# Metric Thickenings and Group Actions

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# Introduction

Today, we're going to cover groups acting on metric spaces and the spaces we build on top of them: Vietoris–Rips complexes.

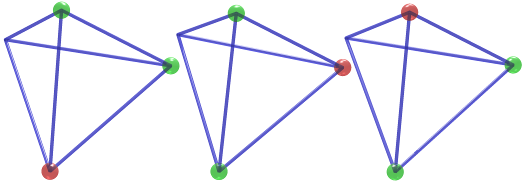
- First, we'll informally review some important definitions,
- and then cover our main result.
- If we have time, we'll talk about additional extensions of our results applied to the Vietoris Rips complexes of real projective *n*-space.

# Preliminaries

We will briefly cover the following definitions:

- Group Actions
- Metric Spaces
- Simplicial Complexes
- Vietoris–Rips Complexes
- Metric Thickenings
- Quotient Spaces
- Homotopy and Persistent Homology

# Group Actions

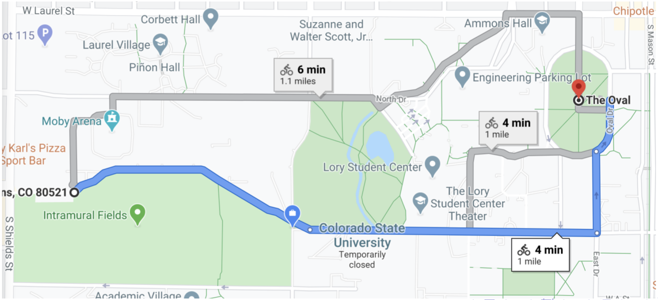


A group G acting on a set S is a map $\phi : G \times S \to S$ with the following properties:

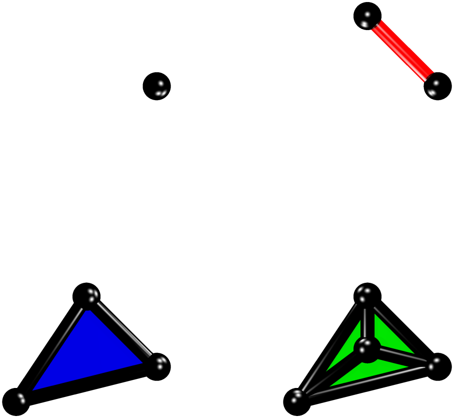
- $\phi(id, s) = s$
- $\phi(gh, s) = \phi(g, \phi(h, s))$

A metric space $(X, d)$ is a set $X$ equipped with a metric $d\colon X \times X \to \mathbb{R}$ satisfying the following properties:

- $d(x, y)$ is a nonnegative real number for all choices of $x$ and $y$ in $X$,
- $d(x, y)$ is zero if and only if $x = y$, and
- $d(x, y) + d(y, z) \geq d(x, z)$.

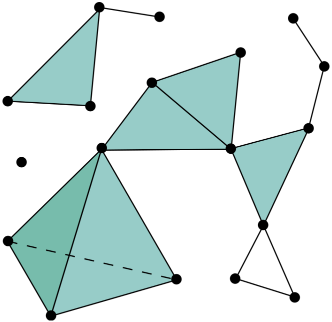


# Simplices



A simplex can be thought of as the convex hull of standard basis vectors in $\mathbb{R}^n$.

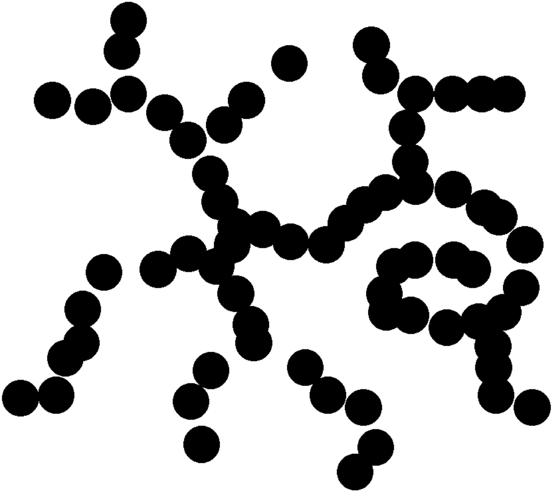
# Simplicial Complexes



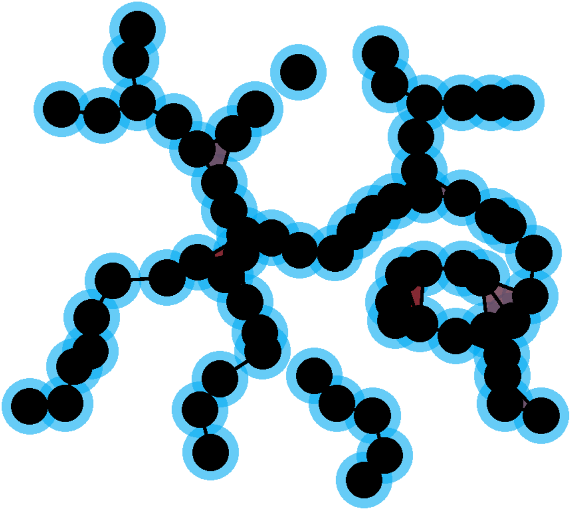
- A simplex on the vertices $v_0, v_1, v_2, \ldots, v_k$ may be thought of as the convex hull of these points when they are placed at the location of the standard basis vectors $e_i$ in Euclidean space.
- A *simplicial complex* is a union of simplices joined together by gluing maps.

https://en.wikipedia.org/wiki/Simplicial_complex

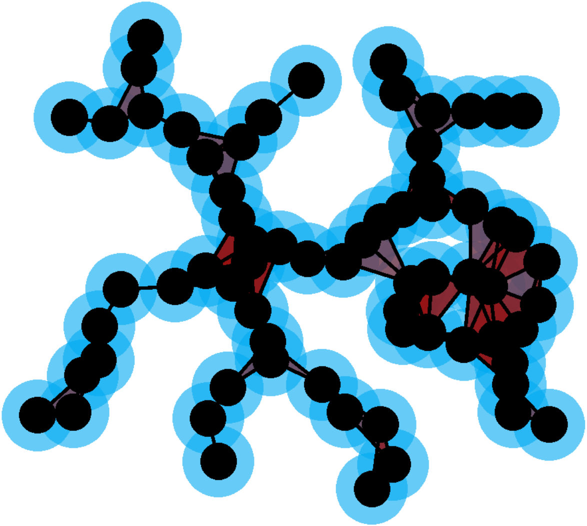
# Vietoris–Rips Complexes



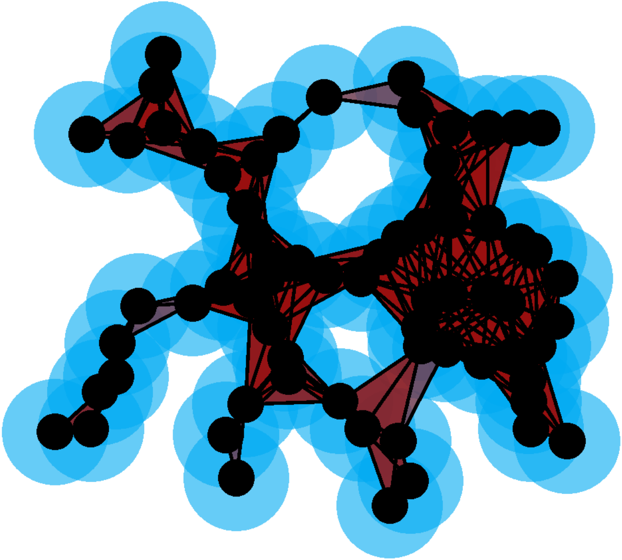
# Vietoris–Rips Complexes



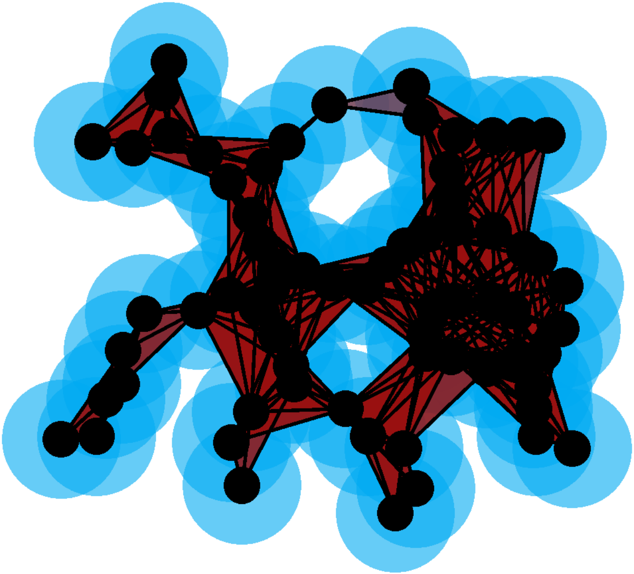
# Vietoris–Rips Complexes



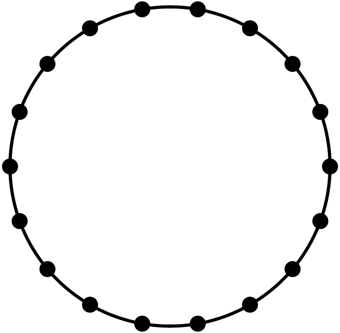
# Vietoris–Rips Complexes



# Vietoris–Rips Complexes

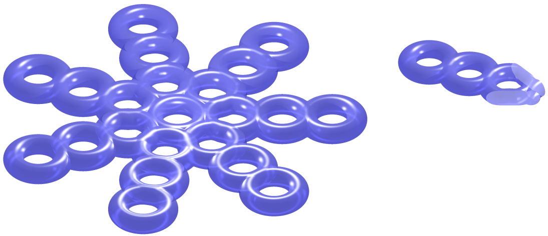


# Vietoris–Rips Metric Thickenings



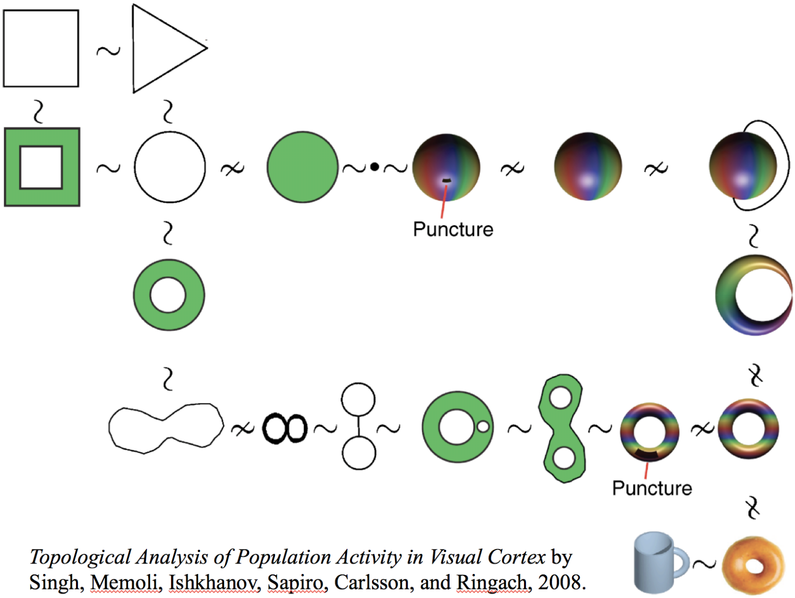
The Vietoris–Rips metric thickening $\mathrm{VR}^m(X;r)$ is a related construction that behaves better than the simplicial complex $\mathrm{VR}(X;r)$ when the underlying metric space $X$ is not finite.

# Quotient Spaces



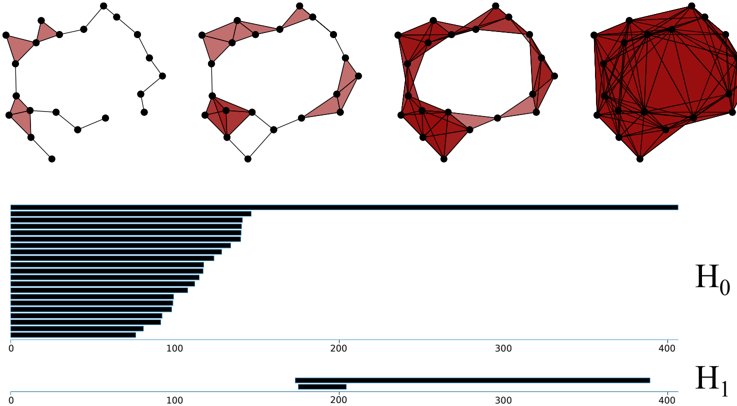
The group action here is the group $C_7$.

# Homotopy




*Topological Analysis of Population Activity in Visual Cortex* by Singh, Memoli, Ishkhanov, Sapiro, Carlsson, and Ringach, 2008.

# Persistent Homology




- Input: Increasing spaces. Output: barcode.
- Significant features persist.
- Cubic computation time in the number of simplices.

# Introduction

- In applied and computational topology, Vietoris–Rips complexes have been used to recover the "shape" of a dataset.
- If $X$ is a sufficiently nice sample from an unknown underlying space $M$, then one can recover the homotopy types, homology groups, or approximate persistent homology of $M$ from $X$.

# Group Action Types

We list increasingly stringent properties that the action of $G$ on $X$ could satisfy:

- The action of $G$ on $X$ is *free* if $g \cdot x = x$ for any $x \in X$ implies that $g$ is the identity element in $G$.
- The action of $G$ on $X$ is a *covering space action* if every point $x \in X$ has a neighborhood $U \ni x$ such that if $U \cap g \cdot U \neq \emptyset$, then $g$ is the identity element in $G$.
- Let $t > 0$. The action of $G$ on a metric space $X$ is an *t-diameter action* when for any nonnegative integer $k$, $\mathrm{diam}_{X/G}\{[x_0], \ldots, [x_k]\} < t$ implies that there exists a unique choice of elements $g_i \in G$ for $1 \leq i \leq k$ such that $\mathrm{diam}_X\{x_0, g_1 \cdot x_1 \ldots, g_k \cdot x_k\} = \mathrm{diam}_{X/G}\{[x_0], \ldots, [x_k]\}$.

# Main Results

Our main theorem is listed here:

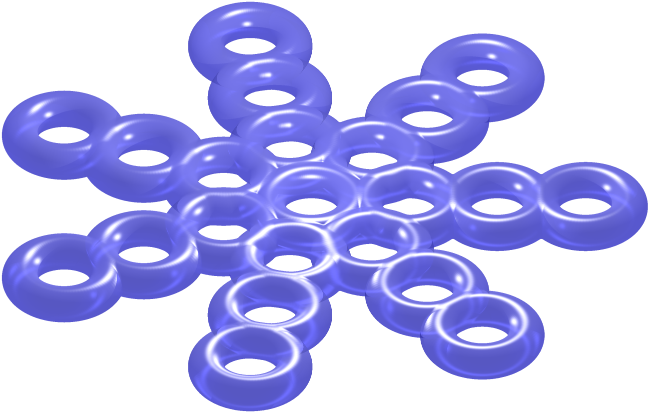
**Proposition**

*Let $G$ be a group acting properly and by isometries on a metric space $X$. If the action is a $t$-diameter action, then*

- $\mathrm{VR}_<(X/G; r)$ *is isomorphic to* $\mathrm{VR}_<(X; r)/G$ *for all* $r \leq t$,
- $\mathrm{VR}_\leq(X/G; r)$ *is isomorphic to* $\mathrm{VR}_\leq(X; r)/G$ *for all* $r < t$,
- $\mathrm{VR}^m_<(X/G; r)$ *is homeomorphic to* $\mathrm{VR}^m_<(X; r)/G$ *for all* $r \leq t$, *and*
- $\mathrm{VR}^m_\leq(X/G; r)$ *is homeomorphic to* $\mathrm{VR}^m_\leq(X; r)/G$ *for all* $r < t$.

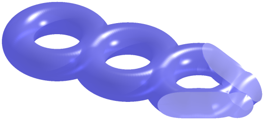
We have analogous results for Čech complexes, omitted here.

# An Example: 22-holed Torus with $D_7$ Symmetry



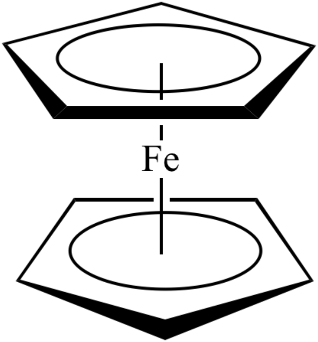
A 22-holed torus in $\mathbb{R}^3$. How would we compute $\mathrm{VR}_{\leq}(X; r)$ for small $r$?

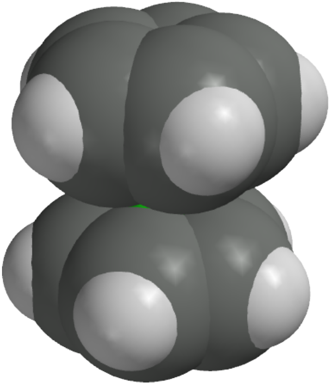
# An Example: Chopped Off Arm



A chopped off section of a 22-holed torus in $\mathbb{R}^3$.

# An Applied Example: Ferrocene, with $C_5$ symmetry



http://www.chem.ucla.edu/∼harding/IGOC/F/ferrocene.html

# Proof of Main Result

- We first consider the case of the Vietoris–Rips simplicial complexes.
- Consider the simplicial map $h \colon \mathrm{VR}(X; r) \to \mathrm{VR}(X/G; r)$ defined by $h(x) = [x]$
- It's well-defined since $G$ acts isometrically.
- If two points in the geometric realization of $\mathrm{VR}(X; r)$ are in the same orbit of the $G$ action, then they have the same image under $h$.
- It follows that $h$ induces a map $\tilde{h} \colon \mathrm{VR}(X; r)/G \to \mathrm{VR}(X/G; r)$.
- We will show that $\tilde{h}$ is a homeomorphism.

# Final Elements of the Proof

We need to show the following two facts.

1. Map $\tilde{h}$ is surjective.
2. Map $\tilde{h}$ is injective.

# Surjectivity of the Map

For surjectivity, note that $\tilde{h}$ is surjective if $h$ is surjective. The map $h$ is surjective because given any simplex

$$\sigma = \{[x_0], \dots, [x_k]\} \in \mathrm{VR}(X/G; r),$$

by the definition of an $r$-diameter action there exists a simplex $\sigma' = \{x_0, g_1 \cdot x_1 \dots, g_k \cdot x_k\} \in \mathrm{VR}(X; r)$ with $h(\sigma') = \sigma$.

# Injectivity of the Map

For injectivity, consider any two points $[\sum \lambda_i x_i], [\sum \lambda'_j x'_j] \in \mathrm{VR}(X;r)/G$ with $\tilde{h}([\sum \lambda_i x_i]) = \tilde{h}([\sum \lambda'_j x'_j])$. This means that

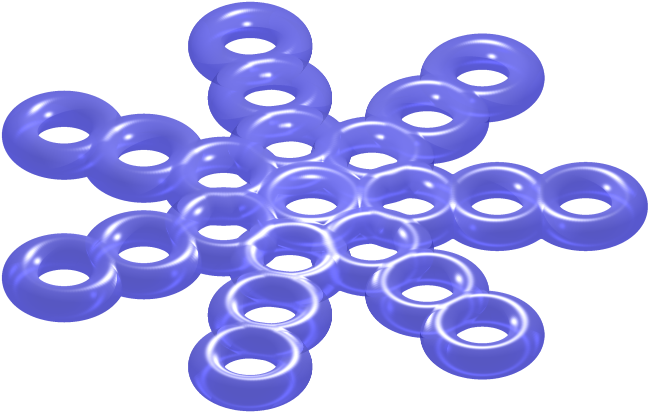
$$\sum \lambda_i [x_i] = h(\sum \lambda_i x_i) = h(\sum \lambda'_j x'_j) = \sum \lambda'_j [x'_j].$$

From the "uniqueness" part of the definition of an $r$-diameter action, given any simplex $\sigma = \{[x_0], \ldots, [x_k]\} \in \mathrm{VR}(X/G;r)$, there exists a unique simplex $\tilde{\sigma} = \{x_0, g_1 \cdot x_1 \ldots, g_k \cdot x_k\} \in \mathrm{VR}(X;r)$ containing $x_0$ with $h(\sigma') = \sigma$ and hence a unique simplex $\sigma'' \in \mathrm{VR}(X;r)/G$ with $\tilde{h}(\sigma'') = \sigma$.

# Modified Proof for Metric Thickenings

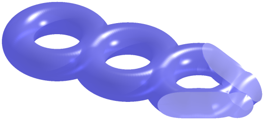
For the case of Vietoris–Rips metric thickenings, we consider the analogous map $h\colon \mathrm{VR}^m(X;r) \to \mathrm{VR}^m(X/G;r)$ defined by $h(\sum \lambda_i x_i) = \sum \lambda_i [x_i]$; this map is well-defined since $G$ acts isometrically. The only additional observation to make in this case is that both $h$ and its inverse are continuous.

# An Example: 22-holed Torus



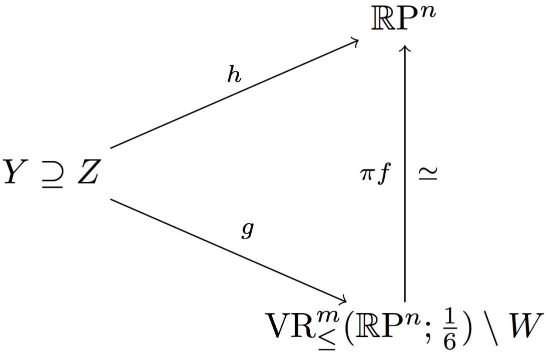
A 22-holed torus in $\mathbb{R}^3$.

# An Example: Chopped Off Arm



A chopped off section of a 22-holed torus in $\mathbb{R}^3$.

# Another Example: Real Projective Space




$$\mathrm{VR}^m_\leq(\mathbb{R}\mathrm{P}^n; \tfrac{1}{6}) = \left(\mathrm{VR}^m_\leq(\mathbb{R}\mathrm{P}^n; \tfrac{1}{6}) \setminus W\right) \cup_g Y \simeq \mathbb{R}\mathrm{P}^n \cup_h Y.$$

# Conclusion

- For $X$ a metric space equipped with a group action by $G$, we show how to relate the Vietoris–Rips complexes $\mathrm{VR}(X; r)$ and $\mathrm{VR}(X/G; r)$ at small scales $r$.
- We use this to identify homotopy types of Vietoris–Rips thickenings of $\mathbb{RP}^n$ at larger scale parameters than were previously known.

Henry Adams, Mark Heim, Chris Peterson. *Metric Thickenings and Group Actions*. Submitted and available at `arXiv:1911.00732`, 2020.