Metric reconstruction via optimal transport

Henry Adams (Colorado State University), Michał Adamaszek (MOSEK ApS), Florian Frick (Carnegie Mellon)

Thanks to my graduate students: Johnathan Bush, Brittany Carr, Mark Heim, Lara Kassab, Joshua Mirth

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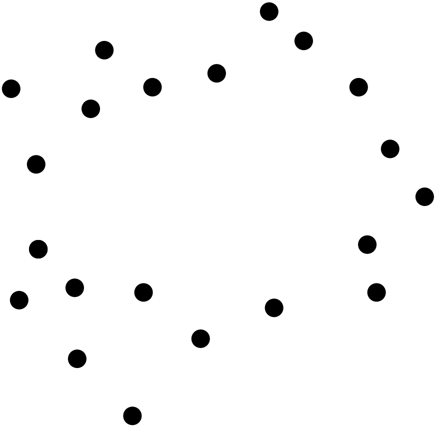
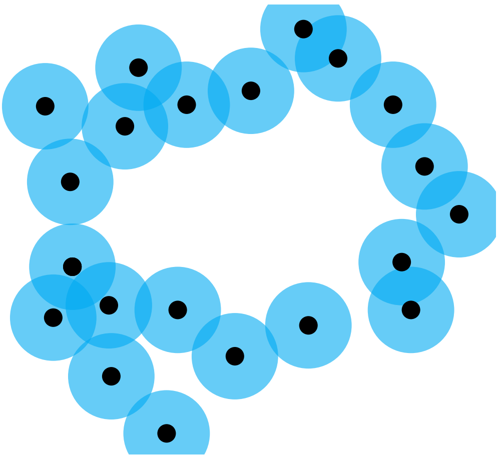
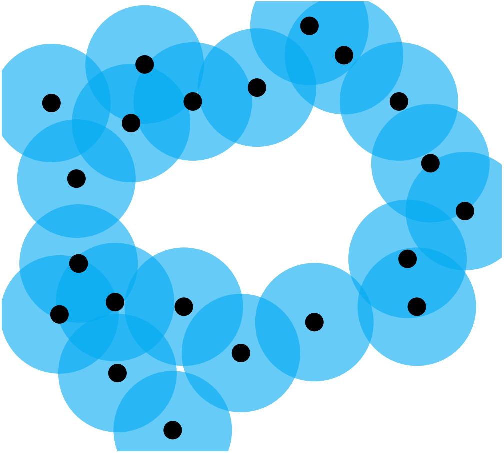
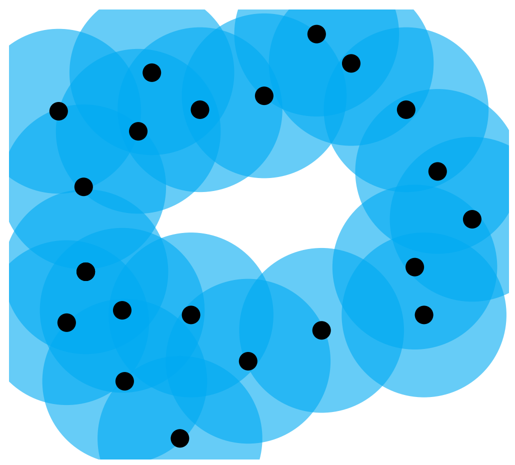
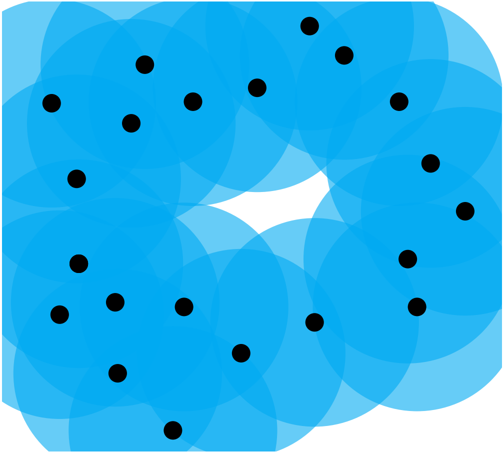
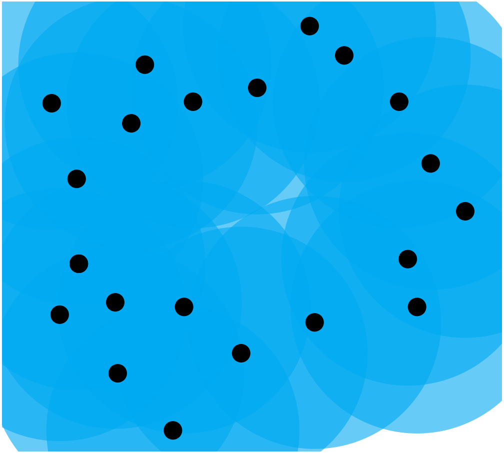


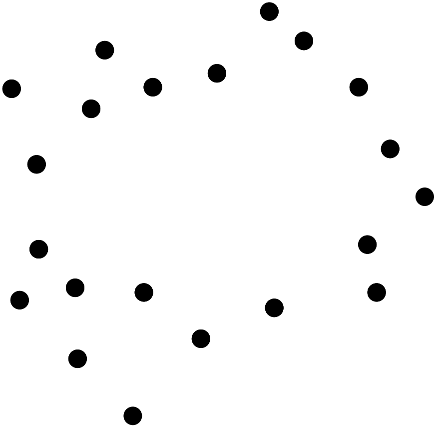








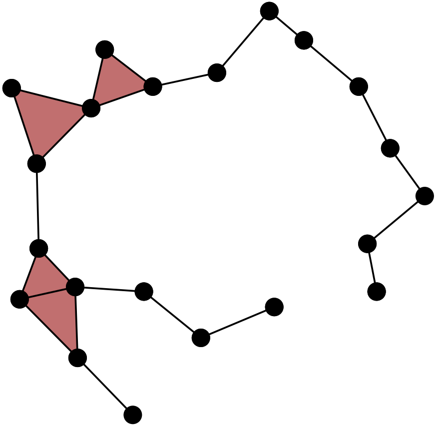




Definition

For X a metric space and scale $r > 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X; r)$ has

- vertex set X
- simplex $\{x_0, \ldots, x_k\}$ when $\mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r$.



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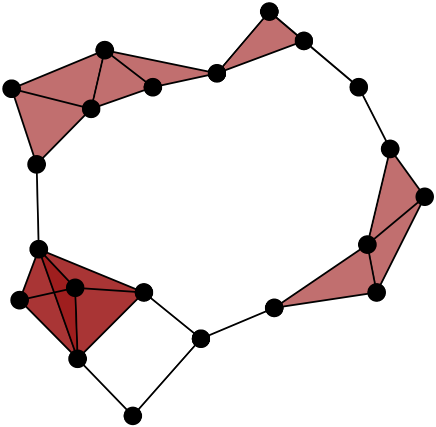

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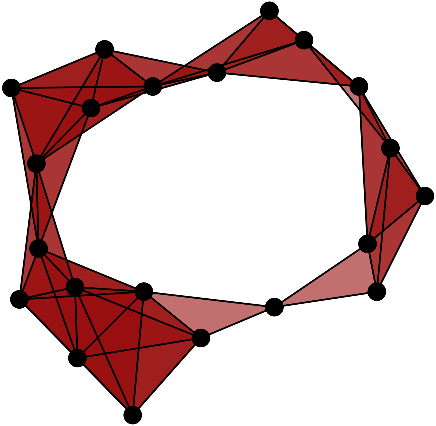

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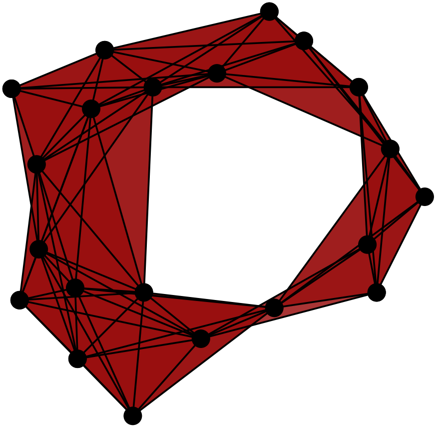

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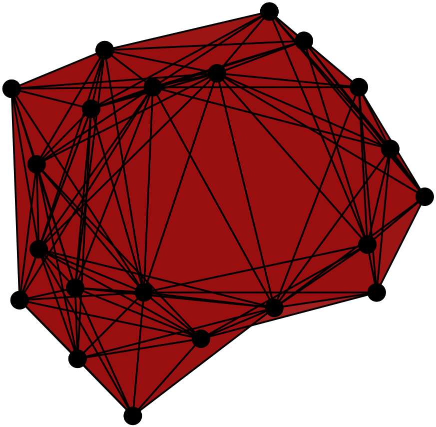

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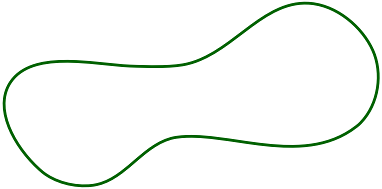

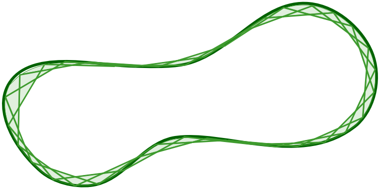

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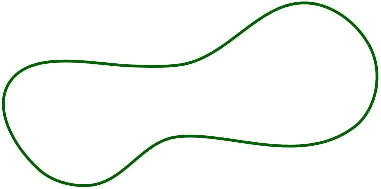

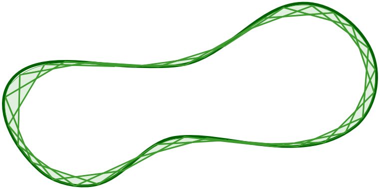

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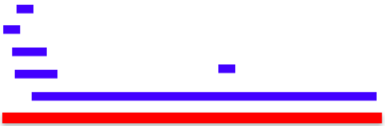
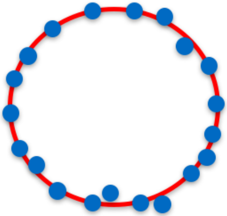
Theorem (Chazal, de Silva, Oudot, 2013)

For X and Y totally bounded metric spaces,

$$d_b(\mathrm{PH}(\mathrm{VR}(X;-)), \mathrm{PH}(\mathrm{VR}(Y;-))) \leq 2d_{\mathrm{GH}}(X,Y).$$





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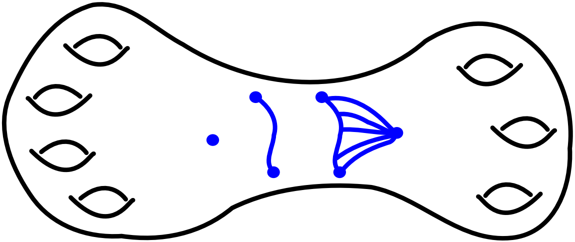
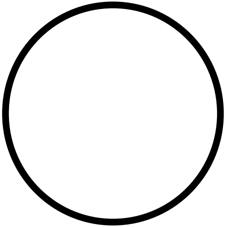
Downsides of the proof:

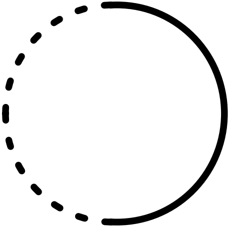
- map $\mathrm{VR}(M;r) \to M$ depends on the choice of a total ordering of all points in M, and
- the inclusion $M \hookrightarrow \mathrm{VR}(M;r)$ is not continuous.



The Vietoris–Rips simplicial complex may not be metrizable!

Metric space M $\rightsquigarrow$ $X \subseteq M$ $\rightsquigarrow$ $\mathrm{VR}(X; r)$



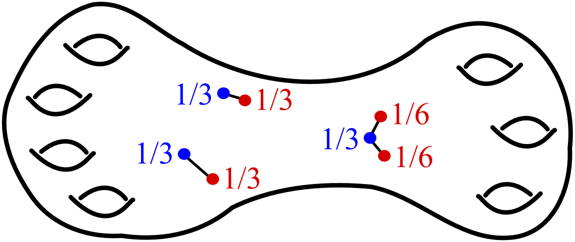


Definition

For X a metric space and $r > 0$, the *Vietoris–Rips metric thickening* is

$$\mathrm{VR}^m(X; r) = \left\{ \sum_{i=0}^{k} \lambda_i x_i \;\middle|\; \lambda_i \geq 0,\ \sum_i \lambda_i = 1,\ x_i \in X,\ \mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r \right\},$$

equipped with the 1-Wasserstein metric.



Note $\mathrm{VR}^m(X; r)$ is a *metric r-thickening* of X.

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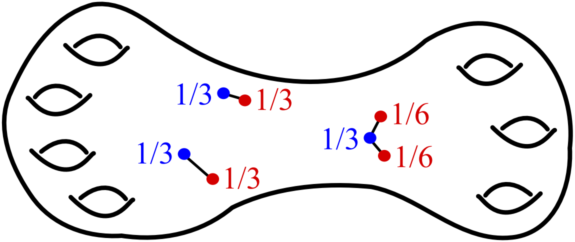
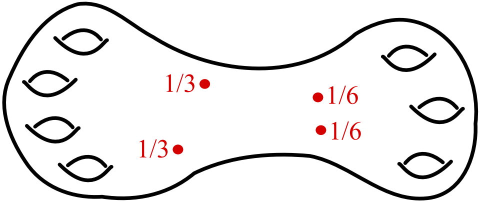


Note $\mathrm{VR}^m(X;r)$ is a *metric r-thickening* of X.

Theorem (Adamaszek, A, Frick)

If M is a Riemannian manifold and r is sufficiently small, then

$$\mathrm{VR}^m(M; r) \simeq M.$$



Proof.

$$\mathrm{VR}^m(M; r) \underset{\iota}{\overset{f}{\rightleftarrows}} M$$

- $f\colon \sum_i \lambda_i \delta_{x_i} \mapsto$ Fréchet mean.
- $f \circ \iota = \mathrm{id}_M$, and $\iota \circ f \simeq \mathrm{id}_{\mathrm{VR}^m(M;r)}$ via a linear homotopy. □

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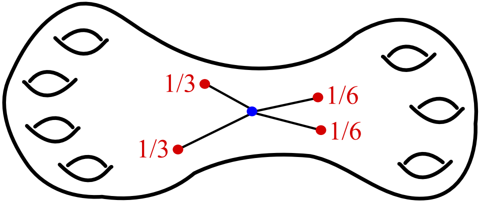

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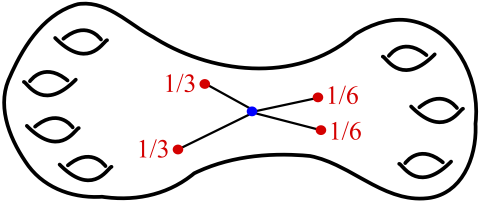

With Joshua Mirth we prove a metric Hausmann's theorem for sets in $\mathbb{R}^n$ with positive reach (including submanifolds).

Metric thickenings, Borsuk–Ulam theorems, and orbitopes

Henry Adams (Colorado State University)
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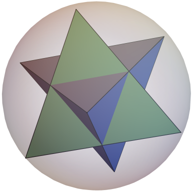
Paper in preparation



Theorem (Borsuk–Ulam)

For $f : S^n \to \mathbb{R}^n$*, there exists a point* $x \in S^n$ *with* $f(x) = f(-x)$*.*

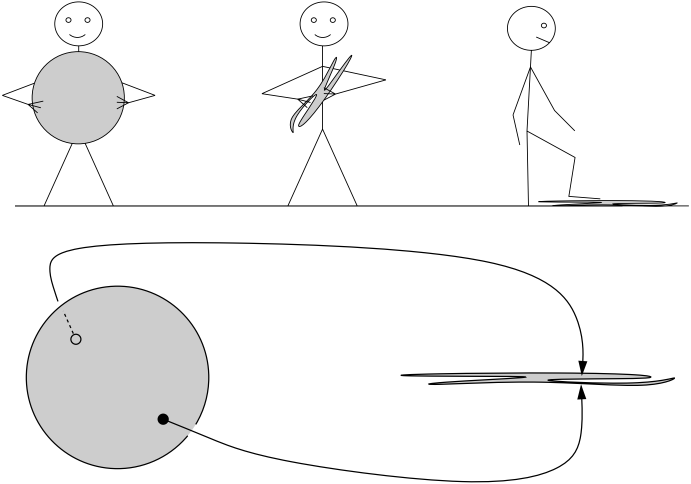


Figure credit: Jiří Matoušek, *Using the Borsuk–Ulam theorem*

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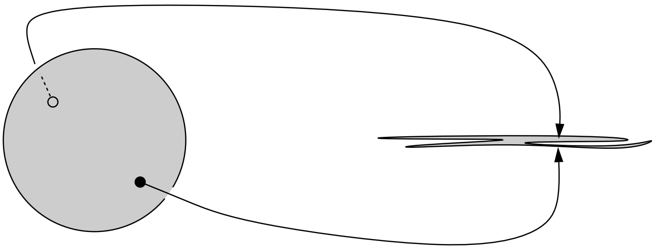

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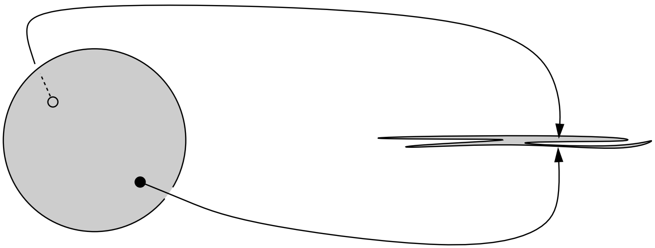

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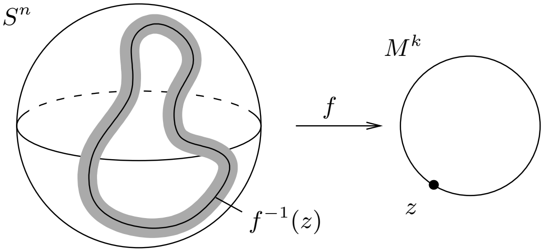

Figure credit: Benjamin Matschke, *Journal of Topology & Analysis*

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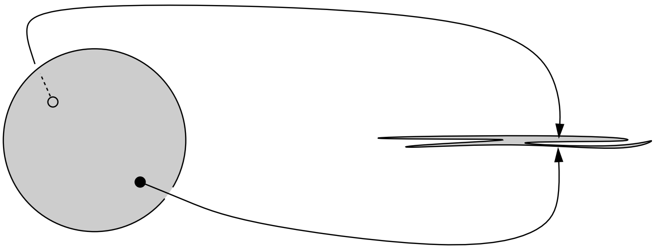

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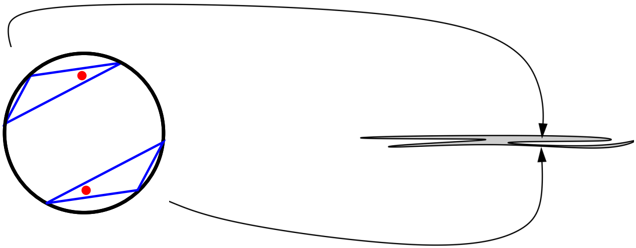

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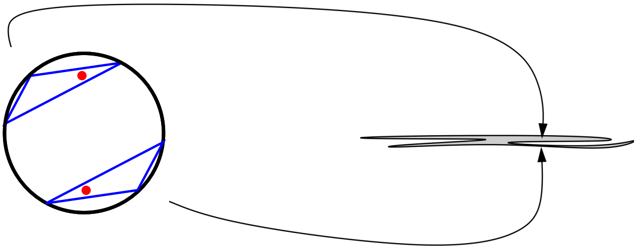
For $f: S^1 \to \mathbb{R}^{2k+1}$, there exists a set $\{x_0, \ldots, x_{2k+1}\}$ of diameter at most $\frac{k}{2k+1}$ such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$.



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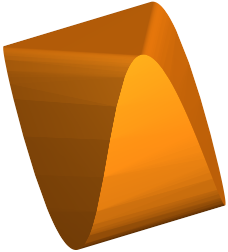

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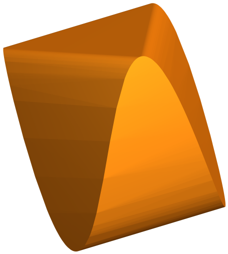
Proof: $S^{2k+1} \simeq \mathrm{VR}^m(S^1; \frac{k}{2k+1}) \xrightarrow{f} \mathbb{R}^{2k+1}$



Sharpness: $f = (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots)$

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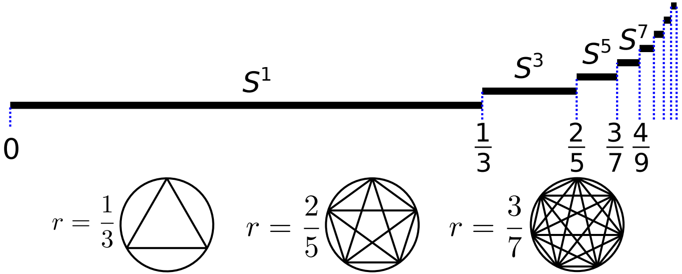

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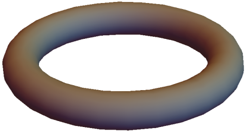

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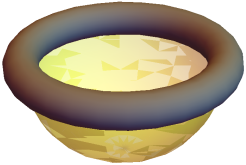

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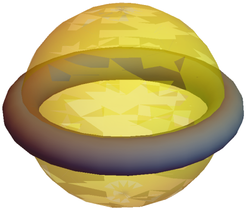

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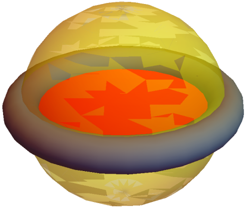

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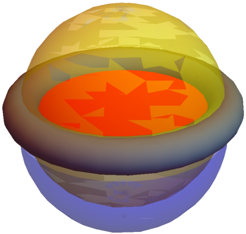

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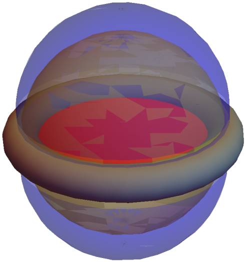

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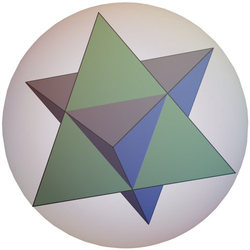
Theorem (A, Bush, Frick)

For $f\colon S^n \to \mathbb{R}^{n+2}$, there exists a set $\{x_0, \ldots, x_{n+2}\}$ of diameter at most r_n such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$.

Proof: S^{n+2} "$\subseteq$" $\mathrm{VR}^m(S^n; r_n) \xrightarrow{f} \mathbb{R}^{n+2}$



r_n is the side-length of an inscribed simplex

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Thank you!