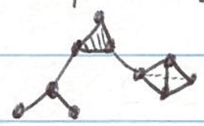
Metric reconstruction via optimal transport
Joint with Michał Adamaszek and Florian Frick

X metric space, $r \geq 0$.

Def The Vietoris–Rips simplicial complex $VR(X;r)$ has
- vertex set X
- simplex $\{x_0, x_1, \ldots, x_k\} \subseteq X$ when $\mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r$.



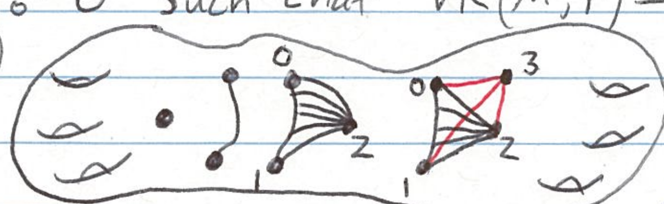
Clique or flag simplicial complex

History Leopold Vietoris (111 years old)
- cohomology theory for metric spaces
- recovers Čech cohomology if X compact
- Vietoris homology is counterpart to Alexander–Spanier cohomology

Ilya Rips
- Geometric group theory
- $VR\left(\begin{smallmatrix}\delta\text{-hyperbolic group} \\ \text{word metric}\end{smallmatrix}; r\right) \simeq *$ for $r \geq 4\delta$

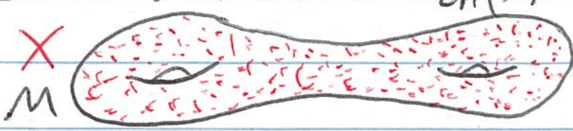
Thm (Hausmann 1995)
M compact Riemannian manifold.
Then $\exists r_0 > 0$ such that $VR(M;r) \simeq M$ $\forall r < r_0$.

Sketch $VR(M;r)$
$\downarrow$
M

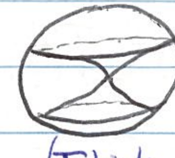


Not canonical
$M \hookrightarrow VR(M;r)$ not continuous

Thm (Latscher 2001) M, r_0 as above.
$\forall r < r_0$ $\exists \delta > 0$ such that if $d_{GH}(X,M) < \delta$, then $VR(X;r) \simeq M$.



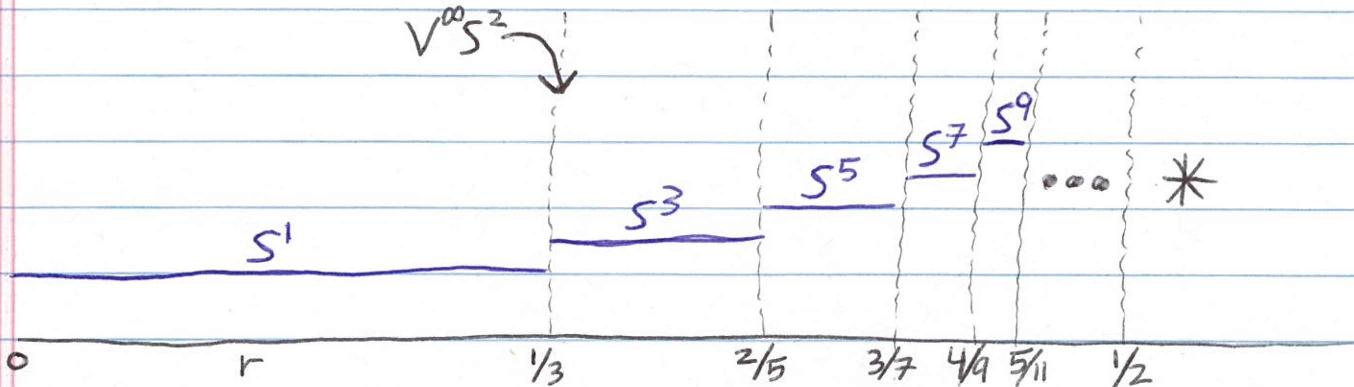
Ex Cyclo-octane molecule C_8H_{16}
(Martin, Thompson, Coutsias, Watson 2010)

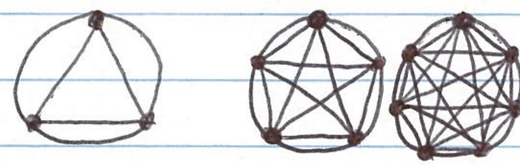


Stability $d_B\big(PH(VR(X;-)), PH(VR(Y;-))\big) \leq 2 \cdot d_{GH}(X,Y)$ (Think $Y=M$)

S^1 is circle with geodesic metric, unit circumference.

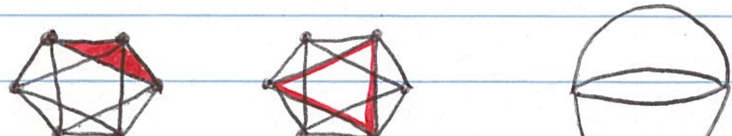
<u>Thm</u> $\mathrm{VR}(S^1;r) \simeq \begin{cases} S^{2\ell+1} & \text{if } \frac{\ell}{2\ell+1} < r < \frac{\ell+1}{2\ell+3} \\ \bigvee^{\infty} S^{2\ell} & \text{if } r = \frac{\ell}{2\ell+1} \end{cases}$ for $\ell \in \mathbb{N}$.





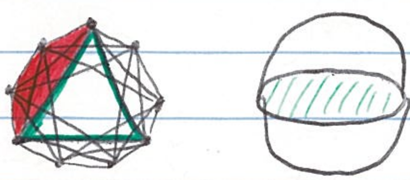
Only connected non-contractible manifold with all VR homotopy types known.
Why care? Stability theorem.

<u>$\mathrm{VR}(\cdots;\frac{1}{3})$</u>



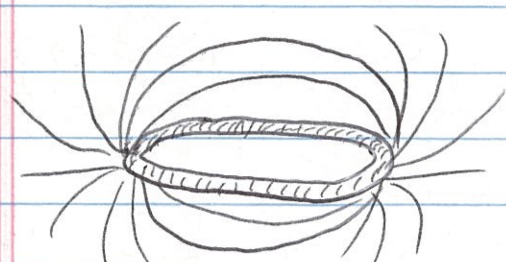
S^2

$\mathrm{VR}(\cdots;\frac{1}{3})$



$S^2 \vee S^2$

$\mathrm{VR}(S^1;\frac{1}{3})$



$$S^3 = S^1 \times D^2 \underset{S^1\times S^1}{\cup} D^2 \times S^1$$

$$S^{2\ell+1} = S^{2\ell-1} \times D^2 \underset{S^{2\ell-1}\times S^1}{\cup} D^{2\ell} \times S^1$$

- Proof combinatorial (colimits, homotopy colimits)
- $\check{C}(S^1;r)$ regime when Nerve lemma fails

Metric reconstruction

metric space M $\rightsquigarrow$ $X \subseteq M$ $\rightsquigarrow$ $VR(X;r) \overset{?}{\simeq} M$

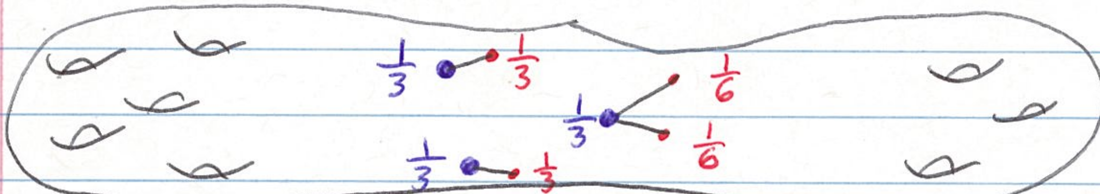
metrizable $\Leftrightarrow$ locally finite

Def X metric space, $r \geq 0$. The Vietoris–Rips thickening is

$$VR^m(X;r) = \left\{ \sum_{i=0}^{k} \lambda_i x_i \;\middle|\; \begin{array}{c} k \in \mathbb{N},\quad x_i \in X,\quad \operatorname{diam}(\{x_0, \ldots, x_k\}) \leq r, \\ \lambda_i \geq 0 \quad \sum \lambda_i = 1 \end{array} \right\},$$

equipped with the 1-Wasserstein metric.

Think of x_i as δ_{x_i}, a Dirac δ-measure.



$$d\left(\sum_{i=0}^{k} \lambda_i x_i, \sum_{j=0}^{k'} \lambda_j' x_j'\right) = \inf_{\{p_{i,j} \geq 0\},\ \sum_j p_{i,j} = \lambda_i,\ \sum_i p_{i,j} = \lambda_j'} \sum p_{i,j} d(x_i, x_j')$$

A matching or transport plan is a joint p.d.f. with given marginals.

Prop $VR^m(X;r)$ is an r-thickening of X.

Extends metric on X, and $d(X, VR^m(X;r)) \leq r$.

Gromov studied in the case X discrete.

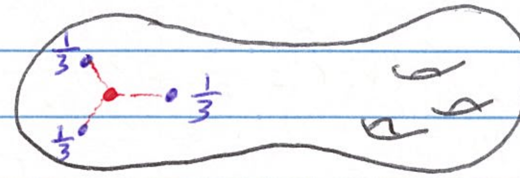
Thm M complete Riemannian manifold, $r_0 \geq 0$ satisfies
- balls of radius r_0 geodesically convex
- $r_0 < \frac{\pi}{4} K^{-1/2}$ (K sectional curvatures)

Then $VR^m(M;r) \simeq M$ for $r < r_0$.

Pf Sketch $VR^m(M;r)$ $\Sigma \lambda_i x_i$

$\downarrow \uparrow$ $\downarrow$

M Karcher or Frechét mean



Linear homotopies (dual space of measures)

Rmk $VR^m(S^1;\frac{1}{3}) \simeq S^3$

Thm $VR^m(S^n;r) \simeq \begin{cases} S^n & r < r_n \\ \Sigma^{n+1} \frac{SO(n+1)}{A_{n+2}} & r = r_n \end{cases}$

r_n = diameter of inscribed regular Δ^{n+1}


$r_1 = 1/3$


r_2

A_{n+2} = alternating group (rotational symmetries of Δ^{n+1})

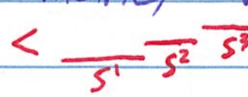
Pf Sketch $VR^m(S^n;r_n) = VR^m(S^n;r_n) \setminus$ interiors of regular Δ^{n+1} $\cup \ \Delta^{n+1} \times \frac{SO(n+1)}{A_{n+2}}$

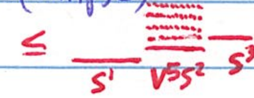
$$\simeq S^n \times C\left(\frac{SO(n+1)}{A_{n+2}}\right) \cup C(S^n) \times \frac{SO(n+1)}{A_{n+2}}$$

$$= S^n * \frac{SO(n+1)}{A_{n+2}}$$

$$= \Sigma^{n+1} \frac{SO(n+1)}{A_{n+2}}.$$

Questions
- Larger r? Lovász' strongly-self-dual polytopes
- Other manifolds? VR(L^∞ tori), flat metric, VR(ellipse)
- Čech





- Morse theory, Morse-Bott theory
- Barvinok-Novik orbitopes, Borsuk-Ulam type theorems
- $VR_< \simeq VR^m_<$?
- Applied topology, AATRN seminar, YouTube