Metric Reconstruction Via Optimal Transport[*]

Michał Adamaszek[†], Henry Adams[‡], and Florian Frick[§]

Abstract. Given a sample of points X in a metric space M and a scale $r > 0$, the Vietoris–Rips simplicial complex $\mathrm{VR}(X; r)$ is a standard construction to attempt to recover M from X up to homotopy type. A deficiency of this approach is that the Vietoris–Rips complex $\mathrm{VR}(X; r)$ is not metrizable if it is not locally finite, and thus does not recover metric information about the metric space M. We attempt to remedy this shortcoming by defining a metric space thickening of X, which we call the *Vietoris–Rips thickening* $\mathrm{VR}^m(X; r)$, via the theory of optimal transport. When M is a complete Riemannian manifold, or alternatively a compact Hadamard space, we show that the Vietoris–Rips thickening satisfies Hausmann's theorem ($\mathrm{VR}^m(X; r) \simeq M$ for r sufficiently small) with a simpler proof than Hausmann's original result: homotopy equivalence $\mathrm{VR}^m(X; r) \to M$ is canonically defined as a center of mass map, and its homotopy inverse is the (now continuous) inclusion map $M \hookrightarrow \mathrm{VR}^m(X; r)$. Furthermore, we describe the homotopy type of the Vietoris–Rips thickening of the n-sphere at the first positive scale parameter r where the homotopy type changes.

Key words. metric thickening, Vietoris–Rips complexes, Wasserstein metric, Karcher mean, homotopy type

AMS subject classifications. 53C23, 54E35, 55P10, 55U10

DOI. 10.1137/17M1148025

1. Introduction. Let X be a set of points sampled from a metric space M; the only information we retain about M is its metric restricted to $X \times X$. In general, it is impossible to reconstruct the homotopy type of M from this data, even if the set X is sufficiently dense (say an ε-net). A remarkable theorem of Latschev [25] states that if M is a closed Riemannian manifold and X is sufficiently close to M in the Gromov–Hausdorff distance, then one can recover the homotopy type of M. Indeed, consider the Vietoris–Rips complex $\mathrm{VR}(X; r)$, which has as its simplices the finite subsets of X of diameter less than $r > 0$. Latschev proves that for certain small values of r, complex $\mathrm{VR}(X; r)$ is homotopy equivalent to M. The Vietoris–Rips complex was introduced independently by Vietoris who defined a homology theory for compact metric spaces (see [41] and [26, Chapter VII, section 5]), and by Rips who showed that torsion-free hyperbolic groups have finite Eilenberg–MacLane spaces [5, Theorem III, eq. 3.21].

[†]Department of Mathematics, University of Copenhagen, Universitetsparken 5,2100 Copenhagen, Denmark (aszek@mimuw.edu.pl).

[‡]Department of Mathematics, Colorado State University, Fort Collins, CO 80523 (adams@math.colostate.edu).

[§]Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213 (frick@cmu.edu).

More recently, Vietoris–Rips complexes have been a commonly used tool in computational topology and persistent homology [16]. If M is a manifold, what properties of M can one recover when given only a finite data set X noisily sampled from M? Latschev's theorem motivates the construction of the Vietoris–Rips complex $\mathrm{VR}(X; r)$ as a proxy for the homotopy type of M, but unfortunately one does not know how to choose r appropriately without knowledge of the (unknown) curvature of M. In practice, computational topologists instead let the scale r vary from small to large, and compute the *persistent homology* of $\mathrm{VR}(X; r)$ (i.e., the homology of $\mathrm{VR}(X; r)$ as r changes) to get a multiresolution summary of the data. The topological perspective has aided the analysis of data arising from image processing [9], conformation spaces of molecules [31, 43], branched polymers [28], and sensor networks [14], for example. Helpful expository introductions to persistent homology include [8] and [18].

Though we started with a metric space X, the Vietoris–Rips complex $\mathrm{VR}(X; r)$ does not come equipped with a natural choice of metric. Indeed, $\mathrm{VR}(X; r)$ is not metrizable if it is not locally finite, meaning it is impossible to equip $\mathrm{VR}(X; r)$ with a metric without changing the homeomorphism type. We use optimal transport to build a family of metric spaces $\mathrm{VR}^m(X; r)$ —the *Vietoris–Rips thickenings*—from the knowledge of pairwise distances in X. In these metric spaces we can take abstract convex combinations of points in X whenever they are at a distance of less than r from one another, just as in $\mathrm{VR}(X; r)$. If $\mathrm{VR}(X; r)$ is not locally finite then $\mathrm{VR}^m(X; r)$ necessarily has a different (metrizable) topology. Furthermore, if M is a closed Riemannian manifold and X is Gromov–Hausdorff close to M, then $\mathrm{VR}^m(X; r)$ is not only homotopy equivalent to M (for appropriate small r), but also Gromov–Hausdorff close to M (Lemma 3.6). We prove the following (see Lemma 3.6, Corollary 6.4, Proposition 6.6, and Theorem 4.2).

Main Theorem 1.1. *Let X be a metric space, and let $r > 0$.*

(1) *Metric space $\mathrm{VR}^m(X; r)$ is an r-thickening of X; in particular, the Gromov–Hausdorff distance between X and $\mathrm{VR}^m(X; r)$ is at most r.*

(2) *If $\mathrm{VR}(X; r)$ is locally finite, then $\mathrm{VR}^m(X; r)$ is homeomorphic to $\mathrm{VR}(X; r)$.*

(3) *If X is discrete, then $\mathrm{VR}^m(X; r)$ is homotopy equivalent to $\mathrm{VR}(X; r)$.*

(4) *If M is a complete Riemannian manifold with curvature bounded from above and below, then $\mathrm{VR}^m(M; r)$ is homotopy equivalent to M for r sufficiently small.*

In the restricted setting where X is discrete, item (1) is stated by Gromov [19, eq. 1.B(c)]. Similar properties are discussed when X is a length space (and $\mathrm{VR}^m(X; r)$ is called a *polyhedral regularization of X*) by Burago, Burago, and Ivanov in [7, Example 3.2.9]. The proof of item (3) relies on the nerve lemma, and a much more general statement is given in Remark 6.5. Item (4) is an analogue of Hausmann's theorem [21, Theorem 3.5] for Vietoris–Rips thickenings, and also holds for compact Hadamard spaces (Remark 4.3). Whereas Hausmann's homotopy equivalence $\mathrm{VR}(M; r) \to M$ relies on the choice of a total ordering of all points in M, our homotopy equivalence $\mathrm{VR}^m(M; r) \to M$ is now canonically defined using Karcher or Fréchet means. Furthermore, the homotopy inverse of our map is the (now continuous) inclusion $M \hookrightarrow \mathrm{VR}^m(M; r)$. We prove that the compositions are homotopy equivalent to the corresponding identity maps by using linear homotopies.

Motivation. We provide the following motivation for our work. In applications of topology to data [8], one is given a sampling of points X from an unknown underlying space M

and would like to use X to recover information of M. There are a variety of theoretical guarantees [2, 4, 12, 25, 32] showing how Vietoris–Rips complexes and related constructions built on a sufficiently dense sampling X can be used to recover information such as the homology groups and homotopy types of M. However, since M is unknown one does not know if the assumptions needed for these results (such as having the Vietoris–Rips scale r be sufficiently small depending on the curvature of M) are satisfied. In practical applications, one often allows the scale r to vary from small to large and computes the persistent homology of $VR(X; r)$. This is reasonable because as X converges to M in the Gromov–Hausdorff distance, the persistent homology of $VR(X; r)$ converges to that of $VR(M; r)$ [11]. However, very little is known about the limiting object, the persistent homology of $VR(M; r)$, even when M is a manifold. Indeed, to the best of our knowledge the only connected noncontractible manifold M for which the persistent homology of $VR(M; r)$ is known at all scales r is the circle [1].[1] When M is an infinite metric space, we believe that the metric thickening $VR^m(M; r)$ is in several ways a more natural object than the simplicial complex $VR(M; r)$. As evidence for this claim, in section 5 we describe the first new homotopy type of Vietoris–Rips thickenings $VR^m(S^n; r)$ of higher-dimensional spheres as scale r increases. We conjecture (Conjecture 6.12) that these homotopy types are closely related to the (unknown) homotopy types of Vietoris–Rips complexes $VR(S^n; r)$ of higher-dimensional spheres, which determine what the persistent homology of a dataset X will converge to when X is a denser and denser sampling of a sphere.

We would like to clarify the relationship between using either the Vietoris–Rips complex $VR(X; r)$ or the Vietoris–Rips thickening $VR^m(X; r)$ in applications of topology to data analysis. For X finite, the persistent homology barcodes for $VR(X; r)$ and for $VR^m(X; r)$ coincide (Corollary 6.10). It is known that for X infinite, the persistent homology intervals for $VR(X; r)$ and $VR^m(X; r)$ can differ at endpoints (open endpoints versus closed endpoints, or vice-versa; see Remark 6.11). It is not known whether the two persistent homology intervals are identical after ignoring endpoints, although we conjecture this to be the case (Conjecture 6.12).[2] When X is infinite, it is difficult to determine the persistent homology of either $VR(X; r)$ or $VR^m(X; r)$. A disadvantage of the thickening is that Latschev's theorem (see [25]) is known only for X finite (Corollary 6.8), although it is conjectured also for X arbitrary (Conjecture 6.9). A second disadvantage of the thickening is that the stability of persistent homology (see [11, Theorem 5.2]) is known only for X and Y finite (Corollary 6.13), although it is conjectured also for X and Y arbitrary (Conjecture 6.14). Advantages of the thickening over the complex are that the thickening is always a metric space, the proof of Hausmann's theorem is more natural, Hausmann's theorem also extends to compact Hadamard spaces and to Euclidean submanifolds [3], there are example spaces whose Vietoris–Rips complexes have uncountable homology but whose thickenings have finite homology (Appendix C), and furthermore, we are able to determine the homotopy types of Vietoris–Rips thickenings of higher-dimensional spheres at larger scale parameters (Theorem 5.4). It is not yet clear

---

[1] And easy consequences thereof, such as when manifold M is an annulus or torus with a particular metric [1, section 10].

[2] And in the unexpected event that the persistent homology intervals for $VR(X; r)$ and $VR^m(X; r)$ can differ drastically, then it is not clear which one should be the primary object of interest.

whether $\mathrm{VR}(X; r)$ or $\mathrm{VR}^m(X; r)$ should be the primary object of study (despite our preferences for the latter), but understanding either space improves one's understanding of the other.

Organization. In section 2 we introduce our primary tool, the Wasserstein or Kantorovich metric on Radon measures of an arbitrary metric space; see Edwards [17] and Kellerer [23, 24]. We define Vietoris–Rips thickenings in section 3 and give some of their basic properties. Section 4 proves metric analogues of Hausmann's theorem and the nerve lemma. In section 5 we leverage this new metric viewpoint to determine the homotopy types of $\mathrm{VR}^m(S^n; r)$, for all spheres S^n, at the first positive scale parameter r where the homotopy type changes. In section 6 we study maps between simplicial complexes and thickenings, including metric analogues of Latschev's theorem and the stability of persistent homology.

Ivan Marin in [29] studies related constructions which produce geometric classifying spaces for a topological group that is furthermore metrizable.

2. Preliminaries.

Topological spaces. We write $Y \simeq Z$ for homotopy equivalent topological spaces Y and Z. Given a topological space Y and a subset $Z \subseteq Y$, let $\overline{Z}$ denote the closure of Z in Y. We denote the n-dimensional sphere by S^n and the closed n-dimensional ball by D^n. Given a topological space Y, let $C(Y)$ be its cone—for example, $C(S^n) = D^{n+1}$. Furthermore, let $\Sigma\, Y$ be the suspension of Y, and let $\Sigma^i\, Y$ be the i-fold suspension of Y. The joining of two topological spaces Y and Z is denoted $Y * Z$.

Metric spaces. Given a metric space (X, d), a point $x \in X$, and a real number $r \geq 0$, we let $B_X(x, r) = \{y \in X \mid d(y, x) < r\}$ (or $B(x, r)$ when the ambient space X is clear from context) denote the open ball of radius r about x. We let $d_{\mathrm{GH}}(X, Y)$ denote the Gromov-Hausdorff distance between metric spaces X and Y.

An r-*thickening* of a metric space X, as defined in [19, eq. 1.B], is a larger metric space $Z \supseteq X$ such that
- the distance function on Z extends that on X, and
- $d(z, X) \leq r$ for all $z \in Z$.

If Z is an r-thickening of X, it follows that $d_{\mathrm{GH}}(X, Z) \leq r$.

Simplicial complexes. Let K be a simplicial complex; we do not notationally distinguish between an abstract simplicial complex and its geometric realization. We let $V(K)$ denote the vertex set of K. If $V' \subseteq V(K)$, then we let $K[V']$ denote the induced simplicial complex on vertex set V'.

Vietoris–Rips and Čech complexes. Let X be a metric space and let $r \geq 0$. The Vietoris–Rips simplicial complex $\mathrm{VR}_<(X; r)$ (resp., $\mathrm{VR}_\leq(X; r)$) has X as its vertex set, and $\{x_0, \ldots, x_k\} \subseteq X$ as a simplex whenever $\mathrm{diam}(\{x_0, \ldots, x_k\}) < r$ (resp., $\mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r$). The Čech simplicial complex $\check{\mathrm{C}}_<(X; r)$ (resp., $\check{\mathrm{C}}_\leq(X; r)$) has X as its vertex set, and $\{x_0, \ldots, x_k\} \subseteq X$ as a simplex whenever $\bigcap_{i=0}^k B_X(x_i, r) \neq \emptyset$ (resp., $\bigcap_{i=0}^k \overline{B_X(x_i, r)} \neq \emptyset$). We write $\mathrm{VR}(X; r)$ or $\check{\mathrm{C}}(X; r)$ when a statement is true for either choice of inequality, $<$ or $\leq$, applied consistently throughout.

The Wasserstein or Kantorovich metric. All of the statements in this subsection follow from [17, 23, 24], and we mainly use the notation from [17]. Let (X, d) be an arbitrary metric space. A measure μ defined on the Borel sets of X is a *Radon measure* if it is inner regular, i.e., $\mu(B) = \sup\{\mu(K) \mid K \subseteq B \text{ is compact}\}$ for all Borel sets B, and if it is locally finite, i.e., every point $x \in X$ has a neighborhood U such that $\mu(U) < \infty$. Let $\mathcal{P}(X)$ denote the set of probability Radon measures such that for some (and hence all) $y \in X$, we have $\int_X d(x, y) \, d\mu < \infty$.

Define a metric on $X \times X$ by setting the distance between $(x_1, x_2), (x_1', x_2') \in X \times X$ to be $d(x_1, x_1') + d(x_2, x_2')$. Given $\mu, \nu \in \mathcal{P}(X)$, let $\Pi(\mu, \nu) \subseteq \mathcal{P}(X \times X)$ be the set of all probability Radon measures π on $X \times X$ such that $\mu(E) = \pi(E \times X)$ and $\nu(E) = \pi(X \times E)$ for all Borel subsets $E \subseteq X$.

Definition 2.1. *The 1-Wasserstein metric on $\mathcal{P}(X)$ is defined by*

$$d_{\mathcal{P}(X)}(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_{X \times X} d(x, y) \, d\pi.$$

In what follows we denote the metric $d_{\mathcal{P}(X)} \colon \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}$ by $d \colon \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}$, as it extends the metric $d \colon X \times X \to \mathbb{R}$.

The infimum in this definition is attained (see [17, page 388], and also [23, 24]). This metric, which gives a solution to the Monge–Kantorovich problem, has many names: the Kantorovich, Wasserstein, optimal transport, or earth mover's metric; see Vershik's survey [40].

A generalization of the Kantorovich–Rubinstein theorem for arbitrary (possibly noncompact) metric spaces, proven in [17, 23, 24], states that the 1-Wasserstein metric satisfies

$$(1) \qquad d(\mu, \nu) = \sup\left\{ \int_X f(x) \, d(\mu - \nu) \,\middle|\, f \colon X \to \mathbb{R}, \ \mathrm{Lip}(f) \leq 1 \right\}.$$

Here $\mathrm{Lip}(f)$ denotes the Lipschitz constant of f. As a consequence, Definition 2.1 indeed defines a metric on $\mathcal{P}(X)$.

Given a point $x \in X$, let $\delta_x \in \mathcal{P}(X)$ be the Dirac probability measure with mass one at x. The map $X \to \mathcal{P}(X)$ defined by $x \mapsto \delta_x$ is an isometry onto its image: for all points $x, y \in X$, we have

$$(2) \qquad\qquad\qquad d(\delta_x, \delta_y) = d(x, y).$$

In [42, page 99], Villani writes "Wasserstein distances incorporate a lot of the geometry of the space. For instance, the mapping $x \mapsto \delta_x$ is an *isometric* embedding ... but there are much deeper links." Our theorems, Theorems 4.2 and 4.4, are evidence that the Wasserstein distances, at small scales, incorporate not only geometry but also homotopy information.

3. The Vietoris–Rips thickening. Consider points $x_0, \ldots, x_k \in X$. Whenever we write $\mu = \sum_{i=0}^{k} \lambda_i \delta_{x_i}$, we assume that $\lambda_i \geq 0$ for all i and $\sum_i \lambda_i = 1$; hence μ is a probability measure. If we require each $\lambda_i > 0$, then this representation is unique. Measure μ is locally finite (as a probability measure it is in fact finite), and it follows that μ is a Radon measure. Furthermore, note that $\mu \in \mathcal{P}(X)$ since for any $y \in X$, we have $\int_X d(x, y) \, d\mu = \sum_{i=0}^{k} \lambda_i d(x_i, y) < \infty$.

Definition 3.1. *Let X be a metric space and let $r \geq 0$. The* Vietoris–Rips thickening *is the following submetric space of $\mathcal{P}(X)$, equipped with the restriction of the 1-Wasserstein metric:*

$$\mathrm{VR}^m_<(X;r) = \left\{ \sum_{i=0}^{k} \lambda_i \delta_{x_i} \in \mathcal{P}(X) \ \bigg| \ k \geq 1, \ \lambda_i \geq 0, \ \sum \lambda_i = 1, \ \mathrm{diam}(\{x_0, \ldots, x_k\}) < r \right\} \quad or$$

$$\mathrm{VR}^m_\leq(X;r) = \left\{ \sum_{i=0}^{k} \lambda_i \delta_{x_i} \in \mathcal{P}(X) \ \bigg| \ k \geq 1, \ \lambda_i \geq 0, \ \sum \lambda_i = 1, \ \mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r \right\}.$$

We include a superscript m to denote "metric." By convention $\mathrm{VR}^m_<(X;0)$ is the empty set, and $\mathrm{VR}^m_\leq(X;0)$ is equal to X as a metric space.

Thus the Vietoris–Rips thickening of a metric space X is the space of all (abstract) convex combinations of nearby points in X with the Wasserstein metric. A qualitative difference to the usual Vietoris–Rips complex is that the natural embedding $X \to \mathrm{VR}^m(X;r)$ is now a continuous map (and even an isometric embedding). In fact, we will naturally consider X to be a subset of $\mathrm{VR}^m(X;r)$ and write $\sum_{i=0}^{k} \lambda_i x_i$ for a point in $\mathrm{VR}^m(X;r)$ instead of $\sum_{i=0}^{k} \lambda_i \delta_{x_i}$.

Given two points $\mu = \sum_{i=0}^{k} \lambda_i x_i$ and $\mu' = \sum_{j=0}^{k'} \lambda'_j x'_j$ in $\mathrm{VR}^m(X;r)$, note that an element $\pi \in \Pi(x, x')$ can be written as

$$\pi = \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq k'}} \pi_{i,j} \delta_{x_i, x'_j} \quad \text{with} \quad \pi_{i,j} \geq 0, \quad \sum_j \pi_{i,j} = \lambda_i, \quad \text{and} \quad \sum_i \pi_{i,j} = \lambda'_j.$$

The cost of a matching in this finite setting is $\int_{X \times X} d(x,y) \, d\pi = \sum_{i,j} \pi_{i,j} d(x_i, x'_j)$. One can analogously define a p-Wasserstein metric on $\mathrm{VR}^m(X;r)$ for any $1 \leq p \leq \infty$.

Remark 3.2. The Vietoris–Rips thickening $\mathrm{VR}^m(X;r)$ need not be homeomorphic to its corresponding complex $\mathrm{VR}(X;r)$. Indeed, as a simplicial complex $\mathrm{VR}(X;r)$ is metrizable if and only if it is locally finite; see Sakai [34, Proposition 4.2.16(2)]. In other words, if $\mathrm{VR}(X;r)$ is not locally finite, then it is impossible to equip it with any metric without changing the homeomorphism type.

Remark 3.3. The Vietoris–Rips thickening $\mathrm{VR}^m_\leq(X;r)$ need not be homotopy equivalent to $\mathrm{VR}_\leq(X;r)$. Indeed, when $r = 0$ note $\mathrm{VR}^m_\leq(X;0)$ is equal to X as a metric space, whereas $\mathrm{VR}_\leq(X;0)$ is equipped with the discrete topology. A less trivial example is that $\mathrm{VR}^m_\leq(S^1; \frac{1}{3}) \simeq S^3$ (Remark 5.5), whereas $\mathrm{VR}_\leq(S^1; \frac{1}{3}) \simeq \bigvee^{\mathfrak{c}} S^2$; see [1]. Appendix C gives other examples where $\mathrm{VR}_\leq(X;r)$ has uncountable homology but $\mathrm{VR}^m_\leq(X;r)$ does not. We leave it as an open question whether with the $<$ convention $\mathrm{VR}^m_<(X;r)$ and $\mathrm{VR}_<(X;r)$ are homotopy equivalent, though this is known to be true in certain cases (Remark 6.5, Proposition 6.6).

Remark 3.4. The metric thickening $\mathrm{VR}^m(X;r)$ is rarely complete or compact, except when X is finite or discrete. Indeed, see Appendix D. This situation could be improved by considering a different space, those measures μ whose support has diameter at most r, with no requirement that the support be finite. Theorem 6.18 of [42] implies that if X is Polish (complete and separable), then this space of measures of bounded support is Polish (indeed, it is a closed subset of a Polish space). Similarly, [42, Remark 6.19] implies that this space of measures of bounded support is compact if X is. Furthermore, for X compact a metric version

of Hausmann's theorem would still be true for this space of measures of bounded support, as outlined in Remark 4.3. We instead consider the Vietoris–Rips thickening $\mathrm{VR}^m(X; r)$ in which all measures have finite support, as it is more directly related to the Vietoris–Rips complex (in which all simplices are finite). Nevertheless, the space of measures of bounded support (without an assumption of finiteness) is natural to consider.

The Vietoris–Rips thickening is only one example of a more general construction.

Definition 3.5. *Let X be a metric space and let K be a simplicial complex with vertex set $V(K) = X$. The* metric thickening K^m *is the following submetric space of $\mathcal{P}(X)$, equipped with the restriction of the* 1-*Wasserstein metric:*

$$K^m = \left\{ \sum_{i=0}^k \lambda_i \delta_{x_i} \in \mathcal{P}(X) \ \Big| \ k \geq 1, \ \lambda_i \geq 0, \ \sum \lambda_i = 1, \ \{x_0, \ldots, x_k\} \in K \right\}.$$

Examples include not only Vietoris–Rips thickenings $\mathrm{VR}^m(X; r) = (\mathrm{VR}(X; r))^m$, but also Čech thickenings $\check{\mathrm{C}}^m(X; r) = (\check{\mathrm{C}}(X; r))^m$, alpha thickenings, and witness thickenings, etc. Alpha simplicial complexes are defined, for example, in [16], and witness complexes in [13, 11]. As before, we write $\sum_{i=0}^k \lambda_i x_i \in K^m$ instead of $\sum_{i=0}^k \lambda_i \delta_{x_i}$.

The following lemma is stated by Gromov (see [19, eq. 1.B(c)]) in the specific case where X is discrete.

Lemma 3.6. *If K^m is a metric thickening of metric space X such that each simplex of K has diameter at most $r \geq 0$, then K^m is an r-thickening of X.*

Proof. It is clear by (2) that the Wasserstein metric on K^m extends the metric on X. Next, consider any point $\sum_i \lambda_i x_i$ in K^m. Note

$$d\left(\sum_i \lambda_i x_i, X \right) \leq d\left(\sum_i \lambda_i x_i, x_0 \right) = \sum_i \lambda_i d(x_i, x_0) \leq r \sum_i \lambda_i = r.$$

This completes the proof. ∎

We may therefore say K^m is a *metric r-thickening* if each simplex of K has diameter at most r. Note $\mathrm{VR}^m(X; r)$ is a metric r-thickening and $\check{\mathrm{C}}^m(X; r)$ is a metric $2r$-thickening.

We briefly relate metric thickenings to configuration spaces. Let X be a metric space and let K be a simplicial complex on vertex set X. Denote the *n-skeleton* of K^m by $S_n(K^m) = \{\sum_{i=0}^k \lambda_i x_i \in K^m \mid k \leq n\}$, and denote the configuration space of $n + 1$ unordered points in X by $C_{n+1}(X)$. If K is the maximal simplicial complex on vertex set X, then we have $S_n(K^m) \setminus S_{n-1}(K^m) \simeq C_{n+1}(X)$, where a deformation retraction is obtained by collapsing each open simplex to its barycenter. A related space is $\exp_{n+1} X$, the set of all finite subsets of X of cardinality at most $n + 1$, as studied by Tuffley in [39].

3.1. Basic properties. We describe a basic result on the continuity of maps between metric thickenings induced from maps on the underlying metric spaces, and a result on the continuity of linear homotopies on metric thickenings.

Let X, Y be metric spaces and let K, L be simplicial complexes with vertex sets $V(K) = X$, $V(L) = Y$. Note that if $f \colon X \to Y$ has the property that $f(\sigma)$ is a simplex in L for each

simplex $\sigma \in K$, then the induced map $\tilde{f}\colon K^m \to L^m$ defined by $\sum_i \lambda_i x_i \mapsto \sum_i \lambda_i f(x_i)$ exists. For example, if $f\colon X \to Y$ has the property that $d(x, x') \leq r$ implies $d(f(x), f(x')) \leq r$, then the induced map $\tilde{f}\colon \mathrm{VR}^m_{\leq}(X; r) \to \mathrm{VR}^m_{\leq}(Y; r)$ exists.

Lemma 3.7. *Let X, Y be metric spaces and let K, L be simplicial complexes with vertex sets $V(K) = X$, $V(L) = Y$. Let $f\colon X \to Y$ be a map of metric spaces such that the induced map $\tilde{f}\colon K^m \to L^m$ on metric thickenings exists. If f is c-Lipschitz, then so is $\tilde{f}$.*

Proof. Let $\varepsilon > 0$. Since f is c-Lipschitz, we have $d(f(x), f(x')) \leq cd(x, x')$ for all $x, x' \in X$. Now suppose $d(\sum \lambda_i x_i, \sum \lambda'_j x'_j) \leq \frac{\varepsilon}{c}$. This means there is some $\pi_{i,j} \geq 0$ with $\sum \pi_{i,j} = 1$, $\sum_j \pi_{i,j} = \lambda_i$, $\sum_i \pi_{i,j} = \lambda'_j$, and $\sum \pi_{i,j} d(x_i, x'_j) \leq \frac{\varepsilon}{c}$. It follows that

$$d\big(\tilde{f}(\sum \lambda_i x_i), \tilde{f}(\sum \lambda'_j x'_j)\big) = d\big(\sum \lambda_i f(x_i), \sum \lambda'_j f(x'_j)\big) \leq \sum_{i,j} \pi_{i,j} d(f(x_i), f(x'_j))$$

$$\leq c \sum_{i,j} \pi_{i,j} d(x_i, x'_j) \leq \varepsilon,$$

and hence $\tilde{f}$ is c-Lipschitz. ∎

Remark 3.8. Note that f continuous need not imply that $\tilde{f}$ is continuous. Indeed, consider

$$X = \{(0,0)\} \cup \{(\tfrac{1}{n}, 0) \mid n \in \mathbb{N}\} \cup \{(\tfrac{1}{n}, 1) \mid n \in \mathbb{N}\} \subseteq \mathbb{R}^2.$$

Define the continuous map $f\colon X \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} 0 & \text{if } y = 0, \\ n^2 & \text{if } (x, y) = (\tfrac{1}{n}, 1). \end{cases}$$

Let $K = \mathrm{VR}(X; \infty)$ and $L = \mathrm{VR}(\mathbb{R}; \infty)$ be the maximal simplicial complexes on their vertex sets. Note that $\mu_n = \frac{n-1}{n}\delta_{(\frac{1}{n}, 0)} + \frac{1}{n}\delta_{(\frac{1}{n}, 1)}$ is a sequence in K^m converging to $\delta_{(0,0)}$, but that $\tilde{f}(\mu_n) = \frac{n-1}{n}\delta_0 + \frac{1}{n}\delta_{n^2}$ is not Cauchy (and hence not convergent) in L^m.

A metric thickening K^m is abstractly the space of convex combinations of points in X, and taking convex combinations (wherever defined) is a continuous operation in the Wasserstein metric. This is made rigorous in the following lemma.

Lemma 3.9. *Suppose K^m is a metric r-thickening and $f\colon K^m \to K^m$ is a continuous map such that $H\colon K^m \times [0,1] \to K^m$ given by $H(\mu, t) = (1-t)\mu + tf(\mu)$ is well-defined. Then H is continuous.*

Proof. Note that H is the composition of the continuous map $(\mu, t) \mapsto (\mu, f(\mu), t)$ with $L\colon K^m \times K^m \times I \to K^m$ defined by $L(\mu, \nu, t) = (1-t)\mu + t\nu$. For any map $g\colon X \to \mathbb{R}$ with Lipschitz constant at most one, and for any $(\mu, \nu, t), (\mu', \nu', t') \in K^m \times K^m \times I$, we have

$$\int_X g(x) \, d((1-t)\mu + t\nu - (1-t')\mu' - t'\nu')$$

$$= (1-t) \int_X g(x) \, d(\mu - \mu') + t \int_X g(x) \, d(\nu - \nu') + (t' - t) \int_X g(x) \, d(\mu' - \nu')$$

$$\leq \max\{d(\mu, \mu'), d(\nu, \nu')\} + |t' - t| d(\mu', \nu').$$

Hence by (1) we have $d(L(t, \mu, \nu), L(t', \mu', \nu')) \leq \max\{d(\mu, \mu'), d(\nu, \nu')\} + |t' - t| d(\mu', \nu')$, and so L is continuous. It follows that H is continuous. $\blacksquare$

4. Metric analogues of Hausmann's theorem and the nerve lemma. In this section we prove metric analogues of Hausmann's theorem, the nerve lemma, and Latschev's theorem. The classical versions of these results confirm that Vietoris–Rips and Čech simplicial complexes recover homotopy types correctly; we show that these results also hold in the setting of metric thickenings.

Let M be a Riemannian manifold. Hausmann's theorem [21, Theorem 3.5] states that there exists some real number $r(M)$ sufficiently small (depending on the scalar curvature of M) such that if $0 < r \leq r(M)$, then the Vietoris–Rips simplicial complex $\mathrm{VR}_<(M; r)$ is homotopy equivalent to M. Hausmann's homotopy equivalence $T \colon \mathrm{VR}_<(M; r) \to M$ is extremely non-canonical; it depends on the choice of an arbitrary total ordering of all of the points in M. Furthermore, if M is not a discrete metric space, then the inclusion $M \hookrightarrow \mathrm{VR}_<(M; r)$ is not continuous and hence cannot be a homotopy inverse. By contrast, we use Karcher means to define a map $g \colon \mathrm{VR}^m(M; r) \to M$ on the metric thickening which is a homotopy equivalence for r sufficiently small (Theorem 4.2). A key feature of our proof is that we give a canonical choice for map g, which furthermore has the continuous inclusion $M \hookrightarrow \mathrm{VR}^m(M; r)$ as a homotopy inverse. A similar proof also works for the Čech thickening in Theorem 4.4.

The following technology regarding Karcher means is from [22]. Let the complete Riemannian manifold M and real number $\rho > 0$ satisfy the following conditions:

   (i) For each $m \in M$, the geodesic ball $B(m, \rho)$ of radius ρ about m is convex, meaning the shortest geodesic between any two points in $B(m, \rho)$ is unique in M and lies in $B(m, \rho)$.

   (ii) The manifold M has sectional curvature bounds $\delta \leq K \leq \Delta$, where if $\Delta > 0$ then we also assume $2\rho < \frac{1}{2}\pi\Delta^{-1/2}$.

Given a probability measure μ with support contained in an open ball B of M of radius at most ρ, the function $P_\mu \colon \overline{B} \to \mathbb{R}$ defined by

$$P_\mu(x) = \frac{1}{2} \int_{y \in M} d(x, y)^2 \, d\mu$$

has a unique minimum in B (see [22, Definition 1.3]). This minimizer is denoted C_μ, and called the *center of mass* or the *Karcher mean*. Furthermore, [22, Corollary 1.6] bounds the variation of the Karcher mean: if μ and ν are probability measures with support in an open ball B of radius ρ, then

$$(3) \qquad d(C_\mu, C_\nu) \leq (1 + c(\delta, \Delta)(2\rho)^2) \int_{(x,y) \in M \times M} d(x, y) \, d(\mu \times \nu).$$

Let K be a simplicial complex on vertex set M such that the vertices of each simplex are contained in an open ball of radius ρ. Note that a point $\mu = \sum_{i=0}^k \lambda_i x_i$ in the metric thickening K^m is a measure with support contained in an open ball of radius at most ρ. Using the Karcher mean, we define a map $g \colon K^m \to M$ by setting $g(\mu) = C_\mu$.

Lemma 4.1. *Let M be a complete Riemannian manifold and let $\rho > 0$ be a real number satisfying* (i) *and* (ii). *Suppose K^m is a metric thickening of M such that each simplex of K has diameter at most $r < \rho$. Then the map $g \colon K^m \to M$ is continuous.*

Proof. Let $\mu = \sum_{i=0}^{k} \lambda_i x_i$ and $\mu' = \sum_{j=0}^{k'} \lambda'_j x'_j$. By the definition of $d(\mu, \mu')$ there exists some $\pi_{i,j} \geq 0$ with $\sum_j \pi_{i,j} = \lambda_i$, $\sum_i \pi_{i,j} = \lambda'_j$, and $\sum_{i,j} \pi_{i,j} d(x_i, x'_j) = d(\mu, \mu')$. Note that if $d(\mu, \mu') < \rho - r$, then there is at least one pair of vertices $x_i \in \{x_0, \ldots, x_k\}$ and $x'_j \in \{x'_0, \ldots, x'_{k'}\}$ with $d(x_i, x'_j) < \rho - r$. By the triangle inequality we have

$$\{x_0, \ldots, x_k\} \cup \{x'_0, \ldots, x'_{k'}\} \subseteq B(x_i, \rho).$$

Hence for all μ, μ' with $d(\mu, \mu') < \rho - r$, we have

$$d(g(\mu), g(\mu')) = d(C_\mu, C_{\mu'}) \leq (1 + c(\delta, \Delta)(2\rho)^2) \int_{M \times M} d(x, y) \; d(\mu \times \mu') \qquad \text{by (3)}$$

$$= \left(1 + c(\delta, \Delta)(2\rho)^2\right) \sum_{i,j} \pi_{i,j} d(x_i, x'_j) \leq (1 + c(\delta, \Delta)(2\rho)^2) d(\mu, \mu').$$

So the map $g \colon K^m \to M$ is continuous. ∎

We can now prove our main result: The Vietoris–Rips thickening of a complete Riemannian manifold M accurately recovers the homotopy type of M for sufficiently small distance parameter r.

Theorem 4.2 (metric Hausmann's theorem). *Let M be a complete Riemannian manifold and let $\rho > 0$ be a real number satisfying* (i) *and* (ii). *For $r < \rho$, the map $g \colon \mathrm{VR}^m(M; r) \to M$ is a homotopy equivalence.*

Proof. The inclusion map $\iota \colon M \to \mathrm{VR}^m(M; r)$ is an isometric embedding, and hence continuous. Since $r < \rho$, the map $g \colon \mathrm{VR}^m(M; r) \to M$ is defined and continuous. We will show that ι and are homotopy inverses. Note $g \circ \iota = \mathrm{id}_M$.

We must show $\iota \circ g \simeq \mathrm{id}_{\mathrm{VR}^m(M;r)}$. Define $H \colon \mathrm{VR}^m(M; r) \times [0, 1] \to \mathrm{VR}^m(M; r)$ by $H(\mu, t) = (1 - t)\mu + t(\iota \circ g)(\mu)$. Note $H(-, 0) = \mathrm{id}_{\mathrm{VR}^m(M;r)}$ and $H(-, 1) = \iota \circ g$, and hence it suffices to show that H is well-defined and continuous.

To see that H is well-defined, let $\mu = \sum_{i=0}^{k} \lambda_i x_i \in \mathrm{VR}^m(M; r)$; it suffices to show that $[x_0, \ldots, x_k, g(\mu)]$ is a simplex in $\mathrm{VR}(M; r)$. For the $<$ case, note $\mu \in \mathrm{VR}^m_<(M; r)$ implies $\{x_0, \ldots, x_k\} \subseteq B(x_i, r)$ for all i, giving $g(\mu) \in \cap_i B(x_i, r)$ by [22, Definition 1.3] as required. For the $\leq$ case, for each i and $\varepsilon > 0$ we have that $\{x_0, \ldots, x_k\} \subseteq B(x_i, r + \varepsilon)$, giving $g(\mu) \in \cap_i B(x_i, r + \varepsilon)$ for all $\varepsilon > 0$ and hence $g(\mu) \in \cap_i \overline{B(x_i, r)}$. In either case we have that $[x_0, \ldots, x_k, g(\mu)] \in \mathrm{VR}(M; r)$, so H is well-defined.

Map H is continuous by Lemma 3.9, and therefore, $g \colon \mathrm{VR}^m(M; r) \to M$ is a homotopy equivalence. ∎

The paper [3] proves a variant of the metric Hausmann's theorem when manifold M is instead a subset of Euclidean space, equipped with the Euclidean metric.

Remark 4.3. The following observation is due to an anonymous referee. Generalize the definition of the Karcher mean to arbitrary metric spaces, and suppose that X is a compact metric space and K is a simplicial complex with vertices in a subset of X, such that for any measure $\mu \in K^m$ the Karcher mean is well-defined, and, in particular, unique. One such example is when X is a Hadamard space, i.e., a globally nonpositively curved space [37, Proposition 4.3]. Then the map g in Lemma 4.1 is well-defined and continuous. To see the

continuity of g, let $\mu_n \in K^m$ be a sequence of measures converging to μ. Then by compactness of X their Karcher means $g(\mu_n)$ have a convergent subsequence with limit point z. Now,

$$\int_X |g(\mu_n) - x|^2 \, d\mu_n(x) \leq \int_X |y - x|^2 \, d\mu_n(x)$$

for all y by definition of the Karcher mean. In particular, by passing to the convergent subsequence and taking limits we see that $\int_X |z - x|^2 \, d\mu(x) \leq \int_X |y - x|^2 \, d\mu(x)$ for all y. Thus z is the Karcher mean of μ. Certainly the sequence $g(\mu_n)$ does not have an accumulation point different from z, as this would contradict the uniqueness of Karcher means. By compactness $g(\mu_n)$ converges to $z = g(\mu)$. This shows the continuity of g. Notice that all we used is that Karcher means of a converging sequence of measures $\mu_n \in K^m$ are eventually contained in a compact set. By the same proof as that of Theorem 4.2, it follows that if X is a compact metric space and $\mathrm{VR}^m(X; r)$ is a metric thickening such that for any measure $\mu \in \mathrm{VR}^m(X; r)$ the Karcher mean is well-defined, then the continuous map $g \colon \mathrm{VR}^m(X; r) \to X$ is a homotopy equivalence.

Recall that the Čech complex $\check{\mathrm{C}}(M; r)$ has vertex set M and a k-simplex $\{x_0, \dots, x_k\}$ if $\bigcap_{i=0}^{k} B_r(x_i) \neq \emptyset$. For r sufficiently small all of these r-balls and their nonempty intersections are contractible, and thus $\check{\mathrm{C}}(M; r)$ is homotopy equivalent to M by the nerve lemma. The metric Čech thickening $\check{\mathrm{C}}^m(M; r)$ is not a simplicial complex and the nerve lemma does not apply. Nonetheless, in a similar fashion to Theorem 4.2 we can show that our metric analogue of the Čech complex is homotopy equivalent to M.

Theorem 4.4 (metric nerve lemma). *Let M be a complete Riemannian manifold and let $\rho > 0$ be a real number satisfying* (i) *and* (ii). *For $r < \frac{\rho}{2}$, the map $g \colon \check{\mathrm{C}}^m(M; r) \to M$ is a homotopy equivalence.*

Proof. We will show that the inclusion $\iota \colon M \to \check{\mathrm{C}}^m(M; r)$ and the Karcher mean map $g \colon \check{\mathrm{C}}^m(M; r) \to M$ are homotopy inverses. Note g is defined and continuous by Lemma 4.1 since $r < \frac{\rho}{2}$. As before define $H \colon \check{\mathrm{C}}^m(M; r) \times [0, 1] \to \check{\mathrm{C}}^m(M; r)$ by $H(\mu, t) = (1 - t)\mu + tg(\mu)$. By Lemma 3.9 it suffices to show that $(1 - t)\mu + tg(\mu) \in \check{\mathrm{C}}^m(M; r)$ for any $\mu \in \check{\mathrm{C}}^m(M; r)$. Let $\mu = \sum_{i=0}^{k} \lambda_i x_i \in \check{\mathrm{C}}_<^m(M; r)$, that is, there is a $y \in M$ with $d(y, x_i) < r$ for all i. Since $x_0, \dots, x_k \in B(y, r)$ we have that $g(\mu) \in B(y, r)$, which implies that $t\mu + (1 - t)g(\mu) \in \check{\mathrm{C}}_<^m(M; r)$. The $\leq$ case follows from the same approximation argument as in the proof of Theorem 4.2. ∎

5. Vietoris–Rips thickenings of spheres. In this section we study the Vietoris–Rips thickenings of n-spheres, and in particular, in Theorem 5.4 we describe the first new homotopy type (after that of the n-sphere) that appears as the scale parameter increases. Our proof proceeds by using the natural embedding $S^n \hookrightarrow \mathbb{R}^{n+1}$ to induce a map $\mathrm{VR}^m(S^n; r) \to \mathbb{R}^{n+1}$ on the metric thickening. In order to show that this induced map is continuous, we first need to study the continuity properties of maps from metric thickenings into Euclidean space.

Maps to Euclidean space. Let X be a metric space and let K be a simplicial complex with $V(K) = X$. We study when a map from X into $\mathbb{R}^n$ induces a continuous map on a metric thickening K^m of X. Given a function $f \colon X \to \mathbb{R}^n$, by an abuse of notation we also

let $f \colon K^m \to \mathbb{R}^n$ denote the map defined by $\sum \lambda_i x_i \mapsto \sum \lambda_i f(x_i)$, where $\sum \lambda_i f(x_i)$ is a linear combination of vectors in $\mathbb{R}^n$.

Remark 5.1. Note that $f \colon X \to \mathbb{R}^n$ continuous need not imply $f \colon K^m \to \mathbb{R}^n$ is continuous. Indeed, as in Remark 3.8, let

$$X = \{(0,0)\} \cup \{(\tfrac{1}{n}, 0) \mid n \in \mathbb{N}\} \cup \{(\tfrac{1}{n}, 1) \mid n \in \mathbb{N}\} \subseteq \mathbb{R}^2.$$

Define the continuous map $f \colon X \to \mathbb{R}$ by $f(x,0) = 0$ and $f(\tfrac{1}{n}, 1) = n^2$. Let $K = \mathrm{VR}(X; \infty)$ be the maximal simplicial complex on vertex set X. Note that $\mu_n = \frac{n-1}{n} \delta_{(\frac{1}{n}, 0)} + \frac{1}{n} \delta_{(\frac{1}{n}, 1)}$ is a sequence in K^m converging to $\delta_{(0,0)}$, but that $f(\mu_n) = \frac{1}{n} n^2 = n$ is not a convergent sequence in $\mathbb{R}$.

The next lemma follows from [42, Theorem 6.9] if X is Polish (or from [35, Theorem 5.11] if X is Euclidean). We have included a proof since for finitely supported measures, the result also holds in the non-Polish case.

Lemma 5.2. *Let K^m be a metric thickening of metric space X. If $f \colon X \to \mathbb{R}^n$ is continuous and bounded, then so is $f \colon K^m \to \mathbb{R}^n$.*

Proof. Let C be such that $\|f(x) - f(y)\| \leq C$ for all $x, y \in X$. Fix a point $\sum \lambda_i x_i \in K^m$ and $\varepsilon > 0$. Using the continuity of f at the finitely many points $x_1, \ldots, x_n$, choose $\delta > 0$ so that $d(x_i, y) \leq \delta$ implies $\|f(x_i) - f(y)\| \leq \varepsilon/2$ for all i. Reducing δ if necessary, we can also assume $\delta \leq \frac{\varepsilon}{2C}$. We will show that $d(\sum \lambda_i x_i, \sum \lambda'_j x'_j) \leq \delta^2$ implies $\|f(\sum \lambda_i x_i) - f(\sum \lambda'_j x'_j)\| \leq \varepsilon$, which proves the continuity of $f \colon K^m \to \mathbb{R}^n$ at $\sum \lambda_i x_i$.

Let $\pi_{i,j}$ be a matching from $\sum \lambda_i x_i$ to $\sum \lambda'_j x'_j$ with $\sum_{i,j} \pi_{i,j} d(x_i, x'_j) \leq \delta^2$. Let $A = \{(i,j) \mid d(x_i, x'_j) \geq \delta\}$ and $B = \{(i,j) \mid d(x_i, x'_j) < \delta\}$. We have

$$\delta \sum_A \pi_{i,j} \leq \sum_A \pi_{i,j} d(x_i, x'_j) \leq \sum_{i,j} \pi_{i,j} d(x_i, x'_j) \leq \delta^2,$$

so $\sum_A \pi_{i,j} \leq \delta$. Hence

$$\left\| f(\sum \lambda_i x_i) - f(\sum \lambda'_j x'_j) \right\| = \left\| \sum_i \lambda_i f(x_i) - \sum_j \lambda'_j f(x'_j) \right\| = \left\| \sum_{i,j} \pi_{i,j} f(x_i) - \sum_{i,j} \pi_{i,j} f(x'_j) \right\|$$

$$\leq \sum_{i,j} \pi_{i,j} \|f(x_i) - f(x'_j)\|$$

$$= \sum_A \pi_{i,j} \|f(x_i) - f(x'_j)\| + \sum_B \pi_{i,j} \|f(x_i) - f(x'_j)\|$$

$$\leq C \sum_A \pi_{i,j} + \frac{\varepsilon}{2} \sum_B \pi_{i,j} \leq C\delta + \varepsilon/2 \leq \varepsilon.$$

To see that $f \colon K^m \to \mathbb{R}^n$ is bounded, note that $f(K^m)$ is contained in $\mathrm{conv}(f(X))$. ∎

A similar result, which is not hard to verify, is that if K^m is a metric thickening of metric space X, and if $f \colon X \to \mathbb{R}^n$ is c-Lipschitz, then so is $f \colon K^m \to \mathbb{R}^n$.

Vietoris–Rips thickenings of spheres. Let A_{n+2} be the alternating group on $n+2$ elements. For example, the group A_3 is isomorphic to $\mathbb{Z}/3\mathbb{Z}$, and the group A_4 is also known as the tetrahedral group. In a noncanonical fashion, we can view A_{n+2} as a subgroup of $\mathrm{SO}(n+1)$, as follows. Fix a regular $(n+1)$-dimensional simplex inscribed in S^n inside $\mathbb{R}^{n+1}$, with the center of the simplex at the origin. The $(n+1)$-simplex has $n+2$ vertices, and A_{n+2} as its group of rotational symmetries. We can, therefore, associate each element $g \in A_{n+2}$ with a rotation matrix in $\mathrm{SO}(n+1)$ that permutes the vertices of the simplex in the same way that g does. After this noncanonical identification as a subgroup, A_{n+2} acts on $\mathrm{SO}(n+1)$ via left multiplication, and we let $\frac{\mathrm{SO}(n+1)}{A_{n+2}}$ be the orbit space of this action.[3] We also think of $\frac{\mathrm{SO}(n+1)}{A_{n+2}}$ as the moduli space of regular $(n+1)$-simplices inscribed in $S^n \subseteq \mathbb{R}^{n+1}$.

Denote by S^n the n-dimensional sphere equipped with either the Euclidean or the geodesic metric, and let r_n be the diameter of an inscribed regular $(n+1)$-simplex in S^n. We will show in Proposition 5.3 and Theorem 5.4 that $\mathrm{VR}^m(S^n; r) \simeq S^n$ for $0 < r < r_n$, that $\mathrm{VR}^m_<(S^n; r_n) \simeq S^n$, and that $\mathrm{VR}^m_\leq(S^n; r_n) \simeq \Sigma^{n+1} \frac{\mathrm{SO}(n+1)}{A_{n+2}}$.

In particular, let S^1 be the circle of unit circumference equipped with the path-length metric; this gives $r_1 = \frac{1}{3}$. In [1] the first two authors show that the Vietoris–Rips simplicial complexes of the circle satisfy $\mathrm{VR}(S^1; r) \simeq S^1$ for $0 < r < \frac{1}{3}$, that $\mathrm{VR}_<(S^1; \frac{1}{3}) \simeq S^1$, that $\mathrm{VR}_\leq(S^1; \frac{1}{3}) \simeq \bigvee^{\mathfrak{c}} S^2$, and that $\mathrm{VR}(S^1; r) \simeq S^3$ for $\frac{1}{3} < r < \frac{2}{5}$. By contrast, in the case of metric thickenings Remark 5.5 gives that

$$\mathrm{VR}^m_\leq(S^1; \tfrac{1}{3}) \simeq \Sigma^2 \frac{\mathrm{SO}(2)}{A_3} = \Sigma^2 S^1 = S^3.$$

We think of the homotopy type $\mathrm{VR}_\leq(S^1; \frac{1}{3}) \simeq \bigvee^{\mathfrak{c}} S^2$ as an unnatural artifact of having the Vietoris–Rips simplicial complex equipped with a nonmetrizable topology. By contrast, we think of the Vietoris–Rips metric thickening $\mathrm{VR}^m_\leq(S^1; \frac{1}{3}) \simeq S^3$ as having the "correct" homotopy type.

Let $f \colon \mathrm{VR}^m_\leq(S^n; r) \to \mathbb{R}^{n+1}$ be the projection map sending a finite convex combination of points in S^n to its corresponding linear combination in $\mathbb{R}^{n+1}$. This map f is continuous by Lemma 5.2. Let $\pi \colon \mathbb{R}^{n+1} \setminus \{\vec{0}\} \to S^n$ be the radial projection map. In addition, let W be the set of all interior points of regular $(n+1)$-simplices inscribed in $\mathrm{VR}_\leq(S^n; r_n)$. More precisely,

$$W = \left\{ \textstyle\sum_{i=0}^{n+1} \lambda_i x_i \;\middle|\; \lambda_i > 0 \text{ for all } i \text{ and } \{x_0, \ldots, x_{n+1}\} \text{ is a regular } (n+1)\text{-simplex} \right\}.$$

Note the closure of W in $\mathrm{VR}_\leq(S^n; r_n)$ is homeomorphic to $D^{n+1} \times \frac{\mathrm{SO}(n+1)}{A_{n+2}}$.

Proposition 5.3. *The maps $\pi f \colon \mathrm{VR}^m(S^n; r) \to S^n$ for $0 < r < r_n$, $\pi f \colon \mathrm{VR}^m_<(S^n; r_n) \to S^n$, and $\pi f \colon \mathrm{VR}^m_\leq(S^n; r_n) \setminus W \to S^n$ exist and are homotopy equivalences.*

Proof. An identical proof works for all three maps, and hence we let Y denote either $\mathrm{VR}^m(S^n; r)$ for $0 < r < r_n$, or $\mathrm{VR}^m_<(S^n; r_n)$, or $\mathrm{VR}^m_\leq(S^n; r_n) \setminus W$. The composition $\pi f \colon Y \to S^n$ is defined because $f(y) \neq \vec{0}$ for $\mu \in Y$ by the proof of [27, Lemma 3]. Map πf is continuous since π and f are.

---

[3]By [20, section 1.3, Exercise 24(b)], the homeomorphism type of $\frac{\mathrm{SO}(n+1)}{A_{n+2}}$ is unchanged if one identifies A_{n+2} as a subgroup of $\mathrm{SO}(n+1)$ using a different inscribed $(n+1)$-simplex.

We will show that the homotopy inverse to πf is the inclusion map $i \colon S^n \hookrightarrow Y$, which is continuous only since Y is equipped with the Wasserstein metric. Clearly $\pi f \circ i = id_{S^n}$. In order to show $i \circ \pi f \simeq id_Y$, define homotopy $H \colon Y \times I \to Y$ by $H(\mu, t) = (1-t)\mu + t \cdot i\pi f(\mu)$. This map is well-defined because if $\mu = \sum_{i=0}^k \lambda_i x_i$, then $\{x_0, \ldots, x_k, \pi f(\mu)\}$ is a simplex in Y; see [3]. Since πf is continuous, map H is continuous by Lemma 3.9. Since $H(-, 0) = id_Y$ and $H(-, 1) = i \circ \pi f$, we have shown $i \circ \pi f \simeq id_Y$. Therefore, $\pi f \colon Y \to S^n$ is a homotopy equivalence. ∎

Theorem 5.4. *We have a homotopy equivalence* $\mathrm{VR}^m_{\leq}(S^n; r_n) \simeq \Sigma^{n+1} \frac{\mathrm{SO}(n+1)}{A_{n+2}}$.

Proof. We will construct the following commutative diagram.

$$
\begin{array}{ccc}
D^{n+1} \times \frac{\mathrm{SO}(n+1)}{A_{n+2}} \supset S^n \times \frac{\mathrm{SO}(n+1)}{A_{n+2}} & \xrightarrow[h(x,y) = x]{h} & S^n \\
\Big\| & & \pi f \Big\downarrow \simeq \\
D^{n+1} \times \frac{\mathrm{SO}(n+1)}{A_{n+2}} \supset S^n \times \frac{\mathrm{SO}(n+1)}{A_{n+2}} & \xrightarrow{g} & \mathrm{VR}^m_{\leq}(S^n; r_n) \setminus W
\end{array}
$$

For $y \in \frac{\mathrm{SO}(n+1)}{A_{n+2}}$ let $\{y_0, \ldots, y_{n+1}\}$ be the $n+2$ vertices of the rotated regular $(n+1)$-simplex parameterized by y, and let

$$
\partial \Delta_y = \left\{ \sum_{i=0}^{n+1} \lambda_i y_i \in \mathrm{VR}^m_{\leq}(S^n; r_n) \setminus W \;\middle|\; \lambda_i = 0 \text{ for some } i \right\}
$$

be the boundary of the corresponding simplex. Note $\pi f|_{\partial \Delta_y} \colon \partial \Delta_y \to S^n$ is bijective. Define map $g \colon S^n \times \frac{\mathrm{SO}(n+1)}{A_{n+2}} \to \mathrm{VR}^m_{\leq}(S^n; r_n) \setminus W$ by letting $g(x, y)$ be the unique point of $\partial \Delta_y$ such that $\pi f(g(x, y)) = x$; that is, $g(x, y) = (\pi f|_{\partial \Delta_y})^{-1}(x)$. Note $\pi f \circ g = h$, meaning the square commutes.

We now have the following sequence of homotopy equivalences:

$$
\begin{aligned}
\mathrm{VR}^m_{\leq}(S^n; r_n) &= \mathrm{VR}^m_{\leq}(S^n; r_n) \setminus W \cup_g \left(D^{n+1} \times \frac{\mathrm{SO}(n+1)}{A_{n+2}} \right) \\
&\simeq S^n \cup_h \left(D^{n+1} \times \frac{\mathrm{SO}(n+1)}{A_{n+2}} \right) \\
&\simeq \left(S^n \times C\left(\frac{\mathrm{SO}(n+1)}{A_{n+2}}\right) \right) \cup_{S^n \times \frac{\mathrm{SO}(n+1)}{A_{n+2}}} \left(C(S^n) \times \frac{\mathrm{SO}(n+1)}{A_{n+2}} \right) \\
&= S^n * \frac{\mathrm{SO}(n+1)}{A_{n+2}} = \Sigma^{n+1} \frac{\mathrm{SO}(n+1)}{A_{n+2}}.
\end{aligned}
$$

Indeed, the first line is by the definition of W and g. The second line follows from the commutative diagram above and the homotopy invariance properties of adjunction spaces (see [6, eq. 7.5.7] or [38, Proposition 5.3.3]). The third line follows from these same properties of adjunction spaces, induced by contractibility of $C(\frac{\mathrm{SO}(n+1)}{A_{n+2}})$. The fourth line uses an equivalent definition for the join of two topological spaces as $Y * Z = Y \times C(Z) \cup_{Y \times Z} C(Y) \times Z$, and the fact that joining with a sphere gives an iterated suspension. ∎

Remark 5.5. The case $n = 1$ is instructive, since the final homotopy type can be recognized as a more common space (the 3-sphere). We have $\frac{\mathrm{SO}(2)}{A_3} = \frac{S^1}{\mathbb{Z}/3\mathbb{Z}} = S^1$, and hence the

commutative diagram implies

$$\mathrm{VR}^m_{\leq}(S^1; r_1) = \mathrm{VR}^m_{\leq}(S^1; r_1) \setminus W \cup_g (D^2 \times S^1) \simeq S^1 \cup_h (D^2 \times S^1)$$
$$\simeq (S^1 \times D^2) \cup_{S^1 \times S^1} (D^2 \times S^1) = S^1 * S^1 = S^3.$$

Remark 5.6. Since $\mathrm{SO}(n+1)$ is a compact Lie group of dimension $\binom{n+1}{2}$, we have

$$\tilde{H}_i\big(\mathrm{VR}^m_{\leq}(S^n; r_n)\big) = \tilde{H}_i\Big(\Sigma^{n+1} \tfrac{\mathrm{SO}(n+1)}{A_{n+2}}\Big) = \tilde{H}_{i-n-1}\Big(\tfrac{\mathrm{SO}(n+1)}{A_{n+2}}\Big) = \begin{cases} 0, & i \leq n+1, \\ \mathbb{Z}, & i = \binom{n+1}{2} + n + 1. \end{cases}$$

For the case $n = 2$, let $T = A_4$ be the tetrahedral group. Let $2T$ be the binary tetrahedral group; it has 24 elements but is isomorphic to neither S_4 nor $T \times \mathbb{Z}/2\mathbb{Z}$. The spherical 3-manifold $\tfrac{\mathrm{SO}(3)}{T} = \tfrac{S^3}{2T}$ has fundamental group isomorphic to the binary tetrahedral group $2T$ with abelianization $\mathbb{Z}/3$. Since $\tfrac{\mathrm{SO}(3)}{T}$ is a 3-dimensional closed orientable manifold, this gives

$$\tilde{H}_i\big(\mathrm{VR}^m_{\leq}(S^2; r_2)\big) = \tilde{H}_{i-3}\Big(\tfrac{\mathrm{SO}(3)}{T}\Big) = \begin{cases} \mathbb{Z}/3, & i = 4, \\ \mathbb{Z}, & i = 6, \\ 0 & \text{otherwise.} \end{cases}$$

Conjecture 5.7. We conjecture that for all n, there exists an $\varepsilon > 0$ such that
- $\mathrm{VR}^m_{\leq}(S^n; r)$ is homotopy equivalent to $\Sigma^{n+1} \tfrac{\mathrm{SO}(n+1)}{A_{n+2}}$ for all $r_n \leq r < r + \varepsilon$, and
- $\mathrm{VR}^m_{<}(S^n; r)$, $\mathrm{VR}_{<}(S^n; r)$, and $\mathrm{VR}_{\leq}(S^n; r)$ are homotopy equivalent to $\Sigma^{n+1} \tfrac{\mathrm{SO}(n+1)}{A_{n+2}}$ for all $r_n < r < r + \varepsilon$.

6. Maps between Vietoris–Rips complexes and thickenings. Let X be a metric space and let K be a simplicial complex on vertex set X. We study some of the basic relationships between the simplicial complex K and the metric thickening K^m. Note there is a natural bijection from the geometric realization of K to K^m, given by $\sum \lambda_i x_i \mapsto \sum \lambda_i \delta_{x_i}$. We will show that the function $K \to K^m$ is continuous, a homeomorphism if X is finite, and a weak equivalence if X is discrete (Propositions 6.1 and 6.6). The reverse function $K^m \to K$ is continuous if and only if K is locally finite (Proposition 6.3). By an abuse of notation, we will let μ denote both $\sum \lambda_i \delta_{x_i}$ and $\sum \lambda_i x_i$.

As corollaries, we deduce that in the restricted setting of finite metric spaces, there are metric analogues of Latschev's theorem and the stability of persistent homology (Corollaries 6.8 and 6.13). We conjecture that these theorems hold also in the case of arbitrary metric spaces (Conjectures 6.9 and 6.14), but this is currently unknown.

One could also compare the spaces considered in this section to the *metric of barycentric coordinates* in [5, section 7A.5] and [15]. A related construction is given in [30].

Proposition 6.1. *If K^m is a metric thickening, then the function $K \to K^m$ is continuous.*

Proof. The simplicial complex K is equipped with the coherent topology, and therefore, a map $K \to K^m$ is continuous if and only if its restriction $\sigma \to K^m$ is continuous for each closed simplex σ in K. If σ is a k-simplex, then its topology is induced from the Euclidean metric after embedding σ into $\mathbb{R}^{k+1}$ in the standard way. After giving σ this metric, one can show that $\sigma \to K^m$ is Lipschitz and hence continuous. ∎

Proposition 6.2. *If K^m is a metric thickening of a finite metric space X, then the function $K \to K^m$ is a homeomorphism.*

Proof. Denote the function by $h\colon K \to K^m$. For any closed simplex σ in K, note $h(\sigma)$ is closed in K^m since a convergent sequence of points in $h(\sigma)$ must converge to a point in $h(\sigma)$. In the proof of Proposition 6.1 we showed that each map $h\colon \sigma \to h(\sigma)$ is Lipschitz after equipping σ with the Euclidean metric, but more is true: the map $h\colon \sigma \to h(\sigma)$ is bi-Lipschitz and hence a homeomorphism. Therefore, $h^{-1}\colon K^m \to K$ is continuous as it is formed by gluing together continuous maps on a finite number of closed sets $h(\sigma)$. ■

Proposition 6.3. *Let K^m be a metric r-thickening. The map $K^m \to K$ is continuous if and only if K is locally finite.*

Proof. Suppose K is locally finite. Let $\{\mu_j\}$ be a sequence converging to $\mu = \sum_{i=0}^k \lambda_i x_i$ in K^m. Define

$$Y = \{x \in X \mid \text{there exists } i \text{ and } x' \in X \text{ with } d(x_i, x'), d(x', x) \leq r\}.$$

Note that Y is finite since local finiteness of K implies there are only a finite number of points x' within distance r from any x_i, and also only a finite number of points x within distance r from any x'. There exists some J such that $j \geq J$ implies $\mu_j \in K[Y]^m$, since for any $\mu' \notin K[Y]^m$ we have $d_{K^m}(\mu', \mu) > r$. Since $\{\mu_j\}_{j \geq J}$ converges to μ in $K[Y]^m$, Proposition 6.2 implies that $\{i(\mu_j)\}_{j \geq J}$ converges to $i(\mu)$ in $|K[Y]|$, and hence also in K.

For the reverse direction, suppose K is not locally finite. Then some point $x \in X$ is contained in an infinite number of simplices, and hence in an infinite number of edges. Let $x_1, x_2, x_3, \dots$ be a sequence of distinct vertices in X such that each $[x, x_i]$ is an edge of K. The sequence $(1 - \frac{1}{i})\delta_x + \frac{1}{i}\delta_{u_i}$ converges to δ_x in K^m. However, this sequence does not converge to x in the geometric realization of K. Indeed, any convergent sequence together with its limit point is compact, but the set $\{x\} \cup \{(1 - \frac{1}{i})x + \frac{1}{i}x_i \mid i \geq 1\}$ is not compact in the geometric realization of K because it is not contained in a finite union of simplices. ■

Combining Propositions 6.1 and 6.3 gives the following.

Corollary 6.4. *If K^m is a metric r-thickening and K is locally finite, then the function $K \to K^m$ is a homeomorphism.*

Remark 6.5. We now develop machinery which will allow us to understand the homotopy type of K^m using the nerve lemma. The argument is similar to that used in [2, Theorem 5.2] to prove a version of Hausmann's theorem for simplicial complexes embedded in Euclidean space.

Let X be an arbitrary metric space and let K be a simplicial complex on vertex set X. For $\sigma \subseteq X$, denote by

$$\text{st}(\sigma) = \left\{ \sum_{x \in \tau} \lambda_x x \in K^m \mid \sigma \subseteq \tau, \ \tau \in K, \ \lambda_x > 0, \ \sum \lambda_i = 1 \right\}$$

the *star* of σ. We will consider the covering of K^m by the sets $\mathcal{U} = \{\text{st}(x) \mid x \in X\}$. For any finite $\sigma \subseteq X$ we have that $\bigcap_{x \in \sigma} \text{st}(x) = \text{st}(\sigma)$. Note $\text{st}(\sigma) = \emptyset$ if and only if $\sigma \notin K$; this

implies that the nerve of $\mathcal{U}$ is isomorphic to K. If $\mathrm{st}(\sigma)$ is nonempty, it is contractible: let $\mu = \sum_{x \in \sigma} \frac{1}{|\sigma|} x$ be the barycenter of σ, then $\mathrm{st}(\sigma) \times [0,1] \longrightarrow \mathrm{st}(\sigma)$ defined by

$$\left(\sum_{x \in \tau} \lambda_x x, t \right) \mapsto t\mu + (1-t) \sum_{x \in \tau} \lambda_x x$$

is a deformation retraction to the point μ. It follows that the sets in $\mathcal{U}$ and their finite intersections are either empty or contractible. Hence whenever the nerve lemma applies to the covering $\mathcal{U}$, we get as a consequence that K is homotopy equivalent to K^m. In particular, since every metric space is paracompact [36, 33], the nerve lemma as stated in Corollary 4G.3 of [20] applies whenever $\mathcal{U}$ is a family of open sets.

If X is discrete, then the sets $\mathrm{st}(x)$ are open, and we obtain the following proposition.

Proposition 6.6. *If K^m is a metric thickening of a discrete metric space X, then $K \simeq K^m$.*

Proof. Fix $x \in X$; we need only show that $\mathrm{st}(x)$ is open. Since X is discrete, there exists some $\varepsilon(x) > 0$ such that $B_X(x, \varepsilon) = \{x\}$. Given $\mu \in \mathrm{st}(x)$, let $\lambda_x > 0$ be the coefficient of x. Note that any point in the open ball $B_{K^m}(\mu, \lambda_x \varepsilon)$ must contain a positive coefficient for x. It follows that $B_{K^m}(\mu, \lambda_x \varepsilon) \subseteq \mathrm{st}(x)$, and hence $\mathrm{st}(x)$ is open in K^m. ∎

Remark 6.7. For $X = \mathbb{R}$ (which is not discrete), the open star of 0 is not open in $\mathrm{VR}_{\leq}^m(\mathbb{R}; r)$. Indeed, note that the sequence $\frac{1}{2}\delta_{1/i} + \frac{1}{2}\delta_{r+1/i}$ converges to $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_r$ in $\mathrm{VR}_{\leq}^m(\mathbb{R}; r)$. The limit point of this sequence is in the open star of 0 even though no point of the sequence is.

We end this section with three corollaries of Proposition 6.2. Latschev's theorem [25] states that if M is a closed Riemannian manifold, then for any r sufficiently small there exists a δ (depending on r) such that if $d_{\mathrm{GH}}(M, X) < \delta$, then $\mathrm{VR}_<(X; r) \simeq M$. We immediately get an analogue of Latschev's theorem for Vietoris–Rips thickenings when X is finite.

Corollary 6.8 (metric Latschev's theorem). *If M is a closed Riemannian manifold, then for any r sufficiently small there exists a δ such that if finite X satisfies $d_{\mathrm{GH}}(M, X) < \delta$, then $\mathrm{VR}_<^m(X; r) \simeq M$.*

Proof. Since X is finite, Proposition 6.2 gives that $\mathrm{VR}_<^m(X; r)$ and $\mathrm{VR}_<(X; r)$ are homeomorphic. Then we have $\mathrm{VR}_<(X; r) \simeq M$ by Latschev's theorem [25]. ∎

Conjecture 6.9. We conjecture that Corollary 6.8 is true even for X infinite.

The next two corollaries relate to persistent homology [16, 10]. If X is a metric space, then $\mathrm{VR}^m(X; -)$, $\mathrm{VR}(X; -)$, $\check{\mathrm{C}}^m(X; -)$, and $\check{\mathrm{C}}(X; -)$ define filtered topological spaces (since $r \leq r'$ implies $\mathrm{VR}^m(X; r) \subseteq \mathrm{VR}^m(X; r')$). We let $\mathrm{PH}_i(\mathrm{VR}^m(X; -))$ denote the persistent homology module of $\mathrm{VR}^m(X; -)$ in homological dimension $i \geq 0$.

Corollary 6.10. *Let X be a finite metric space. For any dimension $i \geq 0$ the persistent homology modules $\mathrm{PH}_i(\mathrm{VR}^m(X; -))$ and $\mathrm{PH}_i(\mathrm{VR}(X; -))$ are isomorphic, and the persistent homology modules $\mathrm{PH}_i(\check{\mathrm{C}}^m(X; -))$ and $\mathrm{PH}_i(\check{\mathrm{C}}(X; -))$ are isomorphic.*

Proof. Note that if $r \leq r'$, then the following diagrams commute.

$$
\begin{array}{ccc}
\mathrm{VR}(X;r) & \hookrightarrow & \mathrm{VR}(X;r') \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
\mathrm{VR}^m(X;r) & \hookrightarrow & \mathrm{VR}^m(X;r')
\end{array}
\qquad
\begin{array}{ccc}
\check{\mathrm{C}}(X;r) & \hookrightarrow & \check{\mathrm{C}}(X;r') \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
\check{\mathrm{C}}^m(X;r) & \hookrightarrow & \check{\mathrm{C}}^m(X;r')
\end{array}
$$

The result now follows from Proposition 6.2. ∎

Remark 6.11. When the metric space X is infinite, it is known that the persistent homology modules $\mathrm{PH}_i(\mathrm{VR}^m_{\leq}(X;-))$ and $\mathrm{PH}_i(\mathrm{VR}_{\leq}(X;-))$ can differ at the endpoints of intervals. For example, $\mathrm{PH}_3(\mathrm{VR}^m_{\leq}(S^1;-))$ consists of the single half-open interval $[\frac{1}{3},\frac{2}{5})$, whereas $\mathrm{PH}_3(\mathrm{VR}_{\leq}(S^1;-))$ consists of the single open interval $(\frac{1}{3},\frac{2}{5})$.

Conjecture 6.12. We conjecture that for M a Riemannian manifold, the persistent homology modules $\mathrm{PH}_i(\mathrm{VR}^m(M;-))$ and $\mathrm{PH}_i(\mathrm{VR}(M;-))$ are identical up to replacing closed interval endpoints with open endpoints, or vice-versa.

Corollary 6.13. *Let X and Y be finite metric spaces and $i \geq 0$. Then*

$$
d_b(\mathrm{PH}_i(\mathrm{VR}^m(X;-)), \mathrm{PH}_i(\mathrm{VR}^m(Y;-))) \leq 2d_{\mathrm{GH}}(X,Y),
$$
$$
d_b(\mathrm{PH}_i(\check{\mathrm{C}}^m(X;-)), \mathrm{PH}_i(\check{\mathrm{C}}^m(Y;-))) \leq 2d_{\mathrm{GH}}(X,Y),
$$

where d_b denotes the bottleneck distance between persistent homology modules.

Proof. This proof follows from Corollary 6.10 and [11, Theorem 5.2]. ∎

Conjecture 6.14. We conjecture that Corollary 6.13 is true even for X and Y infinite.

Appendix A. Metric thickenings and the Gromov–Hausdorff distance. In this section we study how the Vietoris–Rips thickening behaves with respect to the Gromov–Hausdorff distance between metric spaces. The first result along these lines is Lemma 3.6, which implies that the Gromov-Hausdorff distance between X and $\mathrm{VR}^m(X;r)$ is at most r.

If X and Y are isometrically embedded in a common metric space Z, then we let $d_{\mathrm{H}}^Z(X,Y)$ denote the Hausdorff distance between X and Y in Z.

Lemma A.1. *If X and Y are metric spaces with $d_{\mathrm{GH}}(X,Y) = \varepsilon$, then*

$$
d_{\mathrm{GH}}(\mathrm{VR}^m(X;r), \mathrm{VR}^m(Y;r)) \leq r + \varepsilon,
$$
$$
d_{\mathrm{GH}}(\check{\mathrm{C}}^m(X;r), \check{\mathrm{C}}^m(Y;r)) \leq r + \varepsilon.
$$

Proof. Let $\delta > 0$. Let Z be a metric space containing isometric embeddings of X and Y with $d_{\mathrm{H}}^Z(X,Y) \leq \varepsilon + \delta$. Note that $\mathrm{VR}^m(Z;r)$ contains isometric copies of $\mathrm{VR}^m(X;r)$ and $\mathrm{VR}^m(Y;r)$. Given $\mu \in \mathrm{VR}^m(X;r)$, note that by Lemma 3.6 we have

$$
d_{\mathrm{VR}^m(Z;r)}(\mu, \mathrm{VR}^m(Y;r)) \leq d_{\mathrm{VR}^m(Z;r)}(\mu, Y) \leq d_{\mathrm{VR}^m(Z;r)}(\mu, X) + d_{\mathrm{VR}^m(Z;r)}(X,Y)
$$
$$
= d_{\mathrm{VR}^m(X;r)}(\mu, X) + d_{\mathrm{H}}^Z(X,Y) \leq r + \varepsilon + \delta.
$$

The same argument works symmetrically for any $\mu \in \mathrm{VR}^m(Y;r)$, giving

$$
d_{\mathrm{H}}^{\mathrm{VR}^m(Z;r)}(\mathrm{VR}^m(X;r), \mathrm{VR}^m(Y;r)) \leq r + \varepsilon + \delta.
$$

Since such a Z exists for every $\delta > 0$, this shows that $d_{\mathrm{GH}}(\mathrm{VR}^m(X;r), \mathrm{VR}^m(Y;r)) \leq r + \varepsilon$. An identical argument works for Čech complexes. ∎

Remark A.2. The dependence of this bound on r cannot be completely removed. Consider $X = \{\frac{-r-\varepsilon}{2}, \frac{r+\varepsilon}{2}\}$ and $Y = \{\frac{-r+\varepsilon}{2}, \frac{r-\varepsilon}{2}\}$. We have $d_{\mathrm{GH}}(X,Y) = \varepsilon$. Note we have $\mathrm{VR}^m(X;r) = X$ and $\mathrm{VR}^m(Y;r) = [\frac{-r+\varepsilon}{2}, \frac{r-\varepsilon}{2}]$, and hence we have $d_{\mathrm{GH}}(\mathrm{VR}^m(X;r), \mathrm{VR}^m(Y;r)) = \frac{r+\varepsilon}{2}$.

Lemma A.3. *If X and Y are metric spaces, then*

$$d_{\mathrm{GH}}(\mathrm{VR}^m(X;\infty), \mathrm{VR}^m(Y;\infty)) = d_{\mathrm{GH}}(X,Y).$$

Proof. Let Z be a metric space equipped with isometric embeddings of X and Y such that $d_{\mathrm{H}}^Z(X,Y) \leq d_{\mathrm{GH}}(X,Y) + \varepsilon$. Note that $\mathrm{VR}^m(Z;\infty)$ contains isometric copies of $\mathrm{VR}^m(X;\infty)$ and $\mathrm{VR}^m(Y;\infty)$. If $\sum \lambda_i x_i \in \mathrm{VR}^m(X;\infty)$, then there exist points $y_i \in Y$ with $d_Z(x_i, y_i) \leq d_{\mathrm{GH}}(X,Y) + \varepsilon$, and therefore, the distance in $\mathrm{VR}^m(Z;\infty)$ between $\sum \lambda_i x_i$ and $\sum \lambda_i y_i \in \mathrm{VR}^m(Y;\infty)$ is at most $d_{\mathrm{GH}}(X,Y) + \varepsilon$. It follows that

$$d_{\mathrm{H}}^{\mathrm{VR}^m(Z;\infty)}(\mathrm{VR}^m(X;\infty), \mathrm{VR}^m(Y;\infty)) \leq d_{\mathrm{GH}}(X,Y) + \varepsilon.$$

Since such a Z exists for every $\varepsilon > 0$, this shows $d_{\mathrm{GH}}(\mathrm{VR}^m(X;\infty), \mathrm{VR}^m(Y;\infty)) \leq d_{\mathrm{GH}}(X,Y)$.

For the converse direction, let Z be any metric space equipped with isometric embeddings of $\mathrm{VR}^m(X;\infty)$ and $\mathrm{VR}^m(Y;\infty)$. Note that Z also contains isometric embeddings of X and Y. Without loss of generality, there is a point $x \in X$ with $d_Z(x,y) \geq d_{\mathrm{H}}^Z(X,Y)$ for all $y \in Y$ (perhaps after interchanging X with Y). It follows that for the corresponding Dirac delta mass $\delta_x \in \mathrm{VR}^m(X;\infty)$, we have $d_Z(\delta_x, \sum \lambda_i y_i) \geq d_{\mathrm{H}}^Z(X,Y)$ for all $\sum \lambda_i y_i \in \mathrm{VR}^m(Y;\infty)$. Hence $d_{\mathrm{H}}^Z(\mathrm{VR}^m(X;\infty), \mathrm{VR}^m(Y;\infty)) \geq d_{\mathrm{H}}^Z(X,Y) \geq d_{\mathrm{GH}}(X,Y)$. Since this is true for all metric spaces Z equipped with isometric embeddings of $\mathrm{VR}^m(X;\infty)$ and $\mathrm{VR}^m(Y;\infty)$, we obtain $d_{\mathrm{GH}}(\mathrm{VR}^m(X;\infty), \mathrm{VR}^m(Y;\infty)) \geq d_{\mathrm{GH}}(X,Y)$. ∎

Appendix B. Crushings. Let X be a metric space and let $A \subseteq X$ be equipped with the subspace metric. The goal of this section is to prove that if there is a crushing of bounded speed from X onto A, then the inclusion maps $\mathrm{VR}^m(A;r) \hookrightarrow \mathrm{VR}^m(X;r)$ and $\check{\mathrm{C}}^m(A;r) \hookrightarrow \check{\mathrm{C}}^m(X;r)$ are homotopy equivalences.

Recall from [21] that a *crushing* from X onto A is a continuous map $F \colon X \times [0,1] \to X$ satisfying

(i) $F(x,1) = x$, $F(x,0) \in A$, $F(a,t) = a$ if $a \in A$, and

(ii) $d(F(x,t'), F(y,t')) \leq d(F(x,t), F(y,t))$ whenever $t' \leq t$.

For notational convenience, define $f_t \colon X \to X$ by $f_t(x) = F(x,t)$. We say that a crushing has *bounded speed c* if $d(f_t(x), f_{t'}(x)) \leq c|t - t'|$ for all $x \in X$.

Lemma B.1. *If there is a crushing of bounded speed from X onto a subset $A \subseteq X$, then the inclusion maps $\mathrm{VR}^m(A;r) \hookrightarrow \mathrm{VR}^m(X;r)$ and $\check{\mathrm{C}}^m(A;r) \hookrightarrow \check{\mathrm{C}}^m(X;r)$ are homotopy equivalences.*

Proof. Let F be a crushing from X to A of bounded speed c. Note that (ii) implies $d(f_t(x), f_t(y)) \leq d(x,y)$, and hence each f_t is 1-Lipschitz.

Let $\iota \colon A \hookrightarrow X$ denote the inclusion map. Since ι and $f_0 \colon X \to A$ are 1-Lipschitz, the maps $\tilde{\iota} \colon \mathrm{VR}^m(A;r) \hookrightarrow \mathrm{VR}^m(X;r)$ and $\tilde{f}_0 \colon \mathrm{VR}^m(X;r) \to \mathrm{VR}^m(A;r)$ are defined and continuous by

Lemma 3.7. We will show that $\tilde{\iota}$ and $\tilde{f}_0$ are homotopy inverses. Since $f_0 \circ \iota = \mathrm{id}_A$, it follows that $\tilde{f}_0 \circ \tilde{\iota} = \mathrm{id}_{\mathrm{VR}^m(A;r)}$.

We must show $\tilde{\iota} \circ \tilde{f}_0 \simeq \mathrm{id}_{\mathrm{VR}^m(X;r)}$. Consider the map $\tilde{F} \colon \mathrm{VR}^m(X;r) \times [0,1] \to \mathrm{VR}^m(X;r)$ defined by $\tilde{F}(\cdot, t) = \tilde{f}_t$, which is well-defined since each f_t is 1-Lipschitz. Let $\varepsilon > 0$, and suppose $d(\sum \lambda_i x_i, \sum \lambda'_j x'_j) \le \varepsilon$ and $|t - t'| \le \varepsilon$. Then there is some $\pi_{i,j} \ge 0$ with $\sum_j \pi_{i,j} = \lambda_i$, $\sum_i \pi_{i,j} = \lambda'_j$, and $\sum \pi_{i,j} d(x_i, x'_j) \le \varepsilon$. We have

$$d\big(\tilde{f}_t(\sum \lambda_i x_i), \tilde{f}_{t'}(\sum \lambda'_j x'_j)\big)$$

$$= d\big(\sum \lambda_i f_t(x_i), \sum \lambda'_j f_{t'}(x'_j)\big)$$

$$\le \sum_{i,j} \pi_{i,j} d\big(f_t(x_i), f_{t'}(x'_j)\big)$$

$$\le \sum_{i,j} \pi_{i,j} d\big(f_t(x_i), f_t(x'_j)\big) + \sum_{i,j} \pi_{i,j} d\big(f_t(x'_j), f_{t'}(x'_j)\big) \quad \text{by the triangle inequality}$$

$$\le \sum_{i,j} \pi_{i,j} d(x_i, x'_j) + c \sum_{i,j} \pi_{i,j} |t - t'| \quad \text{since } f_t \text{ is 1-Lipschitz and } F \text{ has bounded speed } c$$

$$\le \varepsilon + c|t - t'| \le (1 + c)\varepsilon,$$

and so $\tilde{F}$ is continuous. Since $\tilde{F}(\cdot, 0) = \tilde{\iota} \circ \tilde{f}_0$ and $\tilde{F}(\cdot, 1) = \mathrm{id}_{\mathrm{VR}^m(X;r)}$, it follows that $\tilde{\iota} \circ \tilde{f}_0 \simeq \mathrm{id}_{\mathrm{VR}^m(X;r)}$.

The proof for the Čech case is identical. ∎

Appendix C. Examples where $H_*(\mathrm{VR}_\le(X;r))$ has infinite rank but $H_*(\mathrm{VR}_\le^m(X;r))$ does not.

Example C.1. Let $X = [0,1] \times \{0,1\} \subseteq \mathbb{R}^2$. In [11, section 5.2.1] it is remarked that $\mathrm{VR}_\le(X;1)$ with the standard metric has uncountable 1-dimensional homology.

We now show that $\mathrm{VR}_\le^m(X;1)$ with the Wasserstein metric is contractible. Note that a crushing of bounded speed 1 from X to $\{(0,0),(0,1)\}$ is given by $F \colon X \times [0,1] \to X$ defined via

$$F((s,0),t) = (ts,0) \quad \text{and} \quad F((s,1),t) = (ts,1).$$

Hence by Lemma B.1 we have

$$\mathrm{VR}_\le^m(X;r) \simeq \mathrm{VR}_\le^m(\{(0,0),(0,1)\};r) \simeq \begin{cases} S^0 & \text{if } r < 1, \\ * & \text{if } r \ge 1. \end{cases}$$

Example C.2. Let X be two nonparallel rectangles, namely

$$X = \{(s,0,z) \in \mathbb{R}^3 \mid s \in [0,2], z \in [0,1]\} \cup \{(s, 1 + \tfrac{1}{2}s, z) \in \mathbb{R}^3 \mid s \in [0,2], z \in [0,1]\},$$

equipped with the ℓ^1 metric. In [11, Proposition 5.9] it is shown that $\mathrm{VR}_\le(X;r)$ with the standard metric has uncountable 1-dimensional homology for all $r \in [1,2]$.

We now show that $\mathrm{VR}_\le^m(X;r)$ with the Wasserstein metric is either homotopy equivalent to S^0 or contractible. Note that a crushing of bounded speed 2 from X onto $\{(0,0,0),(0,1,0)\}$

is given by $F\colon X \times [0,1] \to X$ defined via

$$F((s,0,z),t) = (ts,0,tz) \quad \text{and} \quad F((s,1+\tfrac{1}{2}s,z),t) = (ts,1+\tfrac{1}{2}ts,tz).$$

Hence by Lemma B.1 we have

$$\mathrm{VR}^m_{\leq}(X;r) \simeq \mathrm{VR}^m_{\leq}(\{(0,0,0),(0,1,0)\};r) \simeq \begin{cases} S^0 & \text{if } r < 1, \\ * & \text{if } r \geq 1. \end{cases}$$

Example C.3. Let S^1 be the circle of unit circumference equipped with the path-length metric; this gives $r_1 = \frac{1}{3}$. In [1] it is shown that $\mathrm{VR}_{\leq}(S^1;\frac{1}{3}) \simeq \bigvee^{\mathfrak{c}} S^2$, whereas $\mathrm{VR}^m_{\leq}(S^1;\frac{1}{3}) \simeq S^3$ by Theorem 5.4.

Appendix D. Vietoris–Rips thickenings are usually not complete. The following results show that the Vietoris–Rips thickening of a complete space need not be complete, and that the Vietoris–Rips thickening of a compact space need not be compact.

Let X be a metric space and let K be a simplicial complex on vertex set X. Recall the *n-skeleton* of K^m is denoted $S_n(K^m) = \{\sum_{i=0}^{k} \lambda_i x_i \in K^m \mid k \leq n\}$. The sets $S_n(K^m)$ are a countable closed covering of K^m. If the $S_n(K^m)$ have empty interior and hence are nowhere dense, then by Baire's category Theorem K^m cannot be complete.

Theorem D.1. *If X is a metric space without isolated points, then $\mathrm{VR}^m_{<}(X;r)$ is not complete for all $r > 0$.*

Proof. It suffices to show that the $S_n(K^m)$ have empty interior. Consider a point $\mu = \sum_{i=0}^{k} \lambda_i x_i \in S_n(K^m)$ with $\lambda_i > 0$ and let $\varepsilon > 0$ be arbitrary. Since X has no isolated points, there exist disjoint points $y_0, \ldots, y_{n+1} \in X$ with $d(x_0, y_j) < \min\{\varepsilon, r - \mathrm{diam}(\{x_0, \ldots, x_k\})\}$ for all $0 \leq j \leq n+1$. Note $\mu' = \sum_{j=0}^{n+1} \frac{\lambda_0}{n+2} y_j + \sum_{i=1}^{k} \lambda_i x_i$ satisfies $d(\mu', \mu) < \varepsilon$ and $\mu' \in \mathrm{VR}^m_{<}(X;r) \setminus S_n(K^m)$. ∎

Theorem D.2. *If $X \subseteq \mathbb{R}^n$ is an infinite convex subset, then $\mathrm{VR}^m_{\leq}(X;r)$ is not complete for all $r > 0$.*

Proof. We mimic the above proof. By assumption on X there exist disjoint points $y_0, \ldots, y_{n+1} \in X$ with $d(x_0, y_j) < \varepsilon$ and $\mathrm{diam}(\{x_0, \ldots, x_k, y_0, \ldots, y_{n+1}\}) \leq \min\{\varepsilon, r\}$. ∎

REFERENCES

[1] M. ADAMASZEK AND H. ADAMS, *The Vietoris–Rips complexes of a circle*, Pacific J. Math., 290 (2017), pp. 1–40.
[2] M. ADAMASZEK, F. FRICK, AND A. VAKILI, *On homotopy types of Euclidean Rips complexes*, Discrete Comput. Geom., 58 (2017), pp. 526–542.
[3] H. ADAMS AND J. MIRTH, *Vietoris–Rips thickenings of Euclidean submanifolds*, preprint, https://arxiv.org/abs/1709.02492, 2017.
[4] D. ATTALI AND A. LIEUTIER, *Geometry-driven collapses for converting a Čech complex into a triangulation of a nicely triangulable shape*, Discrete Comput. Geom., 54 (2015), pp. 798–825.

[5] M. R. BRIDSON AND A. HAEFLIGER, *Metric Spaces of Non-Positive Curvature*, Grundlehren Math. Wiss. 319, Springer-Verlag, Berlin, 2011.

[6] R. BROWN, *Topology and Groupoids*, http://www.groupoids.org.uk, 2006.

[7] D. BURAGO, Y. BURAGO, AND S. IVANOV, *A Course in Metric Geometry*, Grad. Stud. Math. 33, American Mathematical Society, Providence, RI, 2001.

[8] G. CARLSSON, *Topology and data*, Bull. Amer. Math. Soc. (N.S.), 46 (2009), pp. 255–308.

[9] G. CARLSSON, T. ISHKHANOV, V. DE SILVA, AND A. ZOMORODIAN, *On the local behavior of spaces of natural images*, Int. J. Comput. Vis., 76 (2008), pp. 1–12.

[10] F. CHAZAL, D. COHEN-STEINER, M. GLISSE, L. J. GUIBAS, AND S. Y. OUDOT, *Proximity of persistence modules and their diagrams*, in Proceedings of the 25th Annual Symposium on Computational Geometry, 2009, pp. 237–246.

[11] F. CHAZAL, V. DE SILVA, AND S. OUDOT, *Persistence stability for geometric complexes*, Geom. Dedicata, (2013), pp. 193–214.

[12] F. CHAZAL AND S. OUDOT, *Towards persistence-based reconstruction in Euclidean spaces*, in Proceedings of the 24th Annual Symposium on Computational Geometry, ACM, New York, 2008, pp. 232–241.

[13] V. DE SILVA AND G. E. CARLSSON, *Topological estimation using witness complexes*, in Proceedings of the First Eurographics Conference on Point-Based Graphics, 2004, pp. 157–166.

[14] V. DE SILVA AND R. GHRIST, *Coverage in sensor networks via persistent homology*, Algebr. Geom. Topol., 7 (2007), pp. 339–358.

[15] C. H. DOWKER, *Topology of metric complexes*, Amer. J. Math., 74 (1952), pp. 555–577.

[16] H. EDELSBRUNNER AND J. L. HARER, *Computational Topology: An Introduction*, American Mathematical Society, Providence, RI, 2010.

[17] D. A. EDWARDS, *On the Kantorovich–Rubinstein theorem*, Expo. Math., 29 (2011), pp. 387–398.

[18] R. GHRIST, *Barcodes: The persistent topology of data*, Bull. Amer. Math. Soc. (N.S.), 45 (2008), pp. 61–75.

[19] M. GROMOV, *Geometric Group Theory, Volume* 2: *Asymptotic Invariants of Infinite Groups*, London Math. Soc. Lecture Note Ser. 182, Cambridge University Press, Cambridge, 1993.

[20] A. HATCHER, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.

[21] J.-C. HAUSMANN, *On the Vietoris–Rips complexes and a cohomology theory for metric spaces*, Ann. of Math. Stud. 138, Princeton University Press, Princeton, NJ, 1995, pp. 175–188.

[22] H. KARCHER, *Riemannian center of mass and mollifier smoothing*, Comm. Pure Appl. Math., 30 (1977), pp. 509–541.

[23] H. G. KELLERER, *Duality theorems for marginal problems*, Z. Wahrsch. Verw. Gebiete, 67 (1984), pp. 399–432.

[24] H. G. KELLERER, *Duality theorems and probability metrics*, in Proceedings of the Seventh Conference on Probability Theory, Braşov, Romania, 1985, pp. 211–220.

[25] J. LATSCHEV, *Vietoris–Rips complexes of metric spaces near a closed Riemannian manifold*, Arch. Math. (Basel), 77 (2001), pp. 522–528.

[26] S. LEFSCHETZ, *Algebraic Topology*, Colloquium Publications, Vol. 27, American Mathematical Society, Providence, RI, 1942.

[27] L. LOVÁSZ, *Self-dual polytopes and the chromatic number of distance graphs on the sphere*, Acta Sci. Math. (Szeged), 45 (1983), pp. 317–323.

[28] R. MACPHERSON AND B. SCHWEINHART, *Measuring shape with topology*, J. Math. Phys., 53 (2012), 073516.

[29] I. MARIN, *Measure Theory and Classifying Spaces*, preprint, http://arxiv.org/abs/1702.01889, 2017.

[30] I. MARIN, *Simplicial Random Variables*, preprint, http://arxiv.org/abs/1703.03987, 2017.

[31] S. MARTIN, A. THOMPSON, E. A. COUTSIAS, AND J.-P. WATSON, *Topology of cyclo-octane energy landscape*, J. Chem. Phys., 132 (2010), 234115.

[32] P. NIYOGI, S. SMALE, AND S. WEINBERGER, *Finding the homology of submanifolds with high confidence from random samples*, Discrete Comput. Geom., 39 (2008), pp. 419–441.

[33] D. ORNSTEIN, *A new proof of the paracompactness of metric spaces*, Proc. Amer. Math. Soc., 21 (1969), pp. 341–342.

[34] K. SAKAI, *Geometric Aspects of General Topology*, Springer, Tokyo, 2013.

[35] F. SANTAMBROGIO, *Optimal Transport for Applied Mathematicians*, Progr. Nonlinear Differential Equations Appl. 87, Birkäuser/Springer, Cham, 2015.

[36] A. H. STONE, *Paracompactness and product spaces*, Bull. Amer. Math. Soc., 54 (1948), pp. 977–982.

[37] K.-T. STURM, *Probability measures on metric spaces of nonpositive*, Contemp. Math. 338, American Mathematical Society, Providence, RI, 2003, pp. 357–390.

[38] T. TOM DIECK, *Algebraic Topology*, European Mathematical Society, Zurich, 2008.

[39] C. TUFFLEY, *Finite subset spaces of S^1*, Algebr. Geom. Topol., 2 (2002), pp. 1119–1145.

[40] A. M. VERSHIK, *Long history of the Monge–Kantorovich transportation problem*, Math. Intelligencer, 35 (2013), pp. 1–9.

[41] L. VIETORIS, *Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammen-hangstreuen Abbildungen*, Math. Ann., 97 (1927), pp. 454–472.

[42] C. VILLANI, *Optimal Transport: Old and New*, Grundlehren Math. Wiss. 338, Springer-Verlag, Berlin, 2008.

[43] A. ZOMORODIAN, *Topological data analysis*, Advances in Applied and Computational Topology, Proc. Sympos. Appl. Math. 70, American Mathematical Society, Providence, RI, 2012, pp. 1–39.