Metric reconstruction via optimal transport

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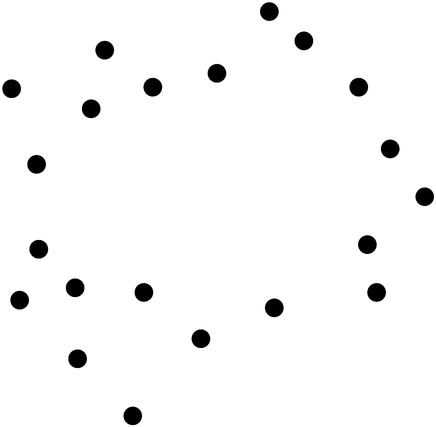
Thanks to my graduate students: Johnathan Bush, Brittany Carr, Mark Heim, Lara Kassab, Joshua Mirth, Alex Williams



Definition

For X a metric space and scale $r > 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X; r)$ has

- vertex set X
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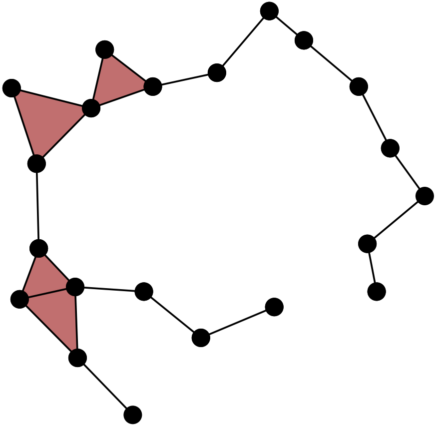

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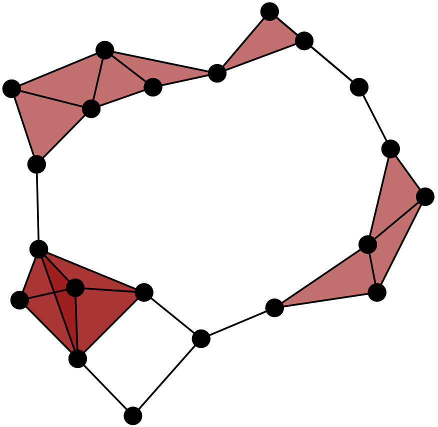

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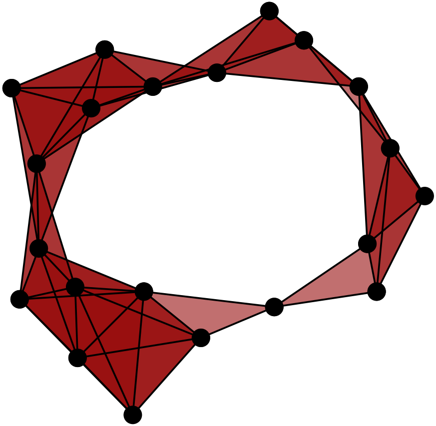

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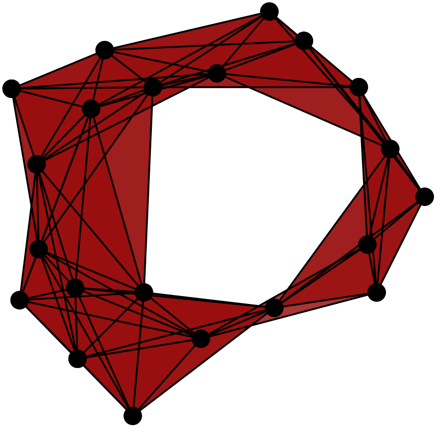

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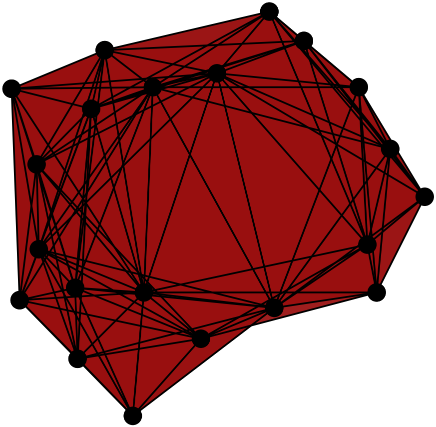

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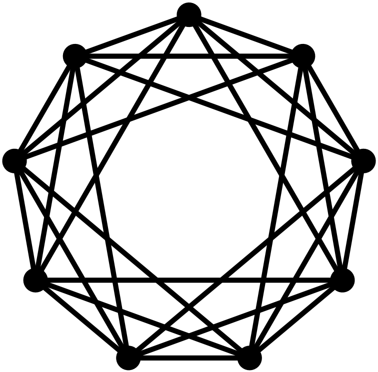

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For M a compact Riemannian manifold and r small,

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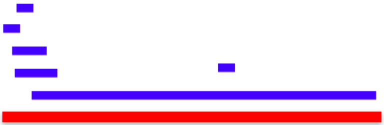
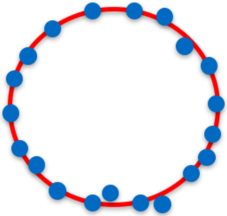
Theorem (Chazal, de Silva, Oudot, 2013)

For X and Y totally bounded metric spaces,

$$d(\mathrm{PH}(\mathrm{VR}(X; -)), \mathrm{PH}(\mathrm{VR}(Y; -))) \leq 2d_{\mathrm{GH}}(X, Y).$$





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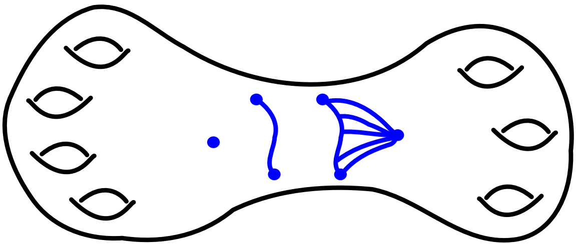
Downsides of the proof:

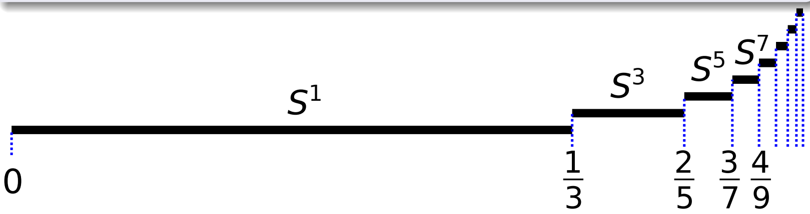
- map $\mathrm{VR}(M;r) \to M$ depends on the choice of a total ordering of all points in M, and
- the inclusion $M \hookrightarrow \mathrm{VR}(M;r)$ is not continuous.



Let S^1 be the circle of unit circumference.

Theorem (Adamaszek, A)

$$\mathrm{VR}(S^1; r) \simeq \begin{cases} S^{2k+1} & \text{if } \frac{k}{2k+1} < r < \frac{k+1}{2k+3} \quad \text{for some } k \in \mathbb{N}. \\ & \end{cases}$$



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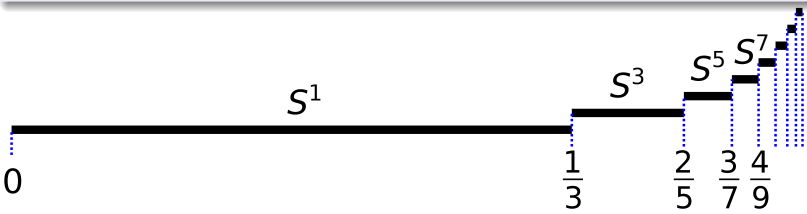

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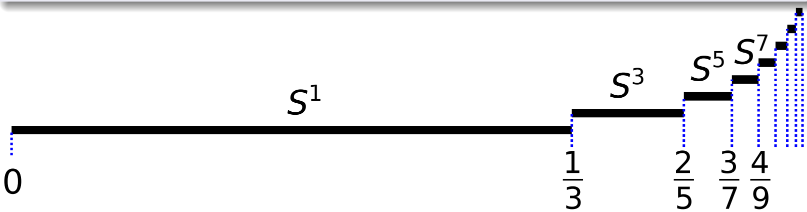
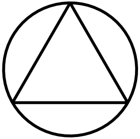
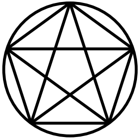
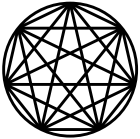


$r = \frac{1}{3}$ 

$r = \frac{2}{5}$ 

$r = \frac{3}{7}$ 

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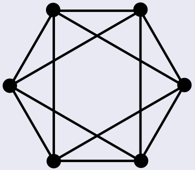
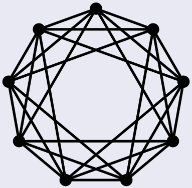


$\mathrm{VR}(6 \text{ points}, \frac{1}{3}) \simeq S^2$



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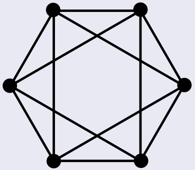

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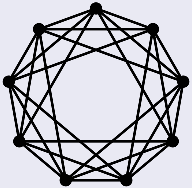

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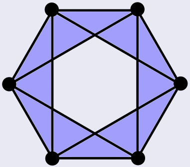

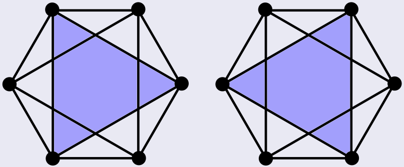

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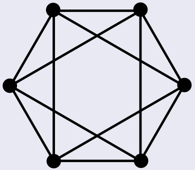

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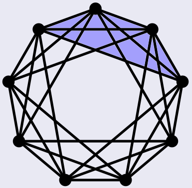

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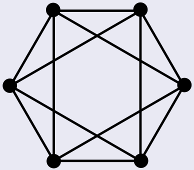

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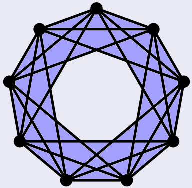

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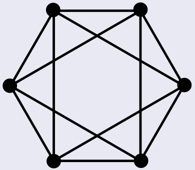

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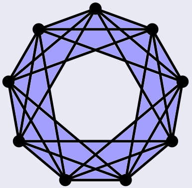

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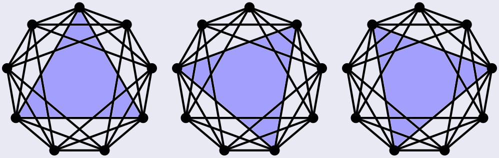

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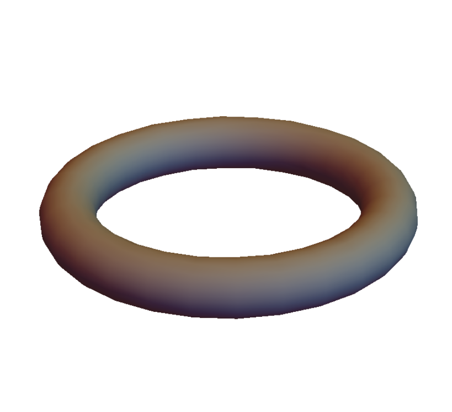

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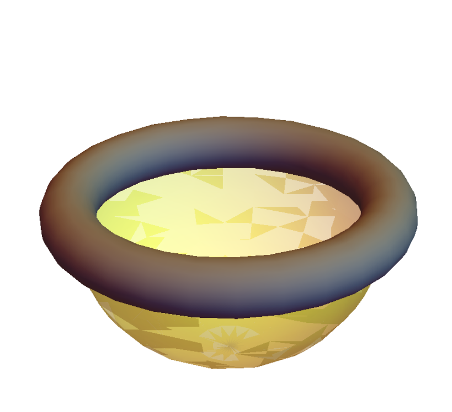

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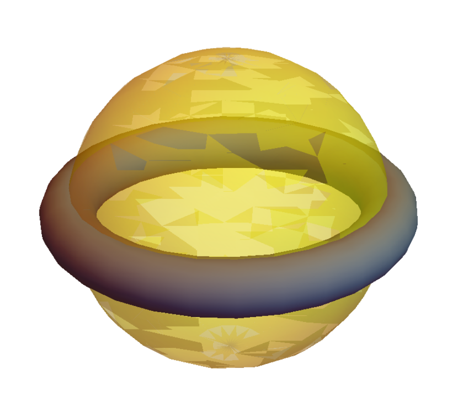

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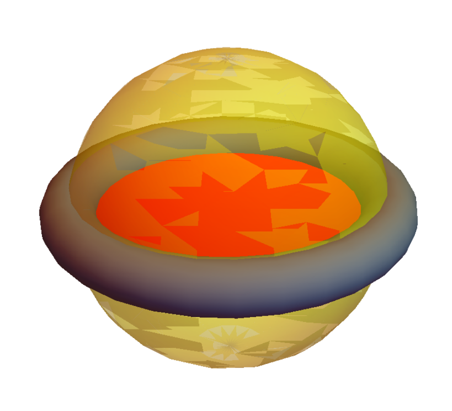

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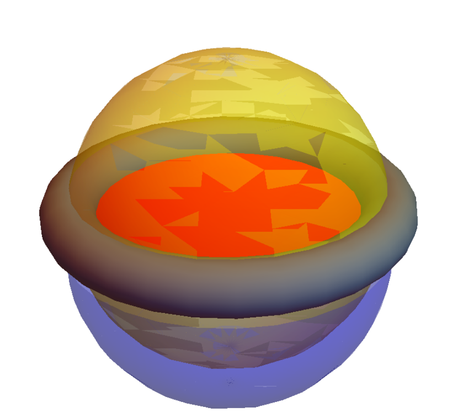

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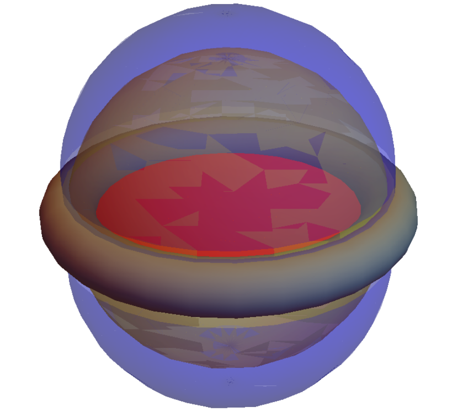

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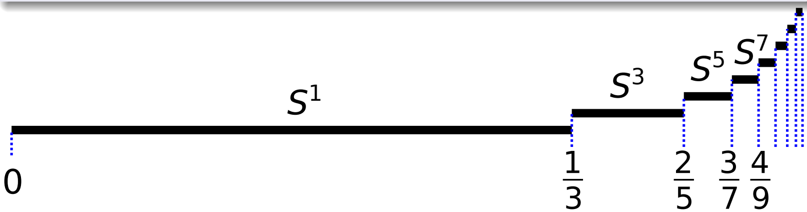
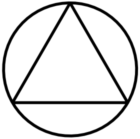
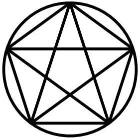
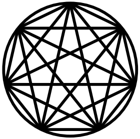
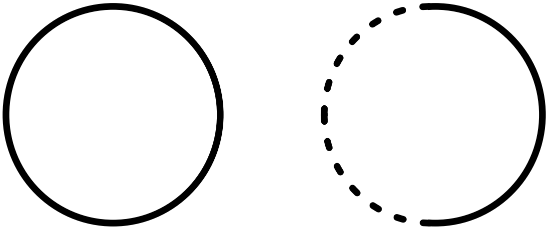


$r = \frac{1}{3}$  $r = \frac{2}{5}$  $r = \frac{3}{7}$ 

Čech/nerve version is analogous! By contrast, $\mathrm{VR}^m(S^1; r) \simeq S^3$.

The Vietoris–Rips simplicial complex may not be metrizable!

Metric space M $\rightsquigarrow$ $X \subseteq M$ $\rightsquigarrow$ $\mathrm{VR}(X; r)$



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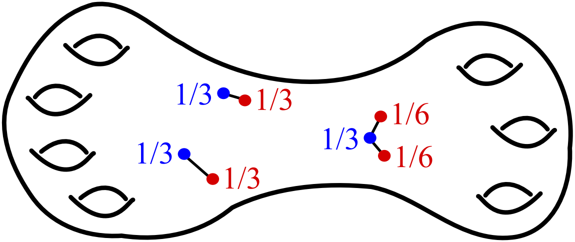
equipped with the 1-Wasserstein metric.



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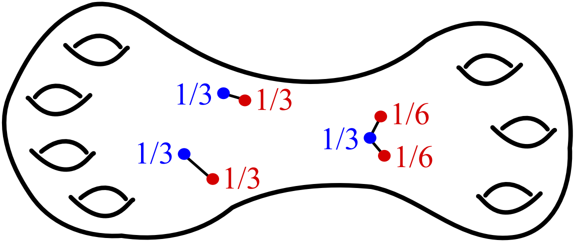

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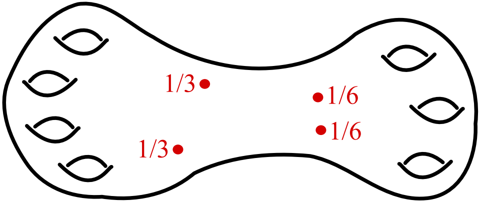
Theorem (Adamaszek, A, Frick)

If M is a Riemannian manifold and r is sufficiently small, then

$$\mathrm{VR}^m(M;r) \simeq M.$$



Proof.

$$\mathrm{VR}^m(M;r) \underset{\iota}{\overset{f}{\rightleftarrows}} M$$

- $f\colon \sum_i \lambda_i \delta_{x_i} \mapsto$ Fréchet mean.
- $f \circ \iota = \mathrm{id}_M$, and $\iota \circ f \simeq \mathrm{id}_{\mathrm{VR}^m(M;r)}$ via a linear homotopy. □

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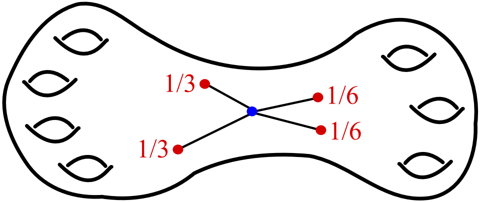

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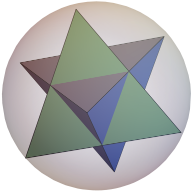
We can now say something about the n-sphere S^n

Theorem (Adamaszek, A, Frick)

$$\mathrm{VR}^m(S^n; r) \simeq \begin{cases} S^n & r < r_c \\ \Sigma^{n+1} \frac{\mathrm{SO}(n+1)}{A_{n+2}} & r = r_c. \end{cases}$$



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r_c = diameter of inscribed regular $(n+1)$-simplex.

Theorem (Unpublished)

$$\check{\mathrm{C}}^m(S^n; r) \simeq \begin{cases} S^n & r < \frac{1}{4} \\ \Sigma^{n+1} \mathbb{RP}^n & r = \frac{1}{4}. \end{cases}$$

Open questions

1. $\mathrm{VR}^m(S^n;r)$ and $\check{\mathrm{C}}^m(S^n;r)$ for larger r? Lovász' strongly self-dual polytopes
2. Other manifolds M?
3. $\mathrm{VR}_<(X;r) \simeq \mathrm{VR}^m_<(X;r)$?
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5. Morse, Morse–Bott, and Bestvina–Brady Morse theories
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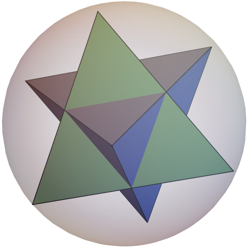

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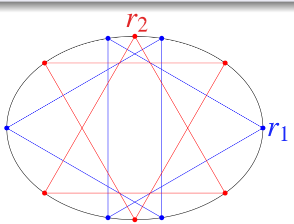

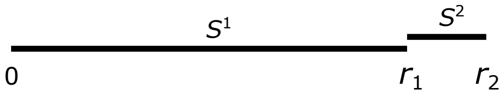

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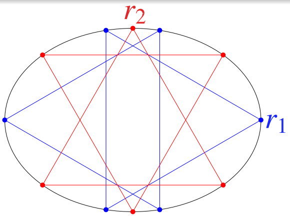

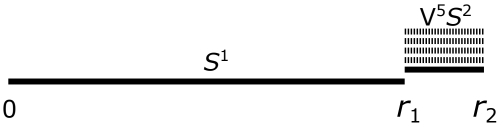

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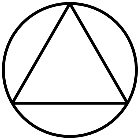
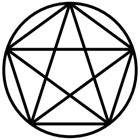
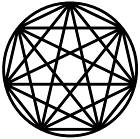
$r = \frac{1}{3}$ 

$r = \frac{2}{5}$ 

$r = \frac{3}{7}$ 

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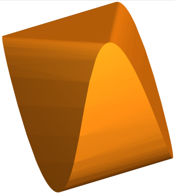

$$f(\theta) = (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots)$$

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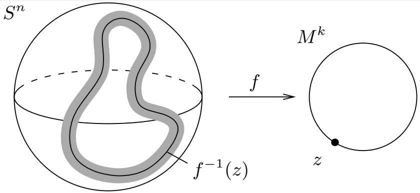

Figure Credit: B. Matschke, *Journal of Topology & Analysis*

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- Michał Adamaszek and Henry Adams, *The Vietoris–Rips complexes of a circle*, Pacific Journal of Mathematics 290 (2017), 1–40.
- Henry Adams, Johnathan Bush, and Florian Frick, *Metric thickenings, Borsuk–Ulam theorems, and orbitopes* (2019), in preparation.
- Jean-Claude Hausmann, *On the Vietoris–Rips complexes and a cohomology theory for metric spaces*, Annals of Mathematics Studies 138 (1995), 175–188.

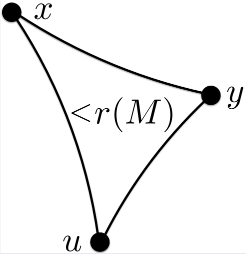
Thank you!

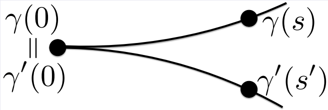
Theorem (Hausman, 1995)

Let M be a Riemannian manifold with $r(M) > 0$.
If $0 < r \leq r(M)$, then $\mathrm{VR}(M; r) \simeq M$.

Definition

Let $r(M)$ be the largest satisfying:
(a) If $d(x, y) < 2r(M)$, then $\exists!$ shortest geodesic between x and y.





- The n-sphere with great circle circumference 1 has $r(S^n) = \frac{1}{4}$.
- $r(M) > 0$ if M has positive injectivity radius and bounded sectional curvature (in particular if M compact).

We can now say something about $\mathrm{VR}^m(S^n; r)$ and $\check{\mathrm{C}}^m(S^n, S^n; r)$

$$\begin{aligned}
\mathrm{VR}^m(S^1; \tfrac{1}{3}) &\simeq \mathrm{VR}^m(S^1; \tfrac{1}{3} - \epsilon) \cup (D^2 \times S^1) \\
&\simeq (S^1 \times D^2) \cup_{S^1 \times S^1} (D^2 \times S^1) \\
&= S^1 * S^1 \\
&= S^3
\end{aligned}$$

$\mathrm{SO}(n+1) =$ group of orientation-preserving isometries of $\mathbb{R}^{n+1}$.
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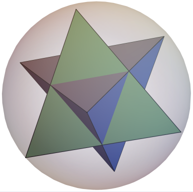

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\check{\mathrm{C}}^m(S^n, S^n; \tfrac{1}{4}) &\simeq \check{\mathrm{C}}^m(S^n, S^n; \tfrac{1}{4} - \epsilon) \cup (C(S^n) \times \mathbb{RP}^n) \\
&\simeq (S^n \times C(\mathbb{RP}^n)) \cup (C(S^n) \times \mathbb{RP}^n) \\
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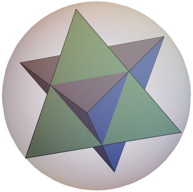
Metric thickenings, Borsuk–Ulam theorems, and orbitopes

Henry Adams (Colorado State University)
Johnathan Bush (Colorado State University)
Florian Frick (Carnegie Mellon)

Paper in preparation



Theorem (Borsuk–Ulam)

For $f : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.

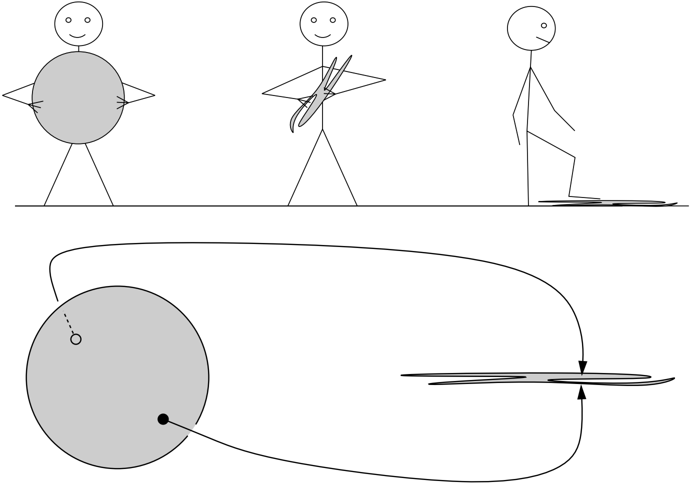


Figure credit: Jiří Matoušek, *Using the Borsuk–Ulam theorem*

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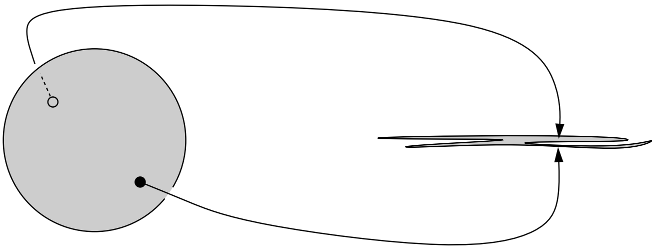
Theorem (Gromov's "waist" theorem)

For $f : S^n \to \mathbb{R}^n$, there exists some $y \in \mathbb{R}^n$ with $\mathrm{vol}_0(f^{-1}(y)) \geq \mathrm{vol}_0(S^0 \subseteq S^n)$.



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Theorem (Gromov's "waist" theorem)

For $f : S^n \to \mathbb{R}^k$ with $k \leq n$, there exists some $y \in \mathbb{R}^n$ with $\mathrm{vol}_{n-k}(f^{-1}(y)) \geq \mathrm{vol}_{n-k}(S^{n-k} \subseteq S^n)$.

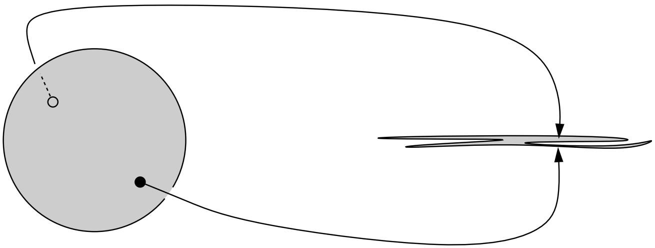

Figure credit: Jiří Matoušek, *Using the Borsuk–Ulam theorem*

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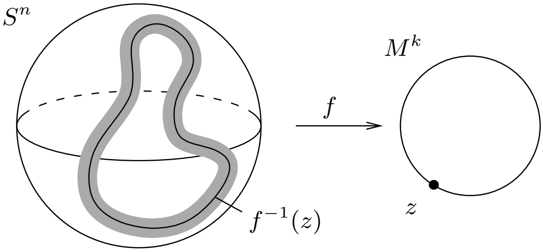

Figure credit: Benjamin Matschke, *Journal of Topology & Analysis*

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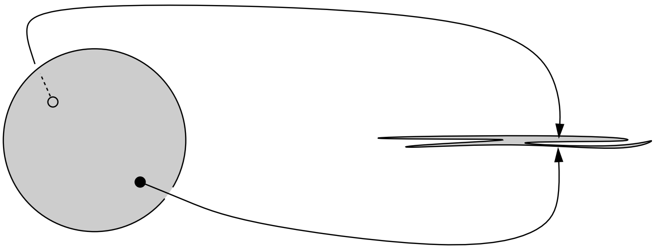

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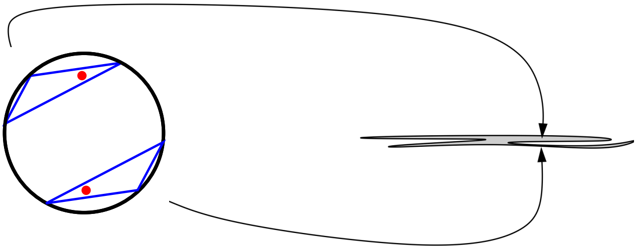

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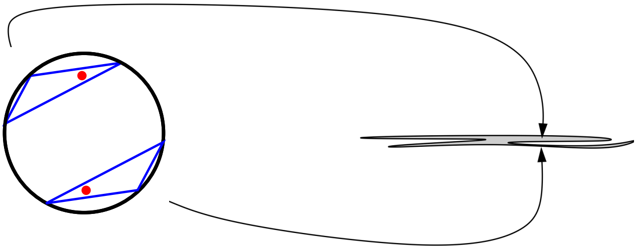

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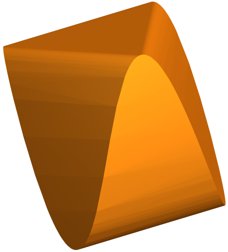

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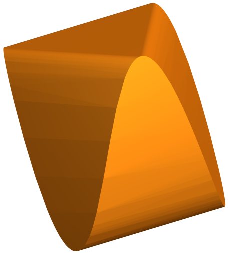

Sharpness: $f = (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots)$

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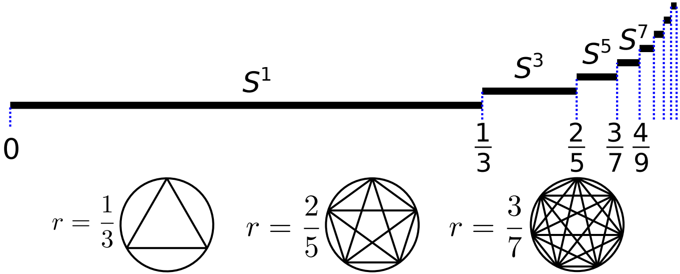

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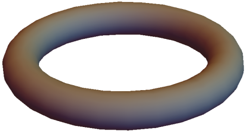

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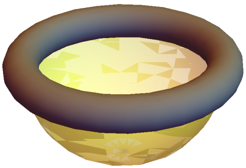

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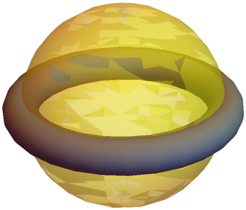

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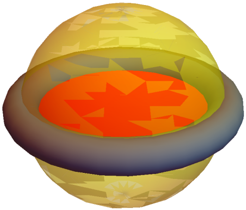

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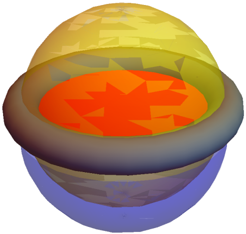

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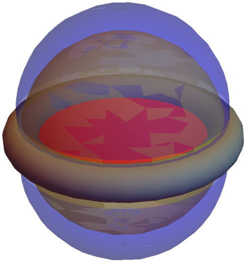

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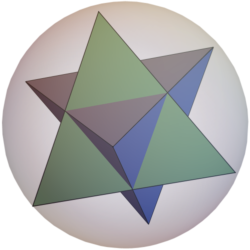
For $f\colon S^n \to \mathbb{R}^{n+2}$, there exists a set $\{x_0, \ldots, x_{n+2}\}$ of diameter at most r_n such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$.

Proof: S^{n+2} " $\subseteq$ " $\mathrm{VR}^m(S^n; r_n) \xrightarrow{f} \mathbb{R}^{n+2}$



r_n is the side-length of an inscribed simplex

Theorem (A, Bush, Frick)

For $f : S^n \to \mathbb{R}^{n+2}$, there exists a point $\sum \lambda_i x_i$ of diameter at most that of an inscribed simplex such that $f(\sum \lambda_i x_i) = f(\sum \lambda_i(-x_i))$.