\det(N) &= \frac{1}{2^{2k}}(-i)^k \det\begin{pmatrix} \underline{1} & 2e^{i\underline{t}} & -e^{-i\underline{t}} & 2e^{2i\underline{t}} & -e^{-2i\underline{t}} & \cdots & 2e^{ki\underline{t}} & -e^{-ki\underline{t}} \end{pmatrix} \\
&= \frac{i^k}{2^k} \det\begin{pmatrix} \underline{1} & e^{i\underline{t}} & e^{-i\underline{t}} & e^{2i\underline{t}} & e^{-2i\underline{t}} & \cdots & e^{ki\underline{t}} & e^{-ki\underline{t}} \end{pmatrix}
\end{aligned}$$

by factoring out column multiples. Defining $\omega := e^{-ki(t_0+t_1+\cdots+t_{2k})}$, we may factor e^{-kit_j} from row j to obtain

$$\begin{aligned}
\det(N) &= \frac{i^k}{2^k}\omega \det\begin{pmatrix} e^{(0+k)i\underline{t}} & e^{(1+k)i\underline{t}} & e^{(-1+k)i\underline{t}} \dots & e^{(k+k)i\underline{t}} & e^{(-k+k)i\underline{t}} \end{pmatrix} \\
&= \frac{i^k}{2^k}\omega \det\begin{pmatrix} e^{ki\underline{t}} & e^{(k+1)i\underline{t}} & e^{(k-1)i\underline{t}} & \dots & e^{2ki\underline{t}} & \underline{1} \end{pmatrix},
\end{aligned}$$

After re-ordering rows by a permutation σ and taking the determinant of the resulting Vandermonde matrix, we have

$$\begin{aligned}
\det(N) &= \operatorname{sign}(\sigma)\frac{i^k}{2^k}\omega \det\begin{pmatrix} \underline{1} & e^{i\underline{t}} & e^{2i\underline{t}} & e^{3i\underline{t}} & \dots & e^{2ki\underline{t}} \end{pmatrix} \\
&= \operatorname{sign}(\sigma)\frac{i^k}{2^k}\omega \prod_{0\le j<l\le 2k}\left(e^{it_l} - e^{it_j}\right).
\end{aligned}$$

Finally, note $\omega = \prod_{0\le j<l\le 2k} e^{-i(t_l+t_j)/2}$ and multiply each term $\left(e^{it_l} - e^{it_j}\right)$ above by the factor $e^{-i(t_l+t_j)/2}$ extracted from ω to obtain

$$\begin{aligned}\det(N) &= \operatorname{sign}(\sigma)\frac{i^k}{2^k}\prod_{0\le j<l\le 2k}\left(e^{i(t_l-t_j)/2} - e^{-i(t_l-t_j)/2}\right)\\ &= \operatorname{sign}(\sigma)\frac{i^k}{2^k}\prod_{0\le j<l\le 2k} 2i\sin\left(\frac{t_l-t_j}{2}\right) = \kappa\prod_{0\le j<l\le 2k}\sin\left(\frac{t_l-t_j}{2}\right)\end{aligned}$$

where $\kappa = \operatorname{sign}(\sigma)\frac{i^k}{2^k}(2i)^{2k^2+k} = \operatorname{sign}(\sigma)\, i^{2(k^2+k)}2^{2k^2} = \operatorname{sign}(\sigma)2^{2k^2}$.

$\square$