THESIS

METRIC THICKENINGS OF EUCLIDEAN SUBMANIFOLDS

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ABSTRACT

METRIC THICKENINGS OF EUCLIDEAN SUBMANIFOLDS

Given a sample X from an unknown manifold M embedded in Euclidean space, it is possible to recover the homology groups of M by building a Vietoris–Rips or Čech simplicial complex on top of the vertex set X. However, these simplicial complexes need not inherit the metric structure of the manifold. Indeed, a simplicial complex is not even metrizable if it is not locally finite. We instead consider metric thickenings of X, called the *Vietoris–Rips* and *Čech thickenings*, which are equipped with the 1-Wasserstein metric in place of the simplicial complex topology. We show that for Euclidean subsets M with positive reach, the thickenings satisfy metric analogues of Hausmann's theorem and the nerve lemma (the metric Vietoris–Rips and Čech thickenings of M are homotopy equivalent to M for scale parameters less than the reach). In contrast to Hausmann's original result, our homotopy equivalence is a deformation retraction, is realized by canonical maps in both directions, and furthermore can be proven to be a homotopy equivalence via simple linear homotopies from the map compositions to the corresponding identity maps.

ACKNOWLEDGEMENTS

DEDICATION

I would like to dedicate this work to…

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Chapter 1

Introduction

The Vietoris–Rips simplicial complex $\mathrm{VR}(X;r)$ of a metric space X at scale parameter $r>0$ has X as its vertex set, and a simplex σ for every finite set of points of diameter less than r. Vietoris–Rips complexes are a natural way to enlarge a metric space. Indeed, Hausmann proves in [16] that given a compact Riemannian manifold M and a sufficiently small scale parameter r, the Vietoris–Rips complex $\mathrm{VR}(M;r)$ is homotopy equivalent to M. In response to a question in Hausmann's paper, Latschev [18] proves furthermore that if $X\subseteq M$ is a sufficiently dense sample, then $\mathrm{VR}(X;r)$ is also homotopy equivalent to M. Latschev's result is a precursor to many theoretical guarantees [4, 5, 7, 8, 21] showing how Vietoris–Rips complexes and related constructions can recover topological information about a shape M from a sufficiently dense sampling X. In applications of topology to data analysis [6, 10] datasets will typically be finite, but nevertheless infinite Vietoris–Rips constructions are important for applications in part because if a dataset X converges to an infinite shape M in the Gromov–Hausdorff distance, then the persistent homology of $\mathrm{VR}(X;r)$ converges to that of the infinite object $\mathrm{VR}(M;r)$ [7].

Despite theoretical guarantees such as Latschev's theorem, the simplicial complex $\mathrm{VR}(X;r)$ does not retain the metric properties of X. In fact, a simplicial complex is metrizable if and only it is locally finite, which $\mathrm{VR}(X;r)$ need not be when X is infinite. Furthermore, if X is not discrete then the natural inclusion map $X\hookrightarrow\mathrm{VR}(X;r)$ is not continuous for any $r>0$. The Vietoris–Rips thickening of X, denoted $\mathrm{VR}^m(X;r)$ and introduced in [2], addresses each of these issues. As a set, $\mathrm{VR}^m(X;r)$ is naturally identified with the geometric realization of $\mathrm{VR}(X;r)$, but it has a completely different topology: the 1-Wasserstein metric. The space $\mathrm{VR}^m(X;r)$ is a *metric thickening* of X, meaning that it is a metric space extending the metric on X. As a result, the inclusion $X\hookrightarrow\mathrm{VR}^m(X;r)$ is continuous for all metric spaces X and scale parameters r. In general, the simplicial complex $\mathrm{VR}(X;r)$ and metric thickening $\mathrm{VR}^m(X;r)$ are neither homeomorphic nor homotopy equivalent, and we argue that the metric thickening is often a more natural object.

In particular, let M be a compact Riemannian manifold. If M is of dimension at least one, then the inclusion $M \hookrightarrow \mathrm{VR}(M;r)$ is not continuous. For r sufficiently small, the homotopy equivalence $\mathrm{VR}(M;r) \xrightarrow{\simeq} M$ in Hausmann's result depends on the choice of a total ordering[1] of the points in X, meaning it is non-canonical as different choices of orderings produce different maps. By contrast, the inclusion $M \hookrightarrow \mathrm{VR}^m(M;r)$ to the metric thickening is continuous, and for r sufficiently small it has as a homotopy inverse the canonical map $\mathrm{VR}^m(M;r) \to M$ defined by Karcher means [2, 17].

In this work we prove a metric analogue of Hausmann's result for subsets of Euclidean space with positive reach. Our main result is the following:

Theorem 4.1.4. *Let X be a subset of Euclidean space $\mathbb{R}^n$, equipped with the Euclidean metric, and suppose the reach τ of X is positive. Then for all $r < \tau$, the metric Vietoris–Rips complex $\mathrm{VR}^m(X;r)$ is homotopy equivalent to X.*

In particular, if M is a compact submanifold of $\mathbb{R}^n$ with positive reach, then its Vietoris–Rips thickening is homotopy equivalent to M for sufficiently small scale parameters. To our knowledge, this is the first version of Hausmann's theorem providing a homotopy equivalence between a Euclidean (and hence typically non-Riemannian) manifold and either its Vietoris–Rips simplicial complex or its Vietoris–Rips metric thickening at sufficiently small scales.

We prove the main theorem by showing that the linear projection of $\mathrm{VR}^m(X;r)$ into $\mathbb{R}^n$ has image contained in the tubular neighborhood of radius τ about X. We then map each point in the tubular neighborhood to its (unique) closest point in X. The composition of these maps produces a homotopy equivalence $\mathrm{VR}^m(X;r) \xrightarrow{\simeq} X$ whose homotopy inverse is the (now continuous) inclusion $X \hookrightarrow \mathrm{VR}^m(X;r)$.

A related construction to the Vietoris–Rips complex is the Čech complex. For $X \subseteq \mathbb{R}^n$, the Čech complex $\check{\mathrm{C}}(X;r)$ is the nerve simplicial complex of the collection of balls $\mathrm{B}(x, r/2)$ with

[1] One could use the axiom of choice to pick such a total order, though constructive total orders may also exist.

centers $x \in X$. The nerve lemma implies[2] that $\check{\mathrm{C}}(X;r)$ is homotopy equivalent to the union of the balls. However, the Čech complex $\check{\mathrm{C}}(X;r)$ need not inherit any metric properties of X, and again is not metrizable if it is not locally finite. We therefore consider the metric Čech thickening $\check{\mathrm{C}}^{\mathrm{m}}(X;r)$ from [2], which is a metric space equipped with the 1-Wasserstein distance that furthermore is a metric thickening of X. In Theorem 4.2.5 we prove that if X is a subset of Euclidean space of positive reach τ, then for all $r < \tau$ the metric Čech thickening $\check{\mathrm{C}}(X;2r)$ is homotopy equivalent to X.[3] The proof mirrors that of Theorem 4.1.4.

In Chapter 2 we review background information on point-set and metric topology, simplicial complexes, and the Wasserstein metric. In Chapter 3 we summarize previous work, in particular, a brief discussion of Hausmann's theorem [16], the metric Vietoris–Rips and Čech thickenings, and the metric analogue of Hausmann's theorem for Riemannian manifolds [2]. Section 4.1 contains our main result, a metric analogue of Hausmann's theorem for Vietoris–Rips thickenings of Euclidean subsets of positive reach, and the lemmas building up to it. We use similar techniques to prove a version for Čech thickenings in Section 4.2. Lastly, in Section 4.3 we give an alternate proof for a metric Hausmann's theorem for Riemannian manifolds [2, Theorem 4.2] using Nash's embedding theorem.

[2]For ambient Čech complexes corresponding to Euclidean balls, though in this paper we also consider intrinsic Čech complexes corresponding to balls in X.

[3]This result does not follow from the nerve lemma since the nerve complex $\check{\mathrm{C}}(X;r)$ and metric thickening $\check{\mathrm{C}}^{\mathrm{m}}(X;r)$ can in general have different homotopy types.

Chapter 2

Preliminaries

2.1 Basic Topology

The main result of this work is to show that two particular topological spaces are homotopy equivalent. This section covers the material necessary to precisely define that statement.

Definition 2.1.1. *A* topological space *is a set X and a collection $\mathscr{T}$ of subsets of X called the open sets of the topology. This collection must satisfy the following three properties:*

1. *X and $\emptyset$ are open.*
2. *If $U_1, \ldots, U_n$ are a finite collection of open sets, then their intersection $\bigcap_{i=1}^n U_i$ is open.*
3. *If U_α for $\alpha \in A$ an arbitrary index set are open, then their union $\bigcup_{\alpha \in A} U_\alpha$ is open.*

It is customary to refer to a topological space $(X, \mathscr{T})$ simply as X, allowing the topology to be inferred from context. If $Y \subseteq X$ a topological space, it can also be considered as a topological space with the *subspace topology*, namely $U \subseteq Y$ is open if there exists an open $V \subseteq X$ such that $U = V \cap X$. The complement of a subspace Y, denoted Y^c, is the portion of X not containing Y, that is, $Y^c = X \setminus Y$. A set whose complement is open is called *closed*. The *closure* of a set U is the smallest closed set containing U, denoted $\overline{U}$.

Mathematical objects are often only as interesting as the maps that can be constructed between them. The maps that are relevant to topological structure are continuous maps.

Definition 2.1.2. *A function f from a topological space X to a topological space Y is* continuous *if for every open $U \subset Y$, the preimage of U under f, written $f^{-1}(U)$, is open as a subset of X.*

Topology is the study of when topological spaces are equivalent. There are several important notions of equivalence. The strongest is homeomorphism:

Definition 2.1.3. *Let X and Y be topological spaces. Then X and Y are* homeomorphic*, denoted $X \cong Y$, if there exist continuous functions $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$, where id_X and id_Y are the identity functions on X and Y, respectively.*

Put another way, two spaces are homeomorphic if there exists a continuous function $f\colon X \to Y$ which is a bijection and has a continuous inverse. Intuitively, two spaces are homeomorphic if it is possible to stretch and bend one into the other without tearing it or creating new holes. However, there is a limit to the extent of stretching permitted by homeomorphism; for example, the unit ball in $\mathbb{R}^2$ is homeomorphic to all of $\mathbb{R}^2$ but is not homeomorphic to a single point (the bijection clearly fails). A weaker equivalence relation on topological spaces is homotopy equivalence. It can be thought of as permitting stretching that "collapses or increases the dimension" of the object.

We first need the notion of a homotopy equivalence of functions:

Definition 2.1.4. *Let $f\colon X \to Y$ and $g\colon X \to Y$ be continuous maps. Then f is homotopic to g, denoted $f \simeq g$, if there exists a continuous function $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$, $H(x,1) = g(x)$.*

Definition 2.1.5. *Let X and Y be topological spaces. Then X is homotopy equivalent to Y, written $X \simeq Y$, if there exists a pair of continuous functions $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.*

Continuing the above example, the unit ball is homotopy equivalent to a point; the map from the ball to the point is the constant map and the map in the reverse direction is an inclusion map. Spaces that are homotopy equivalent to a point are quite important and are referred to as *contractible*.

A given set X may be endowed with a variety of different topologies. It is in general possible for X with two different topologies $\mathscr{T}$ and $\mathscr{S}$ to be neither homeomorphic nor homotopy equivalent. In Section 2.4 we give an explicit example of this.

2.2 Metric spaces

The open sets of a topological space can be defined in many ways, but it is often useful, and in fact one of the main purposes of this work, to define them (when possible) in terms of a metric. For this we need the concept of a metric space.

Definition 2.2.1. *A* metric space*, (X, d), is a set X along with a function $d\colon X \times X \to \mathbb{R}$ such that for all $u, v, w \in X$,*

1. *(Non-degeneracy) $d(u, v) \geq 0$ and $d(u, v) = 0$ if and only if $u = v$,*

2. *(Symmetry) $d(u, v) = d(v, u)$, and*

3. *(Triangle Inequality) $d(u, v) \leq d(u, w) + d(w, v)$.*

The function d is called a distance *or a* metric.

Any metric d on a set X induces a topology called the metric topology. Specifically, an open ball in a metric space X is $\mathrm{B}(x, r) = \{y \in X \mid d(y, x) < r\}$, where $x \in X$ is the ball's center and r is a positive real number, called the radius. Likewise, a closed ball will be denoted $\overline{\mathrm{B}}(x, r) = \{y \in X \mid d(y, x) \leq r\}$. The metric topology on X consists of all sets that can be written as a combination of unions and finite intersections of open balls. The open balls are called the basis for this topology. A topological space X is said to be *metrizable* if there exists a metric $d\colon X \times X \to \mathbb{R}$ that induces the topology of X.

Just as a subset of a general topological space inherits a topology from the containing space, a subset $Y \subseteq X$ of a metric space inherits a metric by restricting d from $X \times X$ to $Y \times Y$. Besides measuring distances between points, it is also possible to use the metric d to measure the distance between a point and a subset, or between two subsets of a given metric space. Given a point $x \in X$ and subset $Y \subseteq X$, the distance between x and Y is the distance between x and its nearest point in Y, precisely, $d(x, Y) = \inf_{y \in Y} d(x, y)$. The distance between two subsets $Y, Y' \subseteq X$ is the distance between their nearest points, $d(Y, Y') = \inf_{y \in Y, y' \in Y'} d(y, y')$. We define the diameter of a set $Y \subseteq X$ to be $\operatorname{diam}(Y) = \sup\{d(y, y') \mid y, y' \in Y\}$. An r-*thickening* of a metric space X is a metric space $Z \supseteq X$ such that the metric on X extends to that on Z, and also $d(x, Z) \leq r$ for all $x \in X$.

2.3 Euclidean Space

Euclidean space is the metric space $(\mathbb{R}^n, d)$ where $\mathbb{R}^n$ denotes the n-fold Cartesian product of the real numbers and d is the usual Euclidean distance, defined as follows: The standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ is defined by

$$\langle (u_1, \ldots, u_n), (v_1, \ldots, v_n) \rangle = u_1 v_1 + \ldots + u_n v_n.$$

We can define the *norm*, $\|\cdot\|$, of an element $x \in \mathbb{R}^n$ by $\|x\| = \langle x, x \rangle^{1/2}$. The metric d is then simply $d(u, v) = \|u - v\|$.

A subset X of $\mathbb{R}^n$ is called *convex* if it contains the entirety of every line segment joining two points in X. More precisely, $X \subseteq \mathbb{R}^n$ is convex if the set $\{\lambda x + (1 - \lambda) y \mid 0 \leq \lambda \leq 1\}$ is a subset of X for any $x, y \in X$. Convex sets possess many convenient properties. In particular, a convex set is contractible (via a straight-line homotopy).

Given a set of points (not necessarily finite) there is a unique minimal convex set containing them. This is called the convex hull. For any $X \subseteq \mathbb{R}^n$ the convex hull of X is defined as

$$\operatorname{conv}(X) = \left\{ x = \sum_{i=0}^{k} \lambda_i x_i \in \mathbb{R}^n \;\middle|\; k \in \mathbb{N},\ x_i \in X,\ \sum_i \lambda_i = 1,\ \lambda_i \geq 0 \right\}.$$

Note that the diameter of X is the same as the diameter of its convex hull.

2.4 Manifolds

Manifolds are a particular type of topological space closely related to Euclidean space in the following sense: given any point x in manifold M, there is a region around x that is essentially Euclidean. We give a cursory overview here; for a full treatment see [19].

Definition 2.4.1. *An n-dimensional* manifold *is a topological space M that is second-countable, Hausdorff, and locally homeomorphic to $\mathbb{R}^n$.*

The first two conditions are technical requirements to prevent various pathological examples. Locally homeomorphic to $\mathbb{R}^n$ means that for every $x \in M$ there exists an open set U containing x such that $U \cong \mathbb{R}^n$. The maps defining the homeomorphisms are called *charts* and a collection of open sets covering M along with their charts is called an *atlas* for the manifold M.[4]

As a first example, note that Euclidean space is itself a manifold, as for each $x \in \mathbb{R}^n$ the open ball $\mathrm{B}(x, r)$ is homeomorphic to $\mathbb{R}^n$. More generally, 1-dimensional manifolds are curves, 2-dimensional manifolds are surfaces, and high-dimensional manifolds can be thought of as higher dimensional analogues of these.

Sometimes the study of manifolds is restricted to a smaller class of objects called *smooth manifolds*. A manifold is called smooth of class C^k if all of the functions $f \colon U \to \mathbb{R}^n$ making $U \cong \mathbb{R}^n$ can be chosen to be *diffeomorphisms*, that is, f and f^{-1} have continuous k-th derivatives. If derivatives of all orders are smooth, a manifold is said to by C^∞. We will not require any of this machinery except in Section 4.3.

We will be interested in two particular types of manifolds: submanifolds of Euclidean space, and, to a lesser extent, Riemannian manifolds. A submanifold of Euclidean space is simply a subset $Y \subseteq \mathbb{R}^n$ such that Y is an m-dimensional manifold (for some $m \leq n$). Note that such a manifold inherits the Euclidean distance from the ambient space, and consequently is endowed with the metric subspace topology.

A Riemannian manifold is a more abstract object. It is a manifold M (often assumed to be smooth) equipped with a tensor g called a *Riemannan metric*.[5] The complete definition is too involved to present here, but see [19, Chapter 13]. The Riemannian metric should be thought of as a generalization of the inner product on $\mathbb{R}^n$. If v is a tangent vector to M at p, the length of v

[4]Again, there are various technical requirements that the different charts and open sets overlap in "nice" ways, which we omit.

[5]Which, confusingly, is not a metric in the sense of Section 2.2.

is defined to be $|v|_g = g_p(v, v)^{1/2}$. In a similar manner, it gives us a way to measure distances on a Riemannian manifold. If $\gamma\colon [0, 1] \to M$ is a smooth curve, its time derivative $\gamma'(t)$ consists of tangent vectors. The length of γ from $\gamma(a)$ to $\gamma(b)$ is then

$$L(\gamma) = \int_a^b |\gamma'(t)|_g \, \mathrm{d}t \tag{2.1}$$

The shortest path between two points p, q on a Riemannian manifold is called a geodesic. The length of this shortest curve, in the sense defined by Equation 2.1, gives a metric on M. Intuitively, a Riemannian metric gives a way of measuring arc-lengths along a curved surface.

An important clarifying remark about metrics on Euclidean submanifolds is the following. Let M be a manifold embedded in Euclidean space $\mathbb{R}^n$ and equipped with the Euclidean distance function (as in Theorem 4.1.4). The usual inner product on $\mathbb{R}^n$ gives a Riemannian metric, and so M can be viewed as a Riemannian manifold with distances given by the Riemannian distance. Assuming M is connected, the Riemannian distance on M is also a metrization of the original manifold topology [19]. The Vietoris–Rips complex (which we will define in Section 2.6), and its homotopy type, may depend upon which of these two metrics one chooses to use. For example, a circle and an ellipse in $\mathbb{R}^2$ with the Riemannian distance function (i.e. the arc-length metric) and equal circumferences have identical Vietoris–Rips complexes. On the other hand, with the Euclidean metric their Vietoris–Rips complexes are not homotopy equivalent [1, 3].

2.5 Simplicial Complexes

We now turn from discussing manifolds to discussing the topological spaces which can be used as approximations thereof.

Definition 2.5.1. *Let V be a set, called the set of vertices. An* abstract simplicial complex *K on vertex set V is a subset of the power set of V with the property that if $\sigma \in K$, then all subsets of σ are in K.*

While this is a purely combinatorial object, every abstract simplicial complex permits a geometric realization, $|K|$, which is a topological space. Intuitively, $|K|$ is a collection of lines, triangles, tetrehdra, and so on in higher dimensions, glued together along their faces. More rigorously, $|K|$ is defined as a set by taking convex linear combinations of vertices:

$$|K| = \left\{ \sum_{i=0}^{k} \lambda_i v_i \;\middle|\; k \in \mathbb{N},\ [v_0, \ldots, v_k] \in K,\ \lambda_i \geq 0,\ \sum_i \lambda_i = 1 \right\}.$$

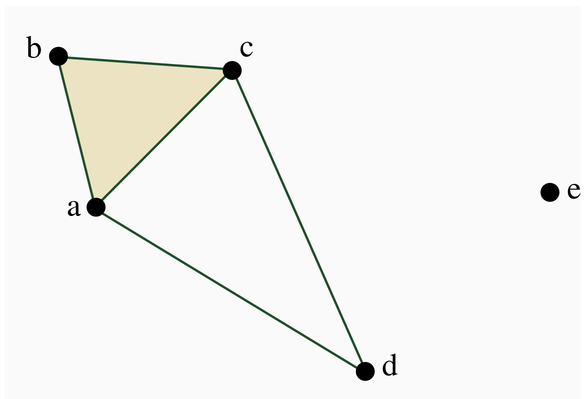

Figure 2.1: A geometric realization of the simplicial complex: $K = \{[abc], [ac], [bc], [ac], [ad], [cd], [a], [b], [c], [d], [e]\}$.

In the simpler case when K is finite, we can put a topology on $|K|$ as follows. Choose an affinely independent set of points in $\mathbb{R}^n$ (for n sufficiently large) to correspond to each of the elements of the vertex set V. Then $|K|$ consists of all convex linear combinations of these points, and $|K|$ is given its topology as a subset of Euclidean space. More generally, one can produce a topology on $|K|$ by viewing it as a subset of $[0,1]^V$, the space of functions $V \to [0,1]$. Indeed note

$$|K| = \left\{ f\colon [0,1] \to V \;\middle|\; \sum_{v \in V} f(v) = 1,\ \mathrm{supp}(f) \in K \right\}.$$

Give $[0,1]^V$ its induced topology as the direct limit of $[0,1]^\tau$ where τ ranges over all finite subsets of V, and equip $|K|$ with the subspace topology [23].

For the rest of this paper we denote both an abstract simplicial complex and its geometric realization by the same symbol K.

2.6 The Vietoris–Rips and Čech simplicial complexes

We are particularly interested in two simplicial complexes, the Vietoris–Rips and the Čech complexes. Both contain n-simplices whenever a set of $n+1$ points in some metric space is close enough together. They differ in regard to what is considered "close enough," but are nonetheless closely related.

Definition 2.6.1. *Let X be a metric space and $r > 0$ a scale parameter. The* Vietoris–Rips simplicial complex *of X with scale parameter r, denoted $\mathrm{VR}_{\leq}(X;r)$, has vertex set X and a simplex for every finite subset $\sigma \subseteq X$ such that $\mathrm{diam}(\sigma) \leq r$. Similarly, $\mathrm{VR}_{<}(X;r)$ contains every finite subset with diameter $< r$.*

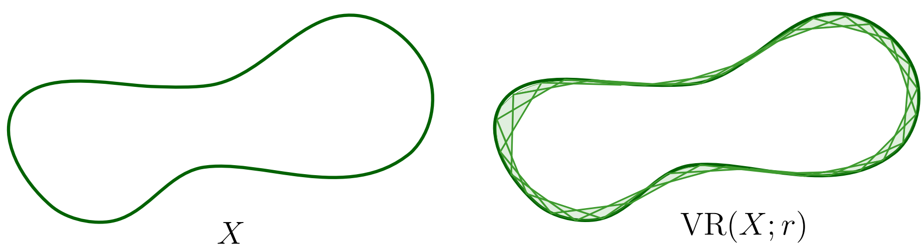

Figure 2.2: A metric space X and (a subset of) its Vietoris–Rips complex.

We will write $\mathrm{VR}(X;r)$ when the distinction between $<$ and $\leq$ is unimportant. The Vietoris–Rips complex is the *clique* or *flag* complex of its 1-skeleton.

Definition 2.6.2. *Let $X \subseteq Y$ be a submetric space and r a scale parameter with $r \geq 0$. The Čech complex of X with scale parameter r, $\check{\mathrm{C}}_{\leq}(X,Y;r)$, has vertex set X and a simplex for every* *finite subset $\sigma \subseteq X$ such that $\bigcap_{x_i \in \sigma} \overline{\mathrm{B}}(x_i, r/2) \neq \emptyset$, where $\overline{\mathrm{B}}(x_i, r/2)$ denotes a closed ball in Y centered at x_i with radius $r/2$.*

Similarly, $\check{\mathrm{C}}_<(X, Y; r)$ contains a simplex for every finite subset σ such that $\bigcap_{x_i \in \sigma} \mathrm{B}(x_i, r/2) \neq \emptyset$.

Again, we will write $\check{\mathrm{C}}(X, Y; r)$ when the distinction between open and closed is unimportant. The Čech complex can be considered as the nerve of the union of balls in Y of radius $r/2$ centered at each of the points in X. Of particular interest are the cases where $Y = \mathbb{R}^n$ and $Y = X$. These are called the called the *ambient* and *intrinsic* Čech complex, respectively. Note that if $X \subseteq \mathbb{R}^n$ then $\check{\mathrm{C}}(X, X; r) \subseteq \check{\mathrm{C}}(X, \mathbb{R}^n; r)$. When it is not necessary to distinguish these two we will write $\check{\mathrm{C}}(X; r)$.

For any $\sigma \in \check{\mathrm{C}}(X, Y; r)$ we have $\mathrm{diam}(\sigma) \leq r$, and so $\check{\mathrm{C}}(X, Y; r)$ is a subcomplex of $\mathrm{VR}(X; r)$. When Y is a geodesic space, the complexes $\mathrm{VR}(X; r)$ and $\check{\mathrm{C}}(X, Y; r)$ have the same 1-skeletons. Whether Y is geodesic or not, the Čech complex can be a proper subset of the Vietoris–Rips complex.

A useful characterisation is the following:

Proposition 2.6.3. *A set of points $x_0, \ldots, x_k \in X$ form a simplex in $\check{\mathrm{C}}_<(X, Y; r)$ if and only if there exists a point $c \in Y$ such that every x_i is contained in the open ball $\mathrm{B}(c, r/2)$. A similar statement is true for $\check{\mathrm{C}}_\leq(X, Y; r)$ with closed balls.*

Proof. Let c be any point in the intersection $\bigcap_{x_i \in \sigma} B(x_i, r/2)$. Then $B(c, r/2)$ contains all of the x_i. Conversely, if all of the x_i are in $B(c, r/2)$, then c is in $\bigcap_{x_i \in \sigma} B(x_i, r/2)$ and hence the intersection is nonempty. □

The Vietoris–Rips and Čech complexes are given the standard topology as simplicial complexes: a subset of the geometric realization is open if and only its intersection with every simplex is open. An important remark is the following. A simplicial complex K is said to be *locally finite*

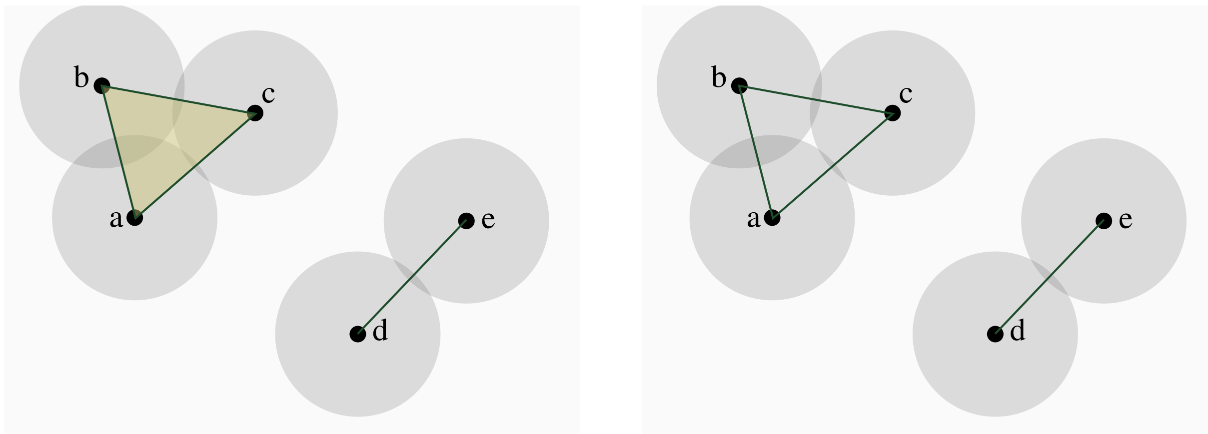

Figure 2.3: The Vietoris–Rips and Čech complexes at the same scale parameter.

if each vertex belongs to only finitely many simplices of K, and a simplicial complex is metrizable if and only if it is locally finite [22, Proposition 4.2.16(2)]. This means that in general, the Vietoris–Rips and Čech simplicial complexes cannot be equipped with a metric without changing their homeomorphism types, even though they were built on top of a metric space.

2.7 Nerve Lemma

The nerve lemma is a standard result in the theory of Čech complexes. In its most general form it says the following:

Lemma 2.7.1 (Nerve Lemma: General Version)**.** *If $\mathcal{U}$ is an open cover of a paracompact space X such that ever nonempty intersection $\cap_{i=1}^{n} U_i$ for $U \in \mathcal{U}$ is contractible, then $\mathcal{N}(U) \simeq X$.*

This general form and its proof can be found in, for example, [15, Corollary 4G.3]. A simpler statement is the following

Lemma 2.7.2 (Nerve Lemma: Convex Version)**.** *Let U_α for $\alpha \in A$ an index set be convex subsets of $\mathbb{R}^n$. Then $\mathcal{N}(U_\alpha) \simeq \cup_{\alpha \in A} U_\alpha$.*

Since intersections of convex regions are convex, and thus contractible, this follows immediately from the first version. In particular, the Čech complex is the nerve of a set of balls (which are convex), so it is homotopy equivalent to the union of the balls. If one takes open balls which cover a topological space, the nerve lemma implies that the Čech complex is homotopy equivalent to the underlying space. This makes it a natural first choice for topological reconstruction.

2.8 Wasserstein metric

In this section we describe a way to put a metric on probability Radon measures. The use of this will be seen in Section 3.2, where we define the metric Vietoris–Rips and Čech thickenings as probability distributions. The metric has many names: the Wasserstein, Kantorovich, optimal transport, or earth mover's metric. Its origin is in the study of image recognition, and it is known to solve the Monge-Kantorovich problem (see [25]).

Let X be a metric space equipped with a distance function $d\colon X \times X \to \mathbb{R}$. The *Borel sets* of X, denoted $\mathscr{B}(X)$, are the σ-algebra generated by the open sets of X. That is, a set B is in $\mathscr{B}(X)$ if it can be produced from open sets via (at most) countable unions and intersections, and taking complements. A measure on $\mathscr{B}(X)$ is a function $\mu\colon \mathscr{B}(X) \to [0, \infty]$, such that

1. $\mu(\emptyset) = 0$

2. If $B_1, \ldots, B_n, \ldots$ are a countable number of disjoint Borel sets, then

$$\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i)$$

Measures in some sense define the "volume" of a set in X. For technical reasons it is not possible to define a measure on all subsets of X [13], but we will not encounter any non-measurable sets here.

A measure μ defined on the Borel sets of X is

- *inner regular* if $\mu(B) = \sup\{\mu(K) \mid K \subseteq B \text{ is compact}\}$ for all Borel sets B,

- *locally finite* if every point $x \in X$ has a neighborhood U such that $\mu(U) < \infty$,

- a *Radon measure* if it is both inner regular and locally finite, and

- a *probability measure* if $\int_X d\mu = 1$.

The last condition implies that the "volume" of the entire space is 1.

The following is from [11]. Let $\mathcal{P}(X)$ denote the set of probability Radon measures such that for some (and hence all) $y \in X$, we have $\int_X d(x, y)\, d\mu < \infty$. Define the L^1 metric on $X \times X$ by setting the distance between $(x_1, x_2), (x_1', x_2') \in X \times X$ to be $d(x_1, x_1') + d(x_2, x_2')$. Given $\mu, \nu \in \mathcal{P}(X)$, let $\Pi(\mu, \nu) \subseteq \mathcal{P}(X \times X)$ be the set of all probability Radon measures π on $X \times X$ such that $\mu(B) = \pi(B \times X)$ and $\nu(B) = \pi(X \times B)$ for all Borel subsets $B \subseteq X$. Note that such an element π is a joint measure on $X \times X$ whose marginals, when restricted to each X factor, are μ and ν.

Definition 2.8.1. *The 1-Wasserstein metric on $\mathcal{P}(X)$ is defined by*

$$d_W(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_{X \times X} d(x, y)\, d\pi.$$

The names *optimal transport* or *earth mover's* metric can be interpreted as follows. One can think of measures μ and ν as "piles of dirt" in X with prescribed mass distributions. The joint measure π with μ and ν as marginals is a transport plan moving the μ pile of dirt to the ν pile. The 1-Wasserstein distance between μ and ν is the infimum, over all transport plans π, of the work involved in moving μ to ν via transport plan π.

2.9 Sets of positive reach

We are interested in the case where metric space X is a subset of $\mathbb{R}^n$ of *positive reach*. In particular, any embedded C^k submanifold (with or without boundary) of $\mathbb{R}^n$ with $k \geq 2$ has positive reach [24]. Consider the set

$$Y = \{y \in \mathbb{R}^n \mid \exists\, x_1 \neq x_2 \in M \text{ with } d(y, x_1) = d(y, x_2) = d(y, X)\}.$$

The closure $\overline{Y}$ of Y is the *medial axis* of X. For any point $x \in X$, the *local feature size at* x is the distance $d(x, \overline{Y})$ from x to the medial axis. The *reach* τ of X is the minimal distance $\tau = d(X, \overline{Y})$ between X and its medial axis.

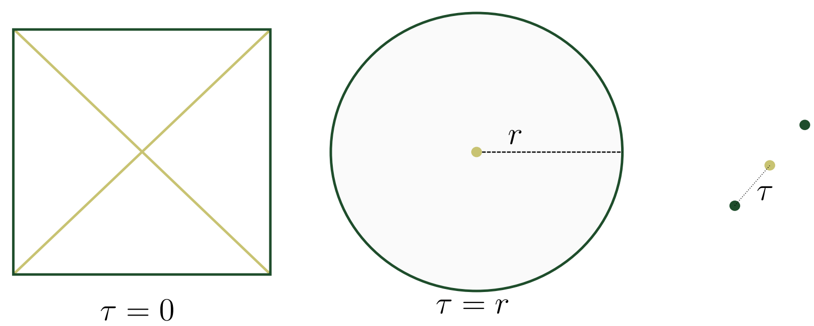

Figure 2.4: Sets with corners (left) have zero reach. Smooth manifolds have positive reach (center). The reach is at most half the distance between non-connected components (right).

For $X \subseteq \mathbb{R}^n$ and $\alpha > 0$ we define its open α-offset (or tubular neighborhood), Tub_α, by

$$\mathrm{Tub}_\alpha = \{x \in \mathbb{R}^n \mid d(x, X) < \alpha\} = \bigcup_{x \in X} \mathrm{B}(x, \alpha).$$

In particular, if X has reach τ, then for every point in Tub_τ there exists a unique nearest point in X. As in [12,21], define $\pi\colon \mathrm{Tub}_\tau \to X$ to be the nearest point projection map, sending an element $x \in \mathrm{Tub}_\tau$ to its unique closest point $\pi(x) \in X$.

Lemma 2.9.1. *The function* $\pi\colon \mathrm{Tub}_\tau \to X$ *is continuous.*

Proof. Let $x, y \in \mathrm{Tub}_\tau$ and $r = \max\{d(x, \pi(x)), d(y, \pi(y))\}$. Then the conditions of [12, Theorem 4.8(8)] are satisfied and so

$$d(\pi(x), \pi(y)) \leq \frac{\tau}{\tau - r} d(x, y). \tag{2.2}$$

Thus π is continuous at x for any $x \in \mathrm{Tub}_\tau$. $\square$

We also state the following proposition, implicit in [21], for any set of positive reach.

Proposition 2.9.2. *Let* $X \subseteq \mathbb{R}^n$ *have reach* $\tau > 0$. *Let* $p \in X$ *and suppose* $x \in \mathrm{Tub}_\tau \setminus X$ *satisfies* $\pi(x) = p$. *If* $c = p + \tau \frac{x-p}{\|x-p\|}$, *then* $\mathrm{B}(c, \tau) \cap X = \emptyset$.

Proof. For any $0 < t < \tau$, let $y_t = p + t\frac{x-p}{\|x-p\|}$. Since $y_t \in \mathrm{Tub}_\tau$, we have $\overline{\mathrm{B}}(y_t, t) \cap X = \{p\}$ and $d(y_t, p) = t$, so $\mathrm{B}(c, t) \cap X = \emptyset$. Note that $\mathrm{B}(c, \tau) = \cup_{0<t<\tau}\mathrm{B}(y_t, t)$. Indeed, to see the inclusion $\subseteq$, suppose that $z \in \mathrm{B}(c, \tau)$, so that $d(z, c) = \tau - \epsilon$ for some $\epsilon > 0$. Let $t = \tau - \frac{\epsilon}{3}$. By the triangle inequality, $d(y_t, z) \leq d(y_t, c) + d(c, z) = \tau - \frac{2\epsilon}{3} < t$, giving $z \in \mathrm{B}(y_t, t)$. The reverse inclusion $\supseteq$ is straightforward. It follows that $\mathrm{B}(c, \tau) \cap X = \emptyset$. $\square$

Chapter 3

Previous Research

The main results in this paper expand upon two previous pieces of work. The nerve lemma (see Section 2.7) shows that Čech complexes can serve as good approximations of a manifold under certain conditions. However, finding points of intersection becomes computationally infeasible in high dimensions, so the Čech complex is often not a desirable object from a computational standpoint. The Vietoris–Rips complex offers the advantage of only requiring pairwise intersections, which are more easily computed. However, Vietoris–Rips complexes are not the nerve of any collection of convex sets, so there is no *a priori* guarantee that they should likewise have the correct homotopy type. Hausmann's theorem [16], though, shows that indeed, under the right conditions Vietoris–Rips complexes do have the correct homotopy type. A survey of this result is presented below.

Our work demonstrates an analogue of Hausmann's theorem for the *metric Vietoris–Rips thickening* of a Euclidean submanifold. The work by Adamaszek, Adams, and Frick [2] introduced the metric thickening of a simplicial complex, which will be the main object of study here. They also prove an analogue of Hausmann's result, in the same context (namely, for Riemannian manifolds) as the original. Our contribution is to expand this to include the case of Euclidean submanifolds, and in Section 4.3 we prove an analogue of their result as a corollary of our Theorem 4.1.4.

3.1 Hausmann – On the Vietoris–Rips Complexes and a Cohomology Theory for Metric Spaces

Vietoris–Rips complexes are used in computational topology as an approximation of the Čech complex that is easier to compute. This is justified in large part by Hausmann's theorem [16, Theorem 3.5], which shows that something much like the nerve lemma is true for Vietoris–Rips complexes under certain conditions. The precise statement is as follows:

Theorem 3.1.1. *Let M be a compact Riemannian manifold*[6] *and $\epsilon > 0$ be sufficiently small*[7]. *Then* $\mathrm{VR}(M;\epsilon) \simeq M$.

We sketch an outline of the proof below:

Hausmann begins by defining a map $T\colon \mathrm{VR}(M;\epsilon) \to M$ inductively on simplices of $\mathrm{VR}(M;\epsilon)$. Let $\sigma \in \mathrm{VR}(M;\epsilon)$. We define a map $T_\sigma\colon \Delta^n \to M$ where where Δ^n is the standard n-simplex with vertices $e_0, \dots, e_{n-1}$. By putting a total order on the points of M we can write each simplex σ uniquely as $\sigma = [x_0, x_1, \dots, x_{n-1}]$ with $x_0 < x_1 < \cdots < x_{n-1}$. Since $\sigma = [x_0, \dots, x_{n-1}]$ with $x_i \in M$, we define $T_\sigma(e_i) = x_i$. Then we interpolate between vertices by defining for $z = \sum_{i=0}^{k} \lambda_i e_i$,

$$x = T_\sigma\left(\frac{1}{1-\lambda_n}\right)\sum_{i=0}^{k-1} \lambda_i e_i$$

and then $T_\sigma(z)$ is the point on the shortest geodesic joining x to x_k with $d(z, T_\sigma(z)) = \lambda_k d(x, x_k)$. This formula then inductively gives a formula for all of σ. There is then a correspondence $\sigma \mapsto T_\sigma$ between simplices of $\mathrm{VR}(M;\epsilon)$ and singular simplices of $\mathrm{VR}(M;\epsilon)$. This gives a map $T\colon \mathrm{VR}(M;\epsilon) \to M$ by $x \mapsto T_\sigma(x)$.

The remainder of the proof relies upon more sophisticated algebraic topology techniques than we have introduced here. We give a brief summary, along with references.

Hausmann's second step is to show that T induces an isomorphism on all homology groups and an isomorphism of the fundamental groups between M and $\mathrm{VR}(M;\epsilon)$. The same holds for the universal cover, $\widetilde{M}$, of M, and the corresponding map $\hat{T}\colon \mathrm{VR}(\widetilde{M};\epsilon) \to \widetilde{M}$. By Whitehead's Theorem and the Hurewicz Theorem [15, Theorem 4.5, Theorem 4.32] he concludes that $\hat{T}$ is a homotopy equivalence.

This allows him to set up the commutative pull-back diagram

[6] The actual condition is slightly more technical and general; for example any Riemannian manifold with positive injectivity radius and bounded sectional curvature is sufficient.

[7] Relating to the curvature condition noted above.

$$\begin{tikzcd}
\mathrm{VR}(\widetilde{M};\epsilon) \arrow[dr, "\breve{T}"'] \arrow[drr, "\hat{T}"] & & \\
& \widetilde{\mathrm{VR}(M;\epsilon)} \arrow[r, "\widetilde{T}"'] \arrow[d, "p_\epsilon"] & \widetilde{M} \arrow[d, "p"] \\
& \mathrm{VR}(M;\epsilon) \arrow[r, "T"] & M
\end{tikzcd}$$

Figure 3.1: The commutative diagram showing T is a homotopy equivalence [16, Figure 3.9]

which implies that $\widetilde{T}$ and hence T are homotopy equivalences.

A few comments on this proof: as mentioned in Chapter 1, it relies on a total ordering of the points of M. While such an order may be constructive, it is in general far from canonical (as simple a space as $\mathbb{S}^1$ does not have a canonical total order). Since the map T depends directly on this ordering, different orderings will generally produce different maps. However, once an ordering is chosen, the map T is quite natural—it sends a simplex to a sort of "geodesic convex hull" of the vertices.

3.2 Metric Reconstruction via Optimal Transport

Our main result consists of an analogue of Hausmann's theorem for metric thickenings of a simplicial complex. These objects were introduced by Adamaszek, Adams, and Frick in [2]. A summary follows.

3.2.1 The Vietoris–Rips and Čech Thickenings

The definitions in this section are from [2, 11].

Given a metric space X and a scale parameter r we will define the *Vietoris–Rips thickening* $\mathrm{VR}^m(X;r)$, which will be a metric space r-thickening of X. As a set, $\mathrm{VR}^m(X;r)$ is the set of all formal convex combinations of points in X with diameter at most r, namely[8]

$$\mathrm{VR}^m_{\leq}(X;r) = \Big\{\sum_{i=0}^k \lambda_i x_i \mid k\in\mathbb{N},\ x_i\in X, \text{and } \mathrm{diam}(\{x_0,\dots,x_k\})\leq r\Big\}$$
$$\mathrm{VR}^m_{<}(X;r) = \Big\{\sum_{i=0}^k \lambda_i x_i \mid k\in\mathbb{N},\ x_i\in X, \text{and } \mathrm{diam}(\{x_0,\dots,x_k\})< r\Big\},$$

with $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$. A useful viewpoint is to consider an element of $\mathrm{VR}^m(X;r)$ as a probability measure. For $x\in X$, let δ_x be the Dirac probability measure defined on any Borel subset $E\subseteq X$ by

$$\delta_x(E) = \begin{cases} 1 & \text{if } x\in E \\ 0 & \text{if } x\notin E. \end{cases}$$

By identifying $x\in X$ with $\delta_x\in\mathcal{P}(X)$, and more generally $x=\sum_{i=0}^k \lambda_i x_i$ with $\sum_{i=0}^k \lambda_i\delta_{x_i}\in\mathcal{P}(X)$, we can view $\mathrm{VR}^m(X;r)$ as a subset of $\mathcal{P}(X)$, the set of all Radon probability measures on X. Hence we can equip the set $\mathrm{VR}^m(X;r)$ with the 1-Wasserstein metric from Section 2.8, namely $d_W(x,x') = \inf_{\pi\in\Pi(x,x')}\int_{X\times X} d(x,x')\,d\pi$ for $x,x'\in\mathrm{VR}^m(X;r)$.

To give a more explicit definition of the metric on $\mathrm{VR}^m(X;r)$, let $x,x'\in\mathrm{VR}^m(X;r)$ with $x=\sum_{i=0}^k \lambda_i x_i$ and $x'=\sum_{i=0}^{k'}\lambda'_i x'_i$ (we cease to distinguish between $x\in X$ and its associated measure, δ_x). Define a matching p between x and x' to be any collection of non-negative real numbers $\{p_{i,j}\}$ such that $\sum_{j=0}^{k'} p_{i,j} = \lambda_i$ and $\sum_{i=0}^k p_{i,j} = \lambda'_j$. It follows as a consequence that $\sum_{i,j} p_{i,j} = 1$, and so matching $\{p_{i,j}\}$ can be thought of as a joint probability distribution with marginals $\{\lambda_i\}_{i=0}^k$ and $\{\lambda'_j\}_{j=0}^{k'}$. Define the *cost of the matching* p to be $\mathrm{cost}(p)=\sum_{i,j} p_{i,j} d(x_i,x'_j)$.

Definition 3.2.1. *The* 1*-Wasserstein metric on* $\mathrm{VR}^m(X;r)$ *is the distance* d_W *defined by*

$$d_W(x,x') = \inf\left\{\mathrm{cost}(p) \mid p \textit{ is a matching between } x \textit{ and } x'\right\}.$$

[8]There is a canonical bijection between the sets underlying $\mathrm{VR}^m(X;r)$ and the geometric realization $|\mathrm{VR}(X;r)|$ of the Vietoris–Rips simplicial complex. However, these two topological spaces will often not be homeomorphic (nor even homotopy equivalent).

Similar to the Vietoris–Rips thickening, we can construct the Čech thickening $\check{\mathrm{C}}^{\mathrm{m}}(X, Y; r)$ equipped with the 1-Wasserstein metric. The construction is exactly the same, except that the elements of $\check{\mathrm{C}}^{\mathrm{m}}(X, Y; r)$ are the convex combinations of vertices from simplices in $\check{\mathrm{C}}(X, Y; r)$ (rather than in $\mathrm{VR}(X; r)$). By [2, Lemma 3.5], both $\mathrm{VR}^m(X; r)$ and $\check{\mathrm{C}}^{\mathrm{m}}(X, Y; r)$ are r-thickenings of the metric space X.

One could alternatively consider a p-Wasserstein metric for $1 \leq p \leq \infty$.

3.2.2 Riemannian Manifolds

The main result of [2] is to show that for a Riemannian manifold M, the Vietoris–Rips thickening of M is homotopic to M itself.

Similar to Hausmann's theorem, assume a manifold M with sufficient conditions on its curvature and on geodesics connecting points in M. In general, the idea is to make sure that small balls contain minimal geodesics connecting every two points, in a manner analogous to them being convex. If ρ is a real number such that balls of radius ρ satisfy this condition, and ρ and the curvature of M are both sufficiently bounded, then we have the following:

Theorem 3.2.2. *If M is a complete Riemannian manifold and $r > 0$ is sufficently small, then $\mathrm{VR}^m(M; r) \simeq M$.*

The most salient feature of the proof is to identify the map which gives a homotopy equivalence. While Hausman's original result does not have an explicit inverse map $M \to \mathrm{VR}(M; \epsilon)$, here the map in that direction is simply the inclusion map $M \hookrightarrow \mathrm{VR}^m(M; r)$. Since $\mathrm{VR}^m(M; r)$ is a metric thickening, this map is continuous. (M can be realized exactly as the vertex set of $\mathrm{VR}^m(M; r)$, and the metric on $\mathrm{VR}^m(M; r)$ restricts to the original metric on M.) In the other direction, there is a map $g \colon \mathrm{VR}^m(M; r) \to M$ given by Karcher means: given a ball $\mathrm{B}(m, \rho) \subseteq M$ and any measurable map $f \colon A \to \mathrm{B}(m, \rho)$ where A is some probability space, the function $P_f \colon \overline{\mathrm{B}}(m, \rho) \to \mathbb{R}$ defined by

$$P_f(m') = \frac{1}{2} \int_A d(m', f(a))^2 \, \mathrm{d}a$$

has a unique minimum [17] called the Karcher mean, C_f. Recall that elements of $\mathrm{VR}^m(M;r)$ can be thought of as probability densities. In particular, if $x = \sum_{i=0}^{k} \lambda_i x_i$ then we can think of x as a mass distribution $f_x \colon \{0, \ldots, k\} \to \mathrm{B}(x_0, \rho,)$, where the i maps to x_i (if $r < \rho$ all the vertices are contained in this ball). Thus we can use the Karcher mean to define a map $g \colon \mathrm{VR}^m(M;r) \to M$ by $x \mapsto C_{f_x}$. That this map is continuous and a homotopy inverse to the inclusion are shown in [2, Section 4].

For the metric Čech complex given the same restrictions on M, the analogous maps also provide homotopy equivalence, giving

Theorem 3.2.3. *If M is a complete Riemannian manifold and $r > 0$ sufficiently small, then $\check{\mathrm{C}}^{\mathrm{m}}(M;r) \simeq M$.*

Chapter 4

Results

4.1 A metric analogue of Hausmann's result

We now present our main theorem, a metric analogue of Hausmann's result for Vietoris–Rips thickenings of subsets of Euclidean space with positive reach. Since in Section 4.2 we will also give an analogous theorem for the metric Čech thickening, we provide some notation now for both cases. Let $X \subseteq \mathbb{R}^n$ be a set of positive reach. Let $K(X;r)$ be either a Vietoris–Rips complex or Čech complex of X with scale parameter r, and let $K^m(X;r)$ be the corresponding metric Vietoris–Rips or Čech thickening. Define $f\colon K^m(X;r) \to \mathbb{R}^n$ to be the linear projection map $f\left(\sum_i \lambda_i x_i\right) = \sum_i \lambda_i x_i \in \mathbb{R}^n$, where the first sum is a formal convex combination of points in X, and the second sum is the standard addition of vectors in $\mathbb{R}^n$. Recall $\pi\colon \mathrm{Tub}_\tau \to X \subseteq \mathbb{R}^n$ is the nearest-point projection map.

Several geometric lemmas are required.

Lemma 4.1.1. *Let $x_0, \ldots, x_k \in \mathbb{R}^n$, let $y \in \operatorname{conv}\{x_0, \ldots, x_k\}$, and let C be a convex set with $y \notin C$. Then there is at least one x_i with $x_i \notin C$.*

Proof. Suppose for a contradiction that we had $x_i \in C$ for all $i = 0, \ldots, k$. Then since C is convex, we'd also have $y \in \operatorname{conv}\{x_0, \ldots, x_k\} \subseteq C$. Hence it must be the case that $x_i \notin C$ for some i. □

Lemma 4.1.2. *For $X \subseteq \mathbb{R}^n$ and $r > 0$, the map $f\colon \mathrm{VR}^m(X;r) \to \mathbb{R}^n$ has its image contained in $\overline{\mathrm{Tub}_r}$.*

Proof. Let $x = \sum_{i=0}^k \lambda_i x_i \in \mathrm{VR}^m(X;r)$; we have

$$\operatorname{diam}(\operatorname{conv}\{x_0, \ldots, x_k\}) = \operatorname{diam}([x_0, \ldots, x_k]) \le r.$$

Since $f(x) \in \operatorname{conv}\{x_0, \ldots, x_k\}$, it follows that $d(f(x), X) \leq d(f(x), x_0) \leq r$, and so $f(x) \in \overline{\operatorname{Tub}_r}$. $\square$

The substance of Lemma 4.1.3 will be that if $[x_0, \ldots, x_k]$ is a simplex in $\operatorname{VR}(X; r)$ and if $x = \sum_i \lambda_i x_i \in \operatorname{VR}^m(X; r)$, then $\pi(f(x))$ will be "close enough" to $x_0, \ldots, x_k$ so that $[x_0, \ldots, x_k, \pi(f(x))]$ is also a simplex in $\operatorname{VR}(X; r)$. This fact will be crucial for defining the homotopy equivalences in our proof of Theorem 4.1.4.

Lemma 4.1.3. *Let $X \subseteq \mathbb{R}^n$ have positive reach τ, let $[x_0, \ldots x_k]$ be a simplex in $\operatorname{VR}(X; r)$ with $r < \tau$, let $x = \sum \lambda_i x_i \in \operatorname{VR}^m(X; r)$, and let $p = \pi(f(x))$. Then the simplex $[x_0, \ldots, x_k, p]$ is in $\operatorname{VR}(X; r)$.*

Proof. We write the proof for $\operatorname{VR}_{\leq}(X; r)$; an analogous proof works for $\operatorname{VR}_{<}(X; r)$. Note $p = \pi(f(x))$ is defined by Lemma 4.1.2 since $\overline{\operatorname{Tub}_r} \subseteq \operatorname{Tub}_\tau$. We may assume $p \neq f(x)$, since otherwise the conclusion follows as $f(x)$ is in the convex hull of the x_i.

Suppose for a contradiction that $d(x_i, p) > r$ for some i; without loss of generality we may assume $i = 0$. Since $d(x_0, f(x)) \leq r$ we have that $f(x) \neq p$. Following [21], let $c = p + \tau \frac{f(x) - p}{\|f(x) - p\|}$, and let $\operatorname{B}(c, \tau)$ be the open ball of radius τ that is tangent to X at p. By Proposition 2.9.2 this open ball does not intersect X, giving $x_0, \ldots, x_k \notin \operatorname{B}(c, \tau)$. Define $T_p^\perp$ to be the line through $f(x)$ and p. Since $f(x)$ is between p and c on $T_p^\perp$, note that $d(x_0, f(x)) \leq r$ implies x_0 is not on $T_p^\perp$. Let $x_0' \neq x_0$ be the closest point on $T_p^\perp$ to x_0. Let $H_{x_0} = \{z \in \mathbb{R}^n \mid \langle z - x_0', x_0 - x_0' \rangle > 0\}$ be the open half-space containing x_0, whose boundary is the hyperplane containing $T_p^\perp$ that's perpendicular to $x_0 - x_0'$. Since $d(x_0, p), d(x_0, c) > r$, it follows that $H_{x_0}^c \cap \overline{\operatorname{B}}(x_0, r) \subseteq \operatorname{B}(c, \tau)$. Since $x_i \in \overline{\operatorname{B}}(x_0, r) \setminus \operatorname{B}(c, \tau)$, this implies that $x_i \in H_{x_0}$ for all i. This contradicts Lemma 4.1.1 since H_{x_0} is convex with $f(x) \notin H_{x_0}$, even though $f(x) \in \operatorname{conv}(\{x_0, \ldots, x_k\})$. Hence it must be the case that $d(x_0, p) \leq r$, and it follows that $[x_0, \ldots, x_k, p]$ is a simplex in $\operatorname{VR}_{\leq}(X; r)$. $\square$

We are now prepared to prove our main result.

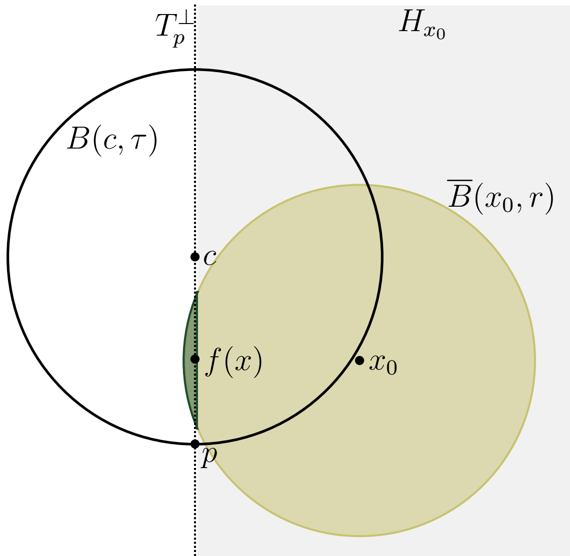

Figure 4.1: Figure for the proof of Lemma 4.1.3. The green shaded region is a subset of $\mathrm{B}(c,\tau)$, forcing all x_i to be in the same half-space.

Theorem 4.1.4. *Let X be a subset of Euclidean space $\mathbb{R}^n$, equipped with the Euclidean metric, and suppose the reach τ of X is positive. Then for all $r<\tau$, the metric Vietoris–Rips thickening $\mathrm{VR}^m(X;r)$ is homotopy equivalent to X.*

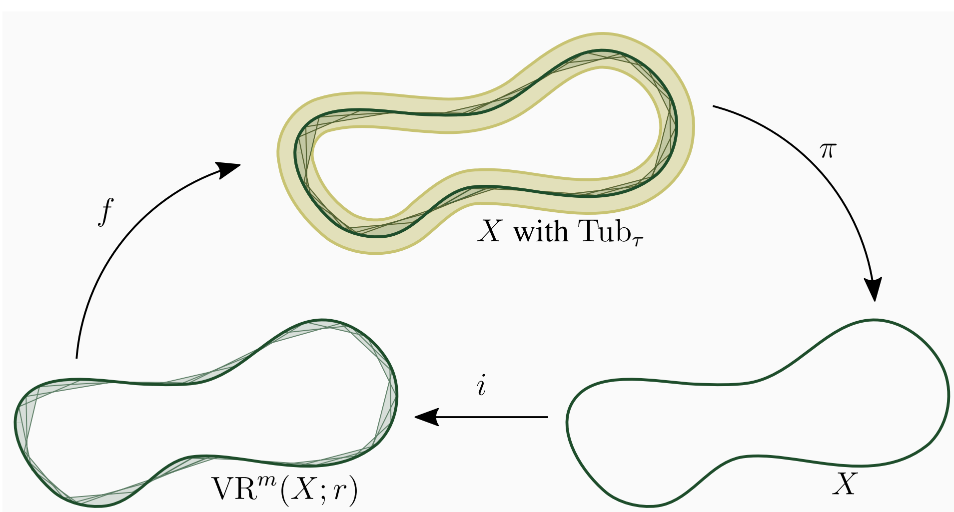

Figure 4.2: The homotopy equivalence between $\mathrm{VR}^m(X;r)$ and X in Theorem 4.1.4.

Proof. By [2, Lemma 5.2], map $f\colon \mathrm{VR}^m(X;r) \to \mathbb{R}^n$ is 1-Lipschitz and hence continuous. It follows from Lemma 4.1.2 that the image of f is a subset of Tub_τ. By Lemma 2.9.1 we have that $\pi\colon \mathrm{Tub}_\tau \to X$ is continuous. Let $\iota\colon X \to \mathrm{VR}^m(X;r)$ be the natural inclusion map.

We will show that ι and $\pi \circ f$ are homotopy inverses. Note that $\pi \circ f \circ i = \mathrm{id}_X$. Consider the map $H\colon \mathrm{VR}^m(X;r) \times I \to \mathrm{VR}^m(X;r)$ defined by $H(x,t) = t \cdot \mathrm{id}_{\mathrm{VR}^m(X;r)} + (1-t)\iota \circ \pi \circ f$. Map H is well-defined by Lemma 4.1.3, and continuous by [2, Lemma 3.8] since $\pi \circ f$ is continuous. It follows that H is a homotopy equivalence from $\iota \circ \pi \circ f$ to $id_{\mathrm{VR}^m(X;r)}$, and hence $\mathrm{VR}^m(X;r)$ is homotopy equivalent to X. □

4.2 A metric analogue of the nerve lemma

We handle the case of Čech thickenings in a similar fashion in this section. Recall we write $\check{\mathrm{C}}^{\mathrm{m}}(X;r)$ for either the ambient Čech complex $\check{\mathrm{C}}^{\mathrm{m}}(X,\mathbb{R}^n;r)$ or the intrinsic Čech complex $\check{\mathrm{C}}^{\mathrm{m}}(X,X;r)$ when the distinction is not important.

Lemma 4.2.1. *Let $[x_0,\ldots,x_k]$ be a simplex in $\check{\mathrm{C}}(X;2r)$. Then for any $x \in \mathrm{conv}([x_0,\ldots,x_k])$, there exists a vertex x_i such that $d(x,x_i) \le r$.*

Proof. We follow the proof of [9, Lemma 2.9] closely. By assumption, balls of radius r centered at the points x_i meet at a common point y. Let $x = \sum_i \lambda_i x_i$ be a point in $\mathrm{conv}([x_0,\ldots,x_k])$. Rewrite this as $\vec{0} = \lambda_0\hat{x}_0 + \cdots + \lambda_k\hat{x}_k$ where $\hat{x}_i = x_i - x$. Also let $\hat{y} = y - x$. Taking the dot product with $\hat{y}$ gives

$$0 = \lambda_0\langle \hat{x}_0, \hat{y}\rangle + \cdots + \lambda_k\langle \hat{x}_k, \hat{y}\rangle.$$

So for some i we have $\langle \hat{x}_i, \hat{y}\rangle \le 0$. In that case,

$$r^2 \ge d(x_i,y)^2 = d(\hat{x}_i,\hat{y})^2 = \|\hat{x}_i\|^2 - 2\langle \hat{x}_i,\hat{y}\rangle + \|\hat{y}\|^2 \ge \|\hat{x}_i\|^2 = \|x_i - x\|^2.$$

□

Lemma 4.2.2. *For $X \subseteq \mathbb{R}^n$ and $r > 0$, the map $f\colon \check{\mathrm{C}}^{\mathrm{m}}(X;2r) \to \mathbb{R}^n$ has its image contained in $\overline{\mathrm{Tub}_r}$.*

Proof. For any point $x = \sum_i \lambda_i x_i \in \check{\mathrm{C}}^{\mathrm{m}}(X; 2r)$ we have that $f(x) \in \mathrm{conv}(\{x_0, \ldots, x_k\})$. The result then follows form Lemma 4.2.1. $\square$

Lemma 4.2.3. *Let $X \subseteq \mathbb{R}^n$ have positive reach τ, let $[x_0, \ldots x_k]$ be a simplex in $\check{\mathrm{C}}(X, \mathbb{R}^n; 2r)$ with $r < \tau$, let $x = \sum \lambda_i x_i \in \check{\mathrm{C}}^{\mathrm{m}}(X, \mathbb{R}^n; 2r)$, and let $p = \pi(f(x))$. Then the simplex $[x_0, \ldots, x_k, p]$ is in $\check{\mathrm{C}}(X, \mathbb{R}^n; 2r)$.*

Proof. We write the proof for $\check{\mathrm{C}}_{\leq}(X, \mathbb{R}^n; 2r)$; an analogous proof works for $\check{\mathrm{C}}_{<}(X, \mathbb{R}^n; 2r)$. Since $[x_0, \ldots, x_k]$ is a simplex in $\check{\mathrm{C}}_{<}(X, \mathbb{R}^n; 2r)$, there exists a ball $\overline{\mathrm{B}}(y, r)$ of radius r centered at some point $y \in \mathbb{R}^n$ such that $x_i \in \overline{\mathrm{B}}(y, r)$ for all i. Also note that $f(x) \in \mathrm{conv}\{x_0, \ldots, x_k\} \subseteq \overline{\mathrm{B}}(y, r)$. Note $p = \pi(f(x))$ is defined by Lemma 4.2.2. We may assume $p \neq f(x)$, since otherwise the conclusion follows from $f(x) \in \overline{\mathrm{B}}(y, r)$. Similarly, we know that $d(p, f(x)) \leq r$ since $d(x_i, f(x)) \leq r$ for some i and since p is the closest point in X to $f(x)$.

Suppose for a contradiction that $p \notin \overline{\mathrm{B}}(y, r)$. Let $c = p + \tau \frac{f(x)-p}{\|f(x)-p\|}$, and let $\mathrm{B}(c, \tau)$ be the open ball with center c and radius τ that is tangent to X at p. By Proposition 2.9.2 every x_i must be in $\overline{\mathrm{B}}(y, r) \setminus \mathrm{B}(c, \tau)$. Let $T_p^{\perp}$ be the line through $f(x)$ and p. We claim that y cannot lie on $T_p^{\perp}$. Indeed if y were on $T_p^{\perp}$ its location would be limited to one of three line segments — one with p between y and $f(x)$, one with y between p and $f(x)$, and one with $f(x)$ between p and y. The first cannot occur as we would have $d(p, y) \leq d(f(x), y) \leq r$. The second cannot occur as we would have $d(p, y) \leq d(p, f(x)) \leq r$. Finally, the third cannot occur because either $d(p, y) < 2\tau - r$ and so the ball $\overline{\mathrm{B}}(y, r)$ is contained in $\mathrm{B}(c, \tau)$ and thus cannot contain any vertex x_i in contradiction of the definition of y, or $d(p, y) \geq 2\tau - r$, in which case $d(f(x), \overline{\mathrm{B}}(y, r) \setminus \mathrm{B}(c, \tau)) > r$ which contradicts Lemma 4.2.1.

Let $y' \neq y$ be the closest point on $T_p^{\perp}$ to y. Let $H_y = \{z \in \mathbb{R}^n \mid \langle z - y', y - y' \rangle > 0\}$ be the open half-space containing y, whose boundary is the hyperplane containing $T_p^{\perp}$ that's perpendicular to $y - y'$. Since $f(x) \in \mathrm{B}(y, r)$ and $p \notin \mathrm{B}(y, r)$, we have $\overline{\mathrm{B}}(y, r) \setminus \mathrm{B}(c, \tau) \subseteq H_y$, which implies $x_i \in H_y$ for all i. This contradicts Lemma 4.1.1 since H_y is convex with $f(x) \notin H_y$, even though

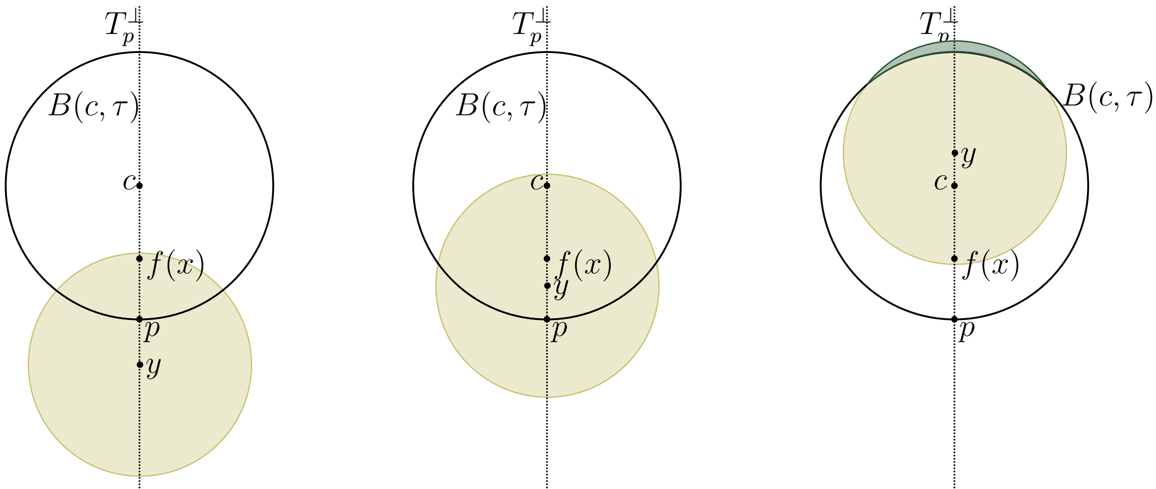

Figure 4.3: (*Left*) If p is between y and $f(x)$, then $\overline{\mathrm{B}}(y,r)$ contains p. (*Middle*) If y is between p and $f(x)$ then again $\overline{\mathrm{B}}(y,r)$ contains p. (*Right*) If $f(x)$ is between y and p, then the green region $\overline{\mathrm{B}}(y,r) \setminus \mathrm{B}(c,\tau)$ is either empty or too far from $f(x)$.

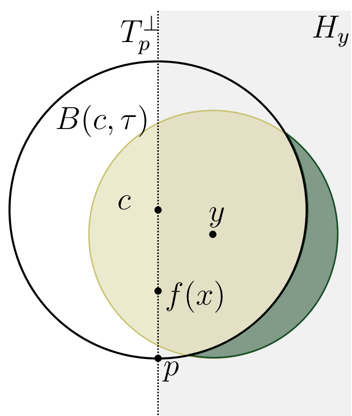

Figure 4.4: Figure for the proof of Lemma 4.2.3. The green region $\overline{\mathrm{B}}(y,r) \setminus \overline{\mathrm{B}}(c,\tau)$ is entirely contained in H_y.

$f(x) \in \mathrm{conv}(\{x_0,\ldots,x_k\})$. Hence it must be the case that $p \in \overline{\mathrm{B}}(y,r)$, and so $[x_0,\ldots,x_k,p]$ is a simplex in $\check{\mathrm{C}}_{\leq}(X,\mathbb{R}^n;2r)$. $\square$

An analogous lemma holds for the intrinsic Čech complex.

Lemma 4.2.4. *Let $X \subseteq \mathbb{R}^n$ have positive reach τ, let $[x_0,\ldots x_k]$ be a simplex in $\check{\mathrm{C}}(X,X;2r)$ with $r<\tau$, let $x=\sum \lambda_i x_i \in \check{\mathrm{C}}^{\mathrm{m}}(X,X;2r)$, and let $p=\pi(f(x))$. Then the simplex $[x_0,\ldots,x_k,p]$ is in $\check{\mathrm{C}}(X,X;2r)$.*

Proof. As in the ambient case, we write the proof for $\check{\mathrm{C}}_{\leq}(X,X;2r)$; an analogous proof works for $\check{\mathrm{C}}_{<}(X,X;2r)$.

Since $[x_0,\ldots,x_k]$ is a simplex in $\check{\mathrm{C}}(X,X;2r)$, there exists a ball $\overline{\mathrm{B}}(y,r)$ of radius r centered at some point $y \in X$ such that $x_i \in \overline{\mathrm{B}}(y,r) \cap X$ for all i. Also note that $f(x) \in \mathrm{conv}\{x_0,\ldots,x_k\} \subseteq \overline{\mathrm{B}}(y,r)$, and again $p = \pi(f(x))$ is well-defined by Lemma 4.2.2. We may assume $p \neq f(x)$, since otherwise the conclusion follows trivially because $p \in X$ and $f(x) \in \overline{\mathrm{B}}(y,r)$, so we would have $p \in \overline{\mathrm{B}}(y,r) \cap X$. Also, we know that $d(p, f(x)) < r$ since $d(x_i, f(x)) \leq r$ for some i and since p is the closest point in X to $f(x)$.

Suppose for a contradiction that $p \notin \overline{\mathrm{B}}(y,r)$. Let $c = p + \tau \frac{f(x)-p}{\|f(x)-p\|}$, and let $\mathrm{B}(c,\tau)$ be the open ball with center c and radius τ that is tangent to X at p. By Proposition 2.9.2 every x_i must be in $\overline{\mathrm{B}}(y,r) \setminus \mathrm{B}(c,\tau)$. As above, let $T_p^{\perp}$ be the line through $f(x)$ and p.

We now claim that y cannot lie on $T_p^{\perp}$. Indeed, since $y \in X$, we would have either $y = p$ contradicting $p \notin \overline{\mathrm{B}}(y,r)$, or $d(y, f(x)) > \tau$ because $y \notin \mathrm{B}(c,\tau)$, contradicting $f(x) \in \overline{\mathrm{B}}(y,r)$.

Let $y' \neq y$ be the closest point on $T_p^{\perp}$ to y. Let $H_y = \{z \in \mathbb{R}^n \mid \langle z-y', y-y'\rangle > 0\}$ be the open half-space containing y, whose boundary is the hyperplane containing $T_p^{\perp}$ that's perpendicular to $y - y'$. Since $f(x) \in \mathrm{B}(y,r)$ and $p \notin \mathrm{B}(y,r)$, we have $\overline{\mathrm{B}}(y,r) \setminus \mathrm{B}(c,\tau) \subseteq H_y$, which implies $x_i \in H_y$ for all i. This contradicts Lemma 4.1.1 since H_y is convex with $f(x) \notin H_y$, even though $f(x) \in \mathrm{conv}(\{x_0,\ldots,x_k\})$. Hence it must be the case that $p \in \overline{\mathrm{B}}(y,r) \cap X$, and so $[x_0,\ldots,x_k,p]$ is a simplex in $\check{\mathrm{C}}(X,X;2r)$. $\square$

The following result is related to the nerve lemma, but it is not a consequence thereof. Indeed, even though the Čech simplicial complex $\check{\mathrm{C}}(X;2r)$ is the nerve of a collection of balls, the metric Čech thickening $\check{\mathrm{C}}^{\mathrm{m}}(X;2r)$ in general need not be homeomorphic nor even homotopy equivalent to the nerve $\check{\mathrm{C}}(X;2r)$.

Theorem 4.2.5. *Let X be a subset of Euclidean space $\mathbb{R}^n$, equipped with the Euclidean metric, and suppose the reach τ of X is positive. Then for all $r < \tau$, the metric Čech thickening $\check{\mathrm{C}}^{\mathrm{m}}(X;2r)$ is homotopy equivalent to X.*

Proof. We follow the same outline as for the Vietoris–Rips case. Map $f\colon \check{\mathrm{C}}^{\mathrm{m}}(X;2r) \to \mathbb{R}^n$ is again continuous by [2, Lemma 5.2]. It follows from Lemma 4.2.2 that the image of f is a subset of Tub_τ. By Lemma 2.9.1 we have that $\pi\colon \mathrm{Tub}_\tau \to X$ is continuous, and let $\iota\colon X \to \mathrm{VR}^m(X;r)$ be the natural inclusion map.

We will show that ι and $\pi \circ f$ are homotopy inverses. Note that $\pi \circ f \circ i = \mathrm{id}_X$. The continuous map $H\colon \check{\mathrm{C}}^{\mathrm{m}}(X;2r) \times I \to \check{\mathrm{C}}^{\mathrm{m}}(X;2r)$ given by $H(x,t) = t \cdot \mathrm{id}_{\check{\mathrm{C}}^{\mathrm{m}}(X;2r)} + (1-t)\iota \circ \pi \circ f$ is well-defined by Lemma 4.2.3 or 4.2.4 and is the necessary homotopy equivalence from $\iota \circ \pi \circ f$ to $id_{\check{\mathrm{C}}^{\mathrm{m}}(X;2r)}$. Hence $\check{\mathrm{C}}^{\mathrm{m}}(X;2r)$ is homotopy equivalent to X. $\square$

In the case of the metric Čech thickening, the bound $r < 2\tau$ is tight. For example, consider the zero sphere $\mathbb{S}^0 = \{-1,1\} \subseteq \mathbb{R}$. The reach of $\mathbb{S}^0$ is $\tau = 1$. At scale parameter $r = 2$ we have that $\check{\mathrm{C}}^{\mathrm{m}}_{\le}(\mathbb{S}^0, \mathbb{R}; 2) \cong [-1,1]$ is contractible, and hence not homotopy equivalent to $\mathbb{S}^0$.

4.3 Independent proofs for Riemannian manifolds

The main result of [2] (namely Theorem 4.2) is that if N is a Riemannian manifold with curvature bounded from above and below, then the thickening $\mathrm{VR}^m(N;r)$ (using the Riemannian metric) is homotopy equivalent to N for r sufficiently small. The proof proceeds by producing a map $\mathrm{VR}^m(N;r) \to N$ using Karcher or Fréchet means, and also provides a bound on scale r which is related the the convexity radius of N. By contrast, the results of our paper imply that if X is a Euclidean submanifold of positive reach, then for all r sufficiently small the Vietoris–Rips thickening $\mathrm{VR}^m(X;r)$ is homotopy equivalent to X. In this section we show that our results can be used to give an independent result for a Riemannian manifold N by using an isometric embedding of N into Euclidean space. Hence in some sense our results are stronger than [2, Theorem 4.2], though for N Riemannian we do not provide as explicit control over which scales r are sufficiently small.

If N is a Riemannian n-manifold with a C^3 positive metric, then a theorem by John Nash proves that N can be embedded isometrically into sufficiently high-dimensional Euclidean space [14, 20].

Corollary 4.3.1. *If N is a compact, n-dimensional Riemannian manifold with a C^k positive metric for $3 \leq k \leq \infty$, then there exists a $\tau > 0$ such that $\mathrm{VR}^m(N; r) \simeq N$ and $\check{\mathrm{C}}^{\mathrm{m}}(N, N; 2r) \simeq N$ for all $0 < r < \tau$.*

Proof. By [20, Theorem 2] there exists a C^k embedding $f\colon N \hookrightarrow \mathbb{R}^d$ with $d \leq \frac{n}{2}(3n + 11)$. The resulting compact C^2-smooth embedded submanifold $f(N)$ has positive reach τ in $\mathbb{R}^d$ [24, Proposition 14]. Thus Theorems 4.1.4 and 4.2.5 imply that $\mathrm{VR}^m(N; r) \simeq N$ and $\check{\mathrm{C}}^{\mathrm{m}}(N, N; 2r) \simeq N$ for all $0 < r < \tau$. $\square$

Chapter 5

Conclusion

Subsets of Euclidean space of positive reach are a class of objects of particular interest in topological data analysis, and in this paper we have shown that Vietoris–Rips and Čech thickenings of these spaces recover the same topological information as the space itself. Moreover, metric Vietoris–Rips and Čech thickenings retain the metric information of their vertex set, in stark contrast with the classical Vietoris–Rips and Čech simplicial complexes, and furthermore the metric thickenings have the advantage of allowing simpler (and explicit) constructions of the maps realizing homotopy equivalences. In addition to subsets of Euclidean space, our result also implies a version of the known results in the case of Riemannian manifolds—the primary object which the past literature has studied.

Several questions, however, remain open. In particular, Latschev's theorem [18] states that if X is Gromov–Hausdorff close to a manifold M, then an appropriate Vietoris–Rips complex of X is homotopy equivalent to the manifold. A metric analogue for Vietoris–Rips thickenings is currently known only when X is finite ([2, Theorem 4.4]), even though we expect the result to also be true for infinite X.

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