Metric Thickenings of Euclidean Submanifolds

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Masters Thesis Defense – October 3, 2017

Introduction

Motivation

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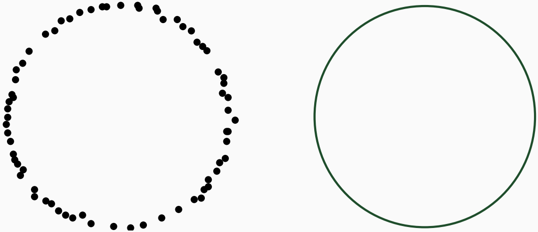

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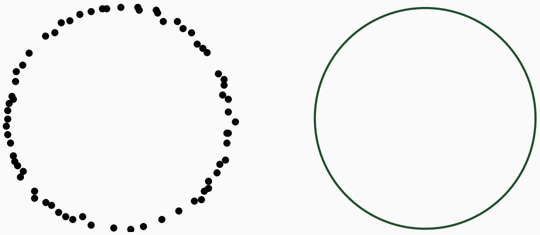

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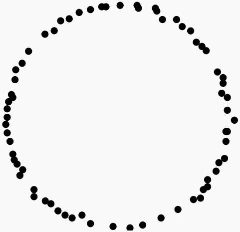

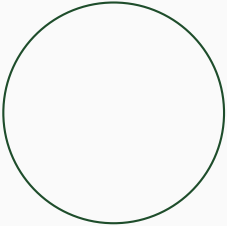

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A reconstruction method should work given a perfect sample.

Background

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For example:

$$V = \{a, b, c, d, e\}$$

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Simplicial Complexes

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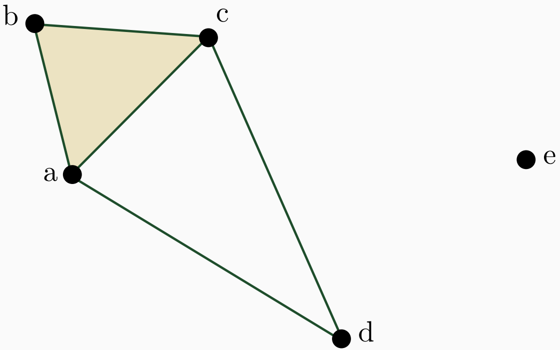

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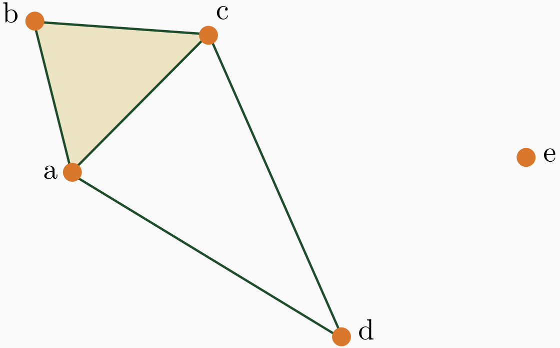

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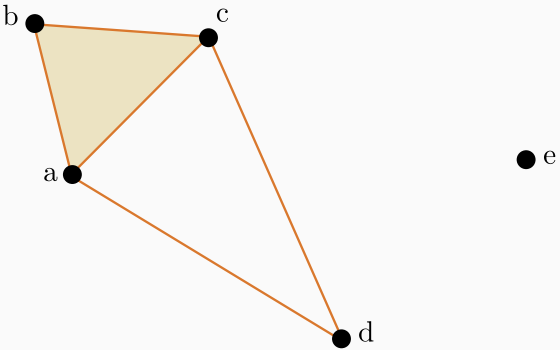

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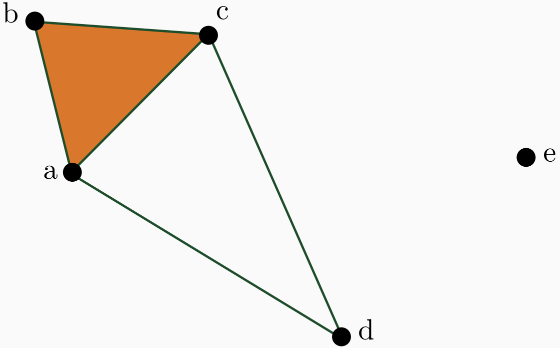

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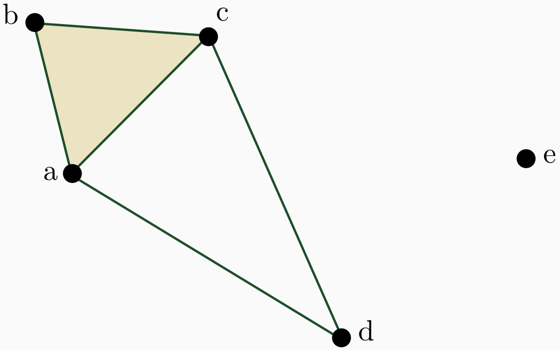
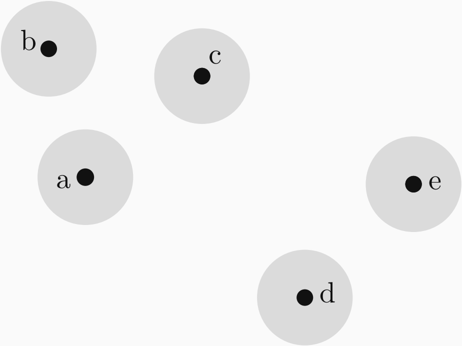


The topology on a finite simplicial complex is the subspace topology of its geometric realization in $\mathbb{R}^n$.

The Vietoris–Rips Complex

Definition

Let X be a metric space and $r > 0$ a scale parameter. The Vietoris–Rips complex, $\mathrm{VR}(X; r)$, of X, has vertex set X and a simplex for every finite subset $\sigma \subseteq X$ such that $\mathrm{diam}(\sigma) \leq r$.



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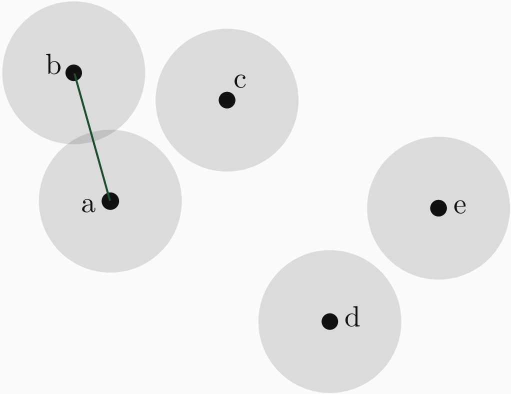

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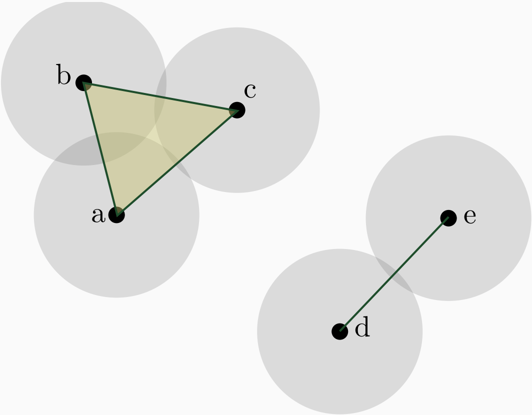

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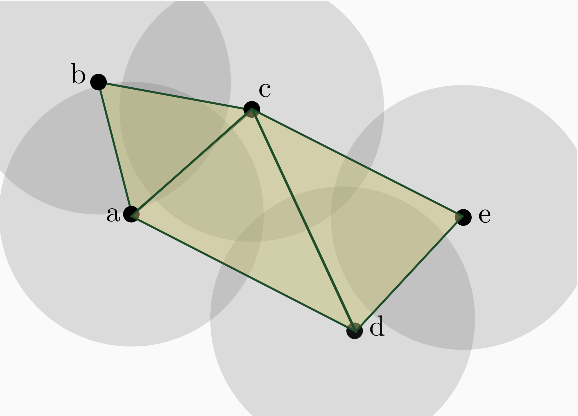

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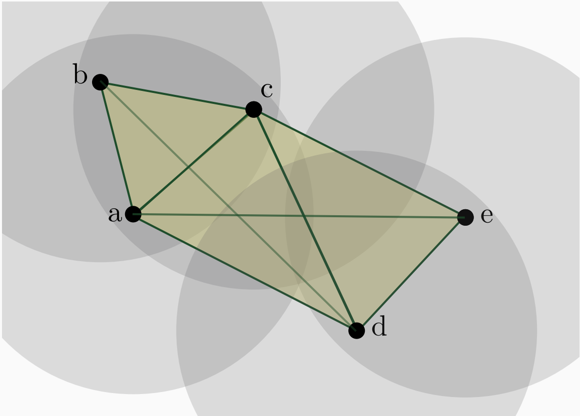


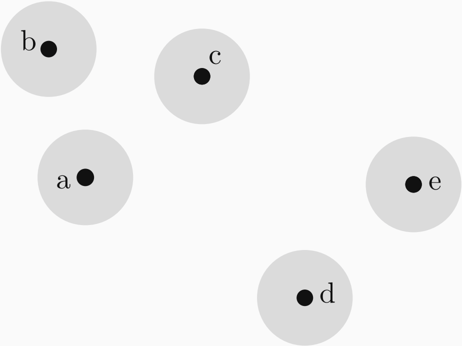
Definition

Let $X \subseteq Y$ be a submetric space and $r > 0$ a scale parameter. The Čech complex $\check{\mathrm{C}}(X, Y; r)$, of X, has vertex set X and a simplex for every finite subset $\sigma \subseteq X$ such that

$$\bigcap_{x_i \in \sigma} \overline{\mathrm{B}}(x_i, r/2) \neq \emptyset$$

.

The Čech Complex



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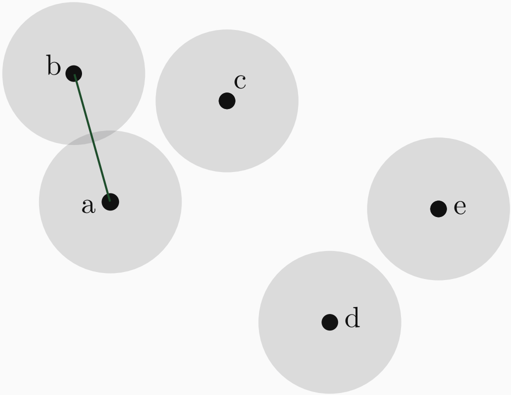

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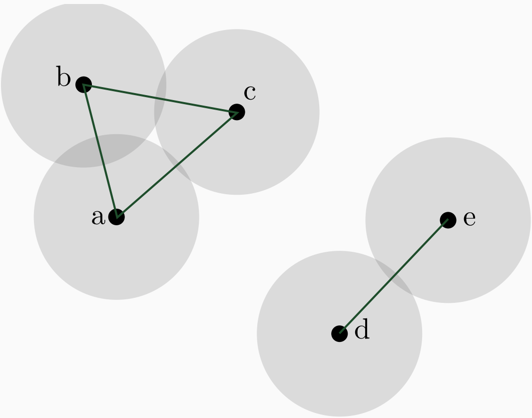

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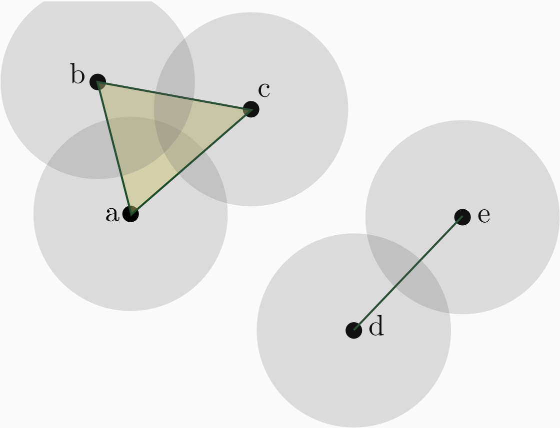

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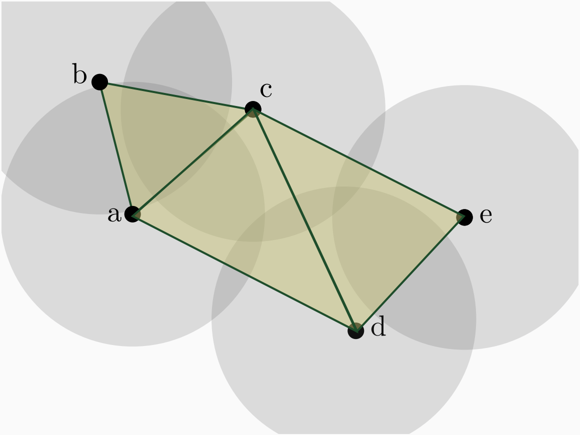

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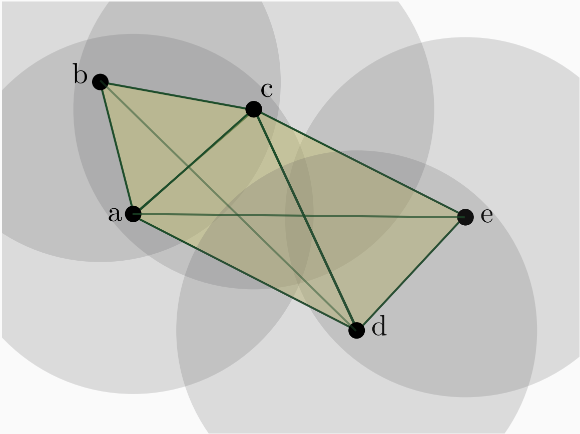

Homotopy Equivalence

Definition

Let $f\colon X \to Y$ and $g\colon X \to Y$ be continuous maps. Then f is homotopic to g, denoted $f \simeq g$, if there exists a continuous function $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$, $H(x,1) = g(x)$.

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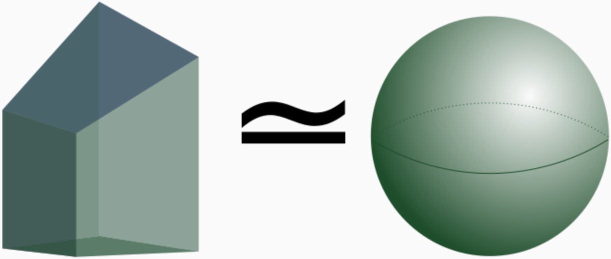
Definition

Let X and Y be topological spaces. Then X is homotopy equivalent to Y, written $X \simeq Y$, if there exists a pair of continuous functions $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

Homotopy equivalence permits "stretching and bending" in a way that allows the dimension to change:



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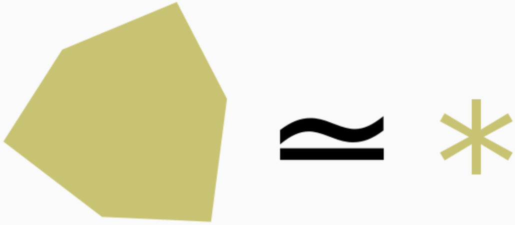

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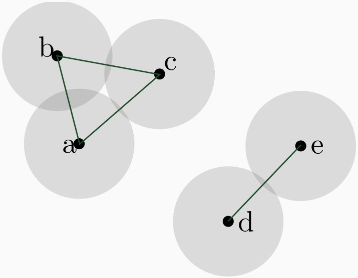
Lemma (Nerve Lemma: Convex Version)

Let U_α for $\alpha \in A$ an index set be convex subsets of $\mathbb{R}^n$. Then $\mathcal{N}(\{U_\alpha\}) \simeq \cup_{\alpha\in A} U_\alpha$.



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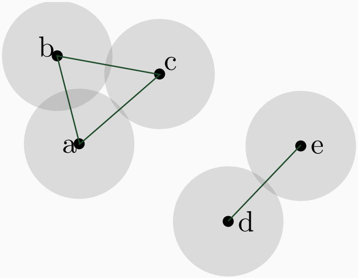

The Čech complex is the nerve of balls of radius $r/2$, so it is homotopy equivalent to the underlying space for a good cover.

Hausmann's Theorem

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  - T depends upon a total order of the points in M.
  - In particular, the inclusion $\iota\colon M \hookrightarrow \mathrm{VR}(M; r)$ does not provide the inverse (in fact, ι is not even continuous.)

Metric Thickenings

Definition (Adamaszek, Adams, Frick)

For a metric space X and $r \geq 0$, the Vietoris–Rips thickening $\mathrm{VR}^m(X;r)$ is the set

$$\mathrm{VR}^m(X;r) = \left\{ \sum_{i=0}^{k} \lambda_i x_i \mid k \in \mathbb{N},\ x_i \in X, \text{ and } \mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r \right\},$$

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- The same construction holds for the Čech thickening, $\check{\mathrm{C}}^{\mathrm{m}}(X;r)$.

Wasserstein Metric

Let $x, x' \in \mathrm{VR}^m(X; r)$ with $x = \sum_{i=0}^{k} \lambda_i x_i$ and $x' = \sum_{i=0}^{k'} \lambda'_i x'_i$. Define a matching p between x and x' to be any collection of non-negative real numbers $\{p_{i,j}\}$ such that $\sum_{j=0}^{k'} p_{i,j} = \lambda_i$ and $\sum_{i=0}^{k} p_{i,j} = \lambda'_j$. Define the cost of the matching p to be $\mathrm{cost}(p) = \sum_{i,j} p_{i,j} d(x_i, x'_j)$.

Definition

The 1-Wasserstein metric on $\mathrm{VR}^m(X; r)$ is the distance d_W defined by

$$d_W(x, x') = \inf \left\{ \mathrm{cost}(p) \;\middle|\; p \text{ is a matching between } x \text{ and } x' \right\}.$$

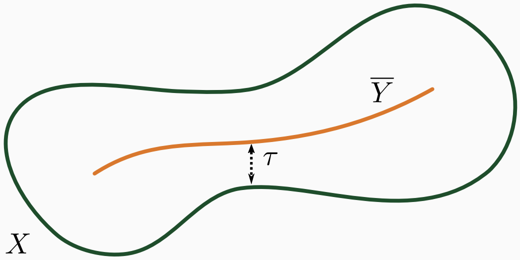
Euclidean Submanifolds

Sets of Positive Reach

The medial axis of $X \subseteq \mathbb{R}^n$ is the closure, $\overline{Y}$, of

$$Y = \{y \in \mathbb{R}^n \mid \exists\, x_1 \neq x_2 \in M \text{ with } d(y, x_1) = d(y, x_2) = d(y, X)\}\,.$$

The reach, τ, of X is the minimal distance $\tau = d(X, \overline{Y})$ between X and its medial axis.



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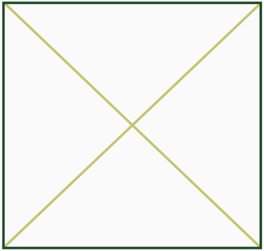

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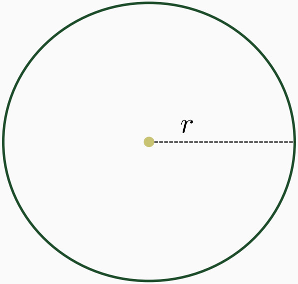
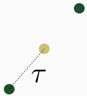
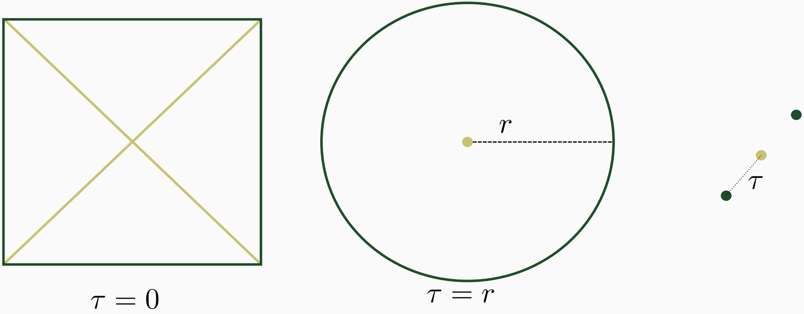


$\tau = r$



- Sets with “corners” have zero reach.

Sets of Positive Reach



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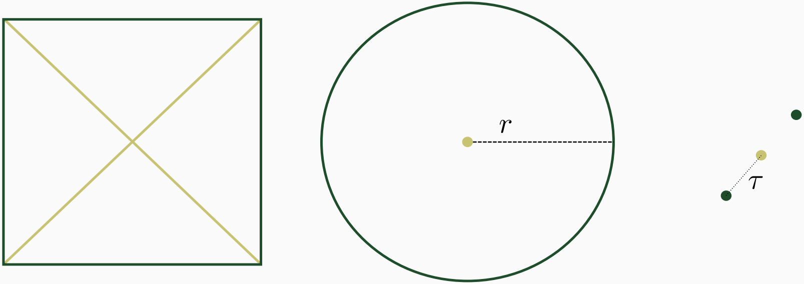


$\tau = 0$ $\tau = r$

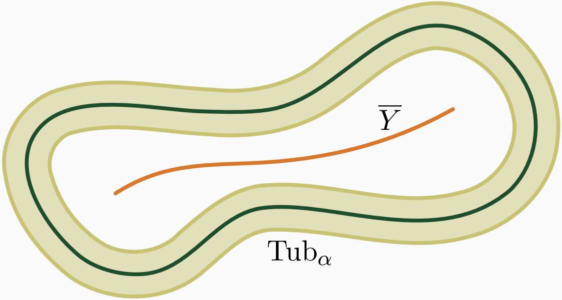
- Sets with "corners" have zero reach.
- Smooth manifolds embedded in $\mathbb{R}^n$ have positive reach.
- Reach is $\leq$ half the distance between non-connected components.

Nearest Point Projection

Define the α-offset of $X \subseteq \mathbb{R}^n$:

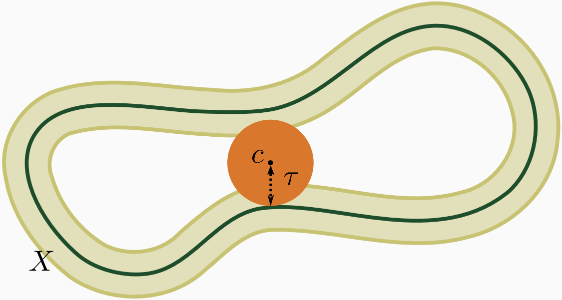
$$\mathrm{Tub}_\alpha = \{x \in \mathbb{R}^n \mid d(x, X) < \alpha\} = \bigcup_{x \in X} \mathrm{B}(x, \alpha).$$

If X has reach τ, then $\pi \colon \mathrm{Tub}_\tau \to X$ where x maps to its nearest point in X is well-defined and continuous [4].



Proposition (Niyogi, Smale, Weinberger)

Let $X \subseteq \mathbb{R}^n$ *have reach* $\tau > 0$*. Let* $p \in X$ *and suppose* $x \in \mathrm{Tub}_\tau \setminus X$ *satisfies* $\pi(x) = p$*. If* $c = p + \tau \frac{x-p}{\|x-p\|}$*, then* $\mathrm{B}(c, \tau) \cap X = \emptyset$.



Results

Main Theorem

Theorem (Metric Hausmann)

Let $X \subseteq \mathbb{R}^n$ and suppose the reach τ of X is positive. Then for all $r < \tau$, the metric Vietoris–Rips thickening $\mathrm{VR}^m(X; r)$ is homotopy equivalent to X.

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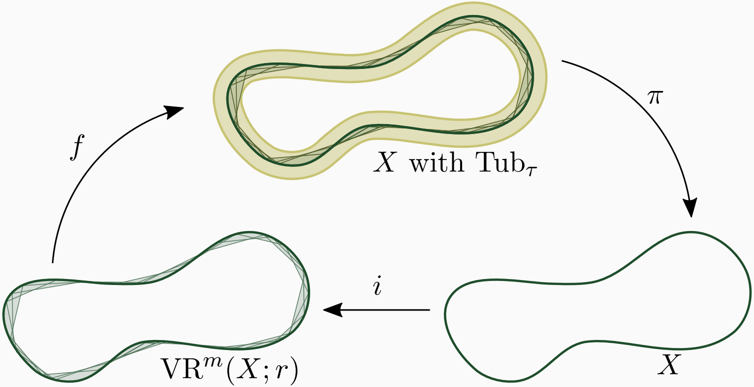
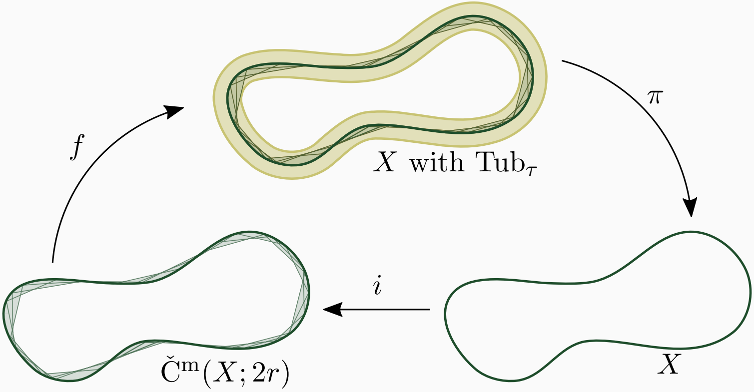


Theorem (Metric Nerve Theorem)

Let X be a subset of Euclidean space $\mathbb{R}^n$, equipped with the Euclidean metric, and suppose the reach τ of X is positive. Then for all $r < \tau$, the metric Čech thickening $\check{\mathrm{C}}^{\mathrm{m}}(X; 2r)$ is homotopy equivalent to X.



Lemmas

Lemma

For $X \subseteq \mathbb{R}^n$ *and* $r > 0$*, the linear projection map* $f \colon \mathrm{VR}^m(X;r) \to \mathbb{R}^n$ *defined by*

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has its image contained in $\overline{\mathrm{Tub}_r}$.

Proof.

Let $x = \sum_{i=0}^k \lambda_i x_i \in \mathrm{VR}^m(X; r)$; we have

$$\mathrm{diam}(\mathrm{conv}\{x_0, \ldots, x_k\}) = \mathrm{diam}([x_0, \ldots, x_k]) \leq r.$$

Since $f(x) \in \mathrm{conv}\{x_0, \ldots, x_k\}$, it follows that $d(f(x), X) \leq d(f(x), x_0) \leq r$, and so $f(x) \in \overline{\mathrm{Tub}_r}$. $\square$

Lemma

Let $x_0, \ldots, x_k \in \mathbb{R}^n$*, let* $y \in \operatorname{conv}\{x_0, \ldots, x_k\}$*, and let* C *be a convex set with* $y \notin C$*. Then there is at least one* x_i *with* $x_i \notin C$*.*

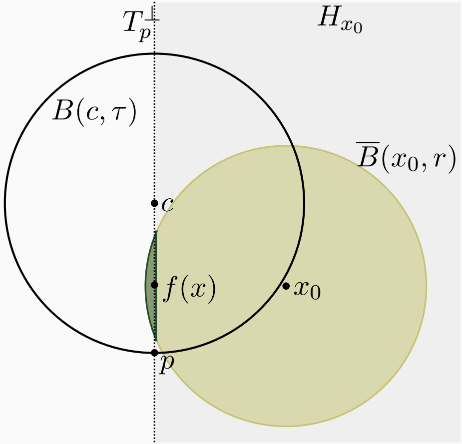
Proof.

Suppose for a contradiction that we had $x_i \in C$ for all $i = 0, \ldots, k$. Then since C is convex, we'd also have $y \in \operatorname{conv}\{x_0, \ldots, x_k\} \subseteq C$. Hence it must be the case that $x_i \notin C$ for some i. □

Lemma

Let $X \subseteq \mathbb{R}^n$ have positive reach τ, let $[x_0, \dots x_k]$ be a simplex in $\mathrm{VR}(X; r)$ with $r < \tau$, let $x = \sum \lambda_i x_i \in \mathrm{VR}^m(X; r)$, and let $p = \pi(f(x))$. Then the simplex $[x_0, \dots, x_k, p]$ is in $\mathrm{VR}(X; r)$.

Proof.



Main Result

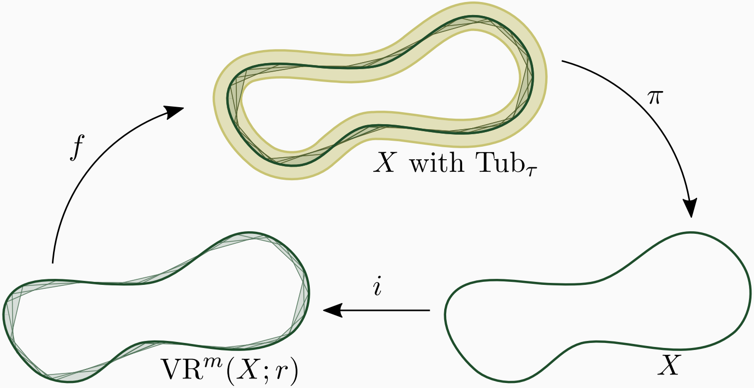
We are now prepared to prove our main result.

Theorem

Let X be a subset of Euclidean space $\mathbb{R}^n$, equipped with the Euclidean metric, and suppose the reach τ of X is positive. Then for all $r < \tau$, the metric Vietoris–Rips thickening $\mathrm{VR}^m(X; r)$ is homotopy equivalent to X.

Proof.

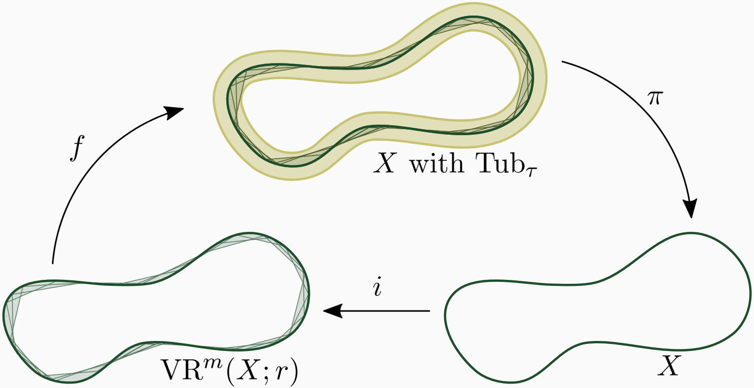
By [1, Lemma 5.2], map $f\colon \mathrm{VR}^m(X;r) \to \mathbb{R}^n$ is 1-Lipschitz and hence continuous. It follows from a previous lemma that the image of f is a subset of Tub_τ. Let $i\colon X \to \mathrm{VR}^m(X;r)$ be the inclusion map. Note that $\pi \circ f \circ i = \mathrm{id}_X$.



□

Proof.

Consider $H\colon \mathrm{VR}^m(X;r) \times I \to \mathrm{VR}^m(X;r)$ defined by $H(x,t) = t \cdot \mathrm{id}_{\mathrm{VR}^m(X;r)} + (1-t) i \circ \pi \circ f$. H is well-defined by the final lemma, and continuous by [1, Lemma 3.8].It follows that H is a homotopy equivalence from $i \circ \pi \circ f$ to $id_{\mathrm{VR}^m(X;r)}$.



□

Čech Result

Theorem

Let X be a subset of Euclidean space $\mathbb{R}^n$, equipped with the Euclidean metric, and suppose the reach τ of X is positive. Then for all $r < \tau$, the metric Čech thickening $\check{\mathrm{C}}^{\mathrm{m}}(X; 2r)$ is homotopy equivalent to X.

Proof.
The proof uses similar techniques to that of the Metric Hausmann's Theorem. □

Conclusion

Conclusions

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- For a Riemannian version see [1]. Or:

Corollary

If N is a smooth, compact, Riemannian manifold, there exists a $\tau > 0$ such that $\mathrm{VR}^m(N; r) \simeq N$ for all $0 < r < \tau$.

Proof.

This follows from the Nash Embedding theorem [8]. □

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  - ◇ Stability under persistent homology [2]?

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