Persistence Stability for Metric Thickenings

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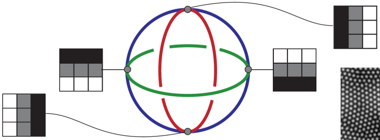
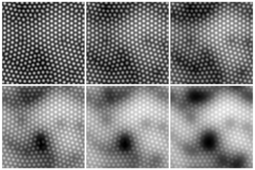
Applied Topology

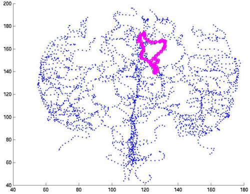
- Premise: data can have shape



Barcodes: The persistent topology of data by Robert Ghrist



Measures of order for nearly hexagonal lattices by Motta et al.



Persistent homology analysis of brain artery trees by Bendich et al.

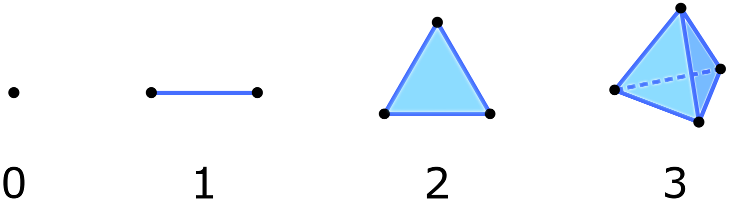
- The field of applied topology uses tools from topology to study the shape of datasets.

Applied Topology

- What is the topology of a dataset in $\mathbb{R}^n$?
- We begin by associating a more interesting topological space to the data points.

Simplicial Complexes

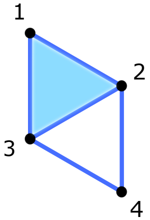
A simplicial complex is formed from simplices.



Simplicial Complexes

An abstract simplicial complex records the vertices that form each simplex. The geometric realization has a topology.

$$\Big\{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\\ \{2,3\},\{2,4\},\{3,4\},\{1,2,3\}\Big\}$$



Starting with a dataset, we can build a simplicial complex using the data points as vertices.

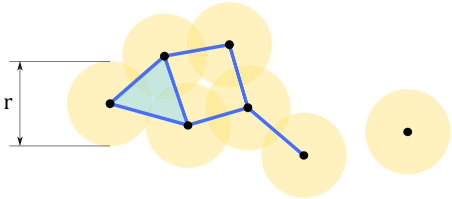
Simplicial Maps

A continuous function between simplicial complexes can be defined by specifying a function on vertices that takes simplices to simplices. This is called a *simplicial map*.

Vietoris–Rips Complexes

Given a vertex set that is a metric space, we can form the *Vietoris–Rips complex*

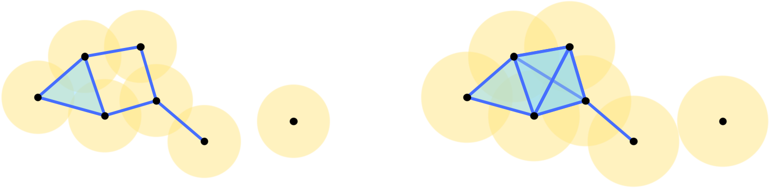
$$\mathrm{VR}(X; r) = \Big\{ \{x_1, \ldots, x_n\} \subseteq X \;\Big|\; \mathrm{diam}\big(\{x_1, \ldots, x_n\}\big) \leq r \Big\}$$



Vietoris–Rips Complexes

The Vietoris–Rips complex grows as the parameter r grows: if $r_1 \leq r_2$, then

$$\mathrm{VR}(X; r_1) \subseteq \mathrm{VR}(X; r_2)$$



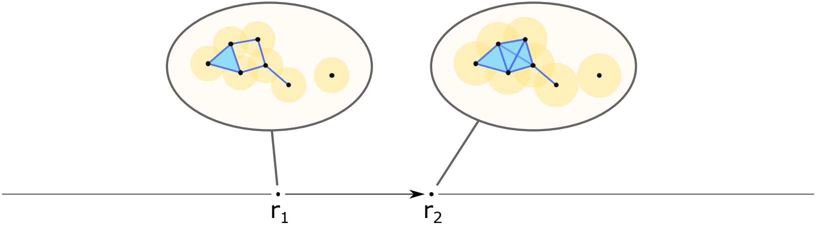
Other simplicial complexes are common as well, including *Čech complexes*.

Filtrations of Vietoris–Rips Complexes

Consider all parameters at once:

$$\mathrm{VR}(X;\_) = \{\mathrm{VR}(X;r) \mid r \in \mathbb{R}\}$$

This is called a *filtration*, and comes with inclusion maps $\mathrm{VR}(X;r_1) \hookrightarrow \mathrm{VR}(X;r_2)$ for all pairs $r_1 \leq r_2$.



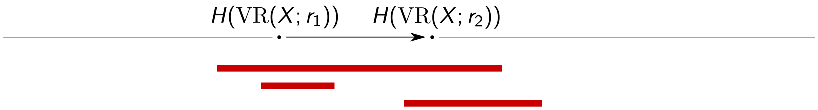
Persistent Homology

Persistent homology is based on homology.

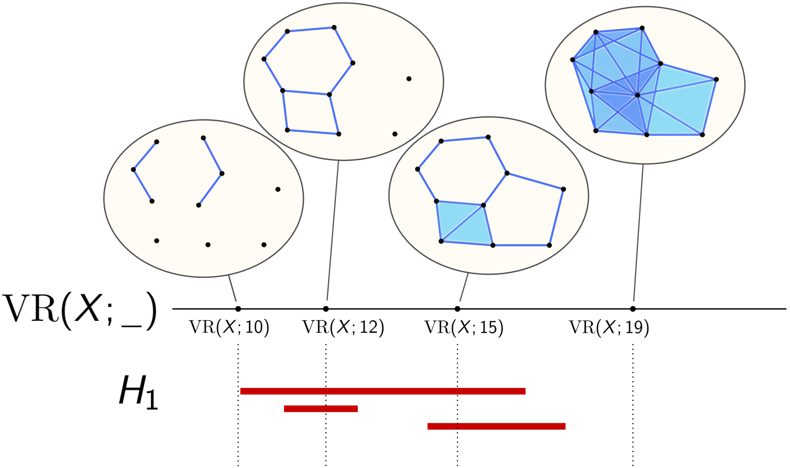
- $H_n(X)$ is a vector space, with dimension equal to the number of n-dimensional holes in X.
- We'll fix n and write $H(X)$.
- H is a functor: given a map $f : X \to Y$, we have a map $H(f) : H(X) \to H(Y)$.

Persistent Homology

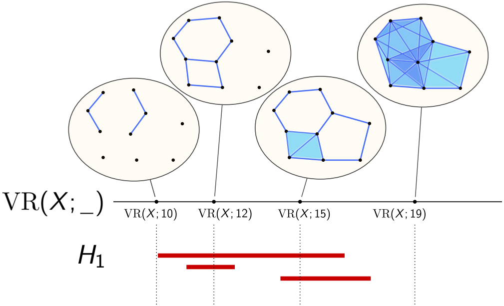
- Applying H to $\mathrm{VR}(X;\_)$ gives a *persistence module* $H(\mathrm{VR}(X;\_))$. This consists of vector spaces and linear maps.
- Persistent homology records the birth and death times of nonzero elements.

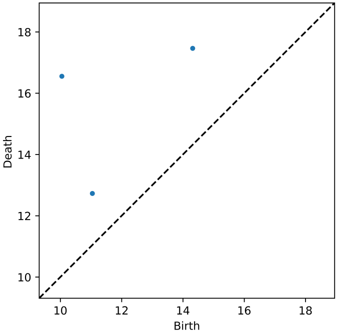


Persistent Homology



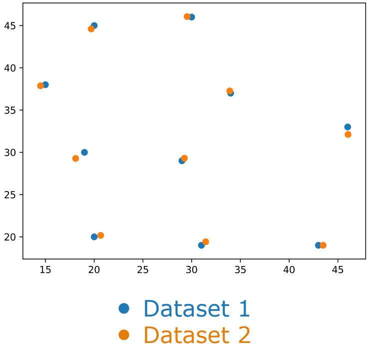
Persistence Diagrams and Barcodes

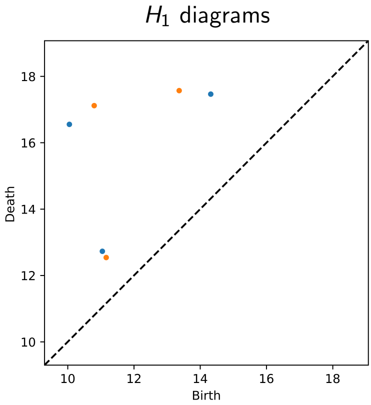




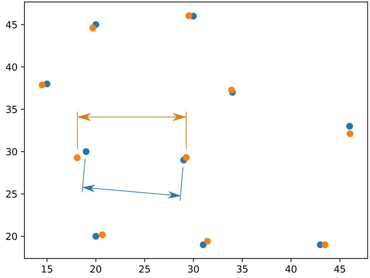
Stability of Persistent Homology

How do small changes to a dataset affect the persistence diagram?



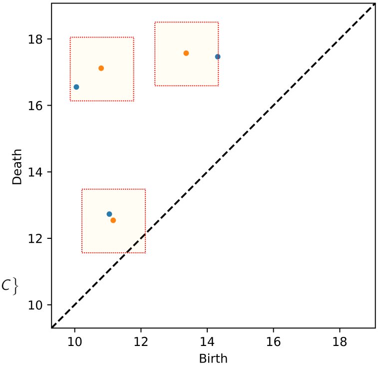


Gromov–Hausdorff Distance and Bottleneck Distance



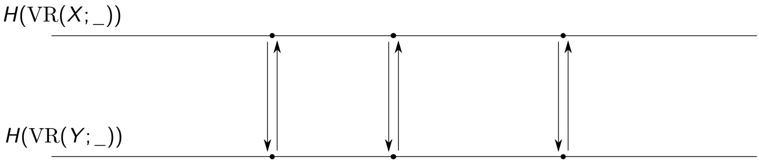
$$d_{GH}(X,Y) = \frac{1}{2} \inf_{C} \sup \left\{ |d_X(x,x') - d_Y(y,y')| : (x,y),(x',y') \in C \right\}$$

$$\mathrm{d_b}(D_1, D_2) = \inf \left\{ \varepsilon \mid \text{there exists an } \varepsilon\text{-matching between } D_1 \text{ and } D_2 \right\}$$

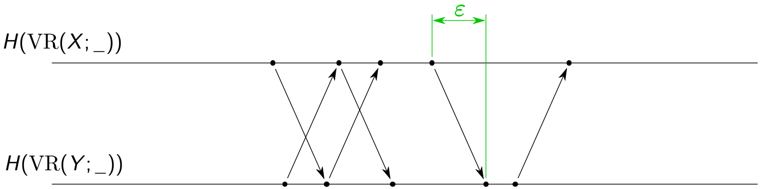


Interleavings

An isomorphism:



An ε-interleaving:



Interleavings

Theorem (Chazal, de Silva, Glisse, and Oudot)

If $\mathbb{U}$ *and* $\mathbb{V}$ *are q-tame persistence modules that are* ε*-interleaved, then* $\mathrm{d_b}\big(\mathrm{dgm}(\mathbb{U}), \mathrm{dgm}(\mathbb{V})\big) \leq \varepsilon$*.*

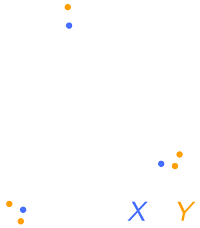
Interleavings for Vietoris–Rips Complexes

Lemma (Chazal, de Silva, and Oudot)

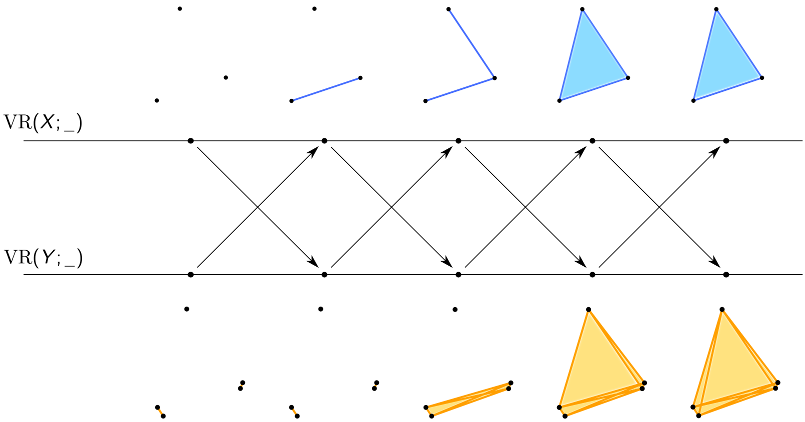
Let X and Y be metric spaces. For any $\varepsilon > 2d_{GH}(X, Y)$, the persistence modules $H(\mathrm{VR}(X; \_))$ and $H(\mathrm{VR}(Y; \_))$ are ε-interleaved.

The interleaving comes from maps on simplicial complexes that commute up to homotopy.

As an example, consider:



Interleavings for Vietoris–Rips Complexes



Stability of Persistent Homology

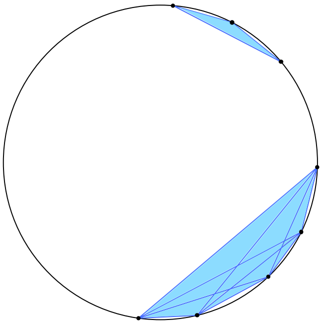
Theorem (Chazal, de Silva, and Oudot)

Let X and Y be totally bounded metric spaces. Then

$$\mathrm{d_b}\Big(\mathrm{dgm}\big(H(\mathrm{VR}(X;\_))\big), \mathrm{dgm}\big(H(\mathrm{VR}(Y;\_))\big)\Big) \leq 2d_{GH}(X, Y)$$

Simplicial Complexes on Infinite Vertex Sets

- $\mathrm{VR}(X; r)$ can be formed for any metric space X, including those with infinitely many points.



- Stability motivates the study of these complexes.
- Difficulties arise: the inclusion $X \hookrightarrow \mathrm{VR}(X; r)$ is not always continuous.

Metric Thickenings

Defined by Adamaszek, Adams, and Frick – an alternate approach for infinite metric spaces.

$$\mathrm{VR}^m(X;r) = \left\{ \sum_{i=1}^{n} \lambda_i \delta_{x_i} \;\middle|\; \lambda_i \geq 0 \text{ for all } i,\ \sum_{i=1}^{n} \lambda_i = 1,\ \{x_1, \ldots, x_n\} \in \mathrm{VR}(X;r) \right\}$$

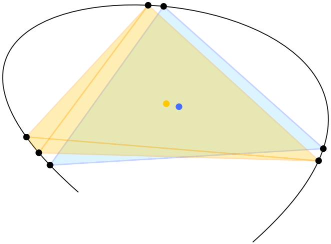
The δ_{x_i} are Dirac delta measures. The space is equipped with the Wasserstein metric.

Metric Thickenings

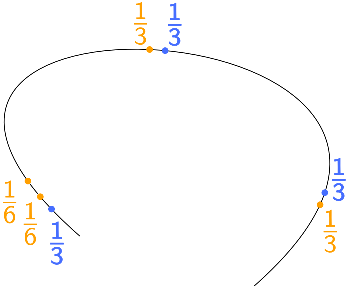
- Metric thickenings are essentially simplicial complexes built on metric spaces, but given different topologies.
- The topologies agree with simplicial complexes in the finite case, but may be different in the infinite case.
- The inclusion $X \hookrightarrow \mathrm{VR}^m(X; r)$ is continuous (for all $r \geq 0$).
- We get similar filtrations: $\mathrm{VR}^m(X; \_)$
- No exact analog of simplicial maps.

A Potentially Different Topology

Simplicial Complexes



Metric Thickenings

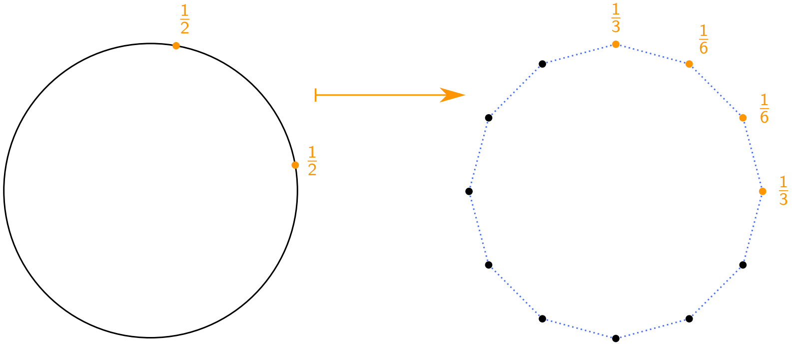


Persistence Stability for Metric Thickenings

- The technique for Vietoris–Rips complexes cannot be used.
- We will still construct interleavings starting with maps on spaces.
- We begin with a metric space X and a finite ε-sample $F \subseteq X$.
- We will define maps between $\mathrm{VR}^m(X; \_)$ and $\mathrm{VR}^m(F; \_)$.

Persistence Stability for Metric Thickenings

Given a map $\varphi_r \colon X \to \mathrm{VR}^m(F; r+\varepsilon)$, we get a continuous *induced map* $\tilde{\varphi}_r \colon \mathrm{VR}^m(X; r) \to \mathrm{VR}^m(F; r+\varepsilon)$ defined by $\tilde{\varphi}_r(\sum_i \lambda_i \delta_{x_i}) = \sum_i \lambda_i \varphi_r(x_i)$.

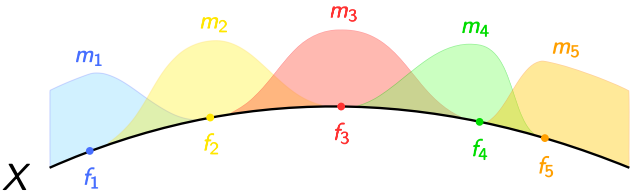


Persistence Stability for Metric Thickenings

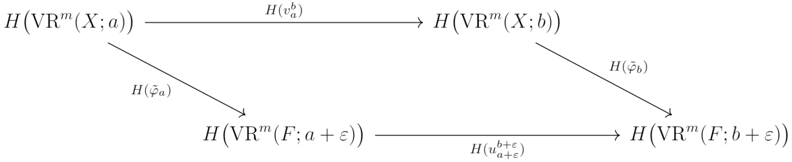
We choose φ_r carefully for arbitrary X and ε-sample F. We will require

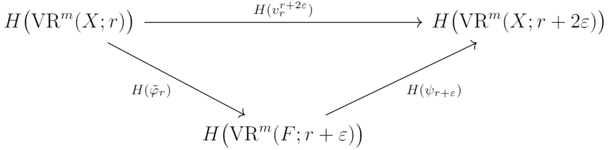
- φ_r is continuous
- $\mathrm{supp}(\varphi_r(x)) \subseteq B_\varepsilon(x)$
- For all $f \in F$, $\varphi_r(f) = \delta_f$

Define $\varphi_r(x) = \sum_{j=1}^{n} m_j(x)\delta_{f_j}$ with appropriate m_j



The induced maps $\tilde{\varphi}_r \colon \mathrm{VR}^m(X; r) \to \mathrm{VR}^m(F; r+\varepsilon)$ can be shown to commute with inclusion maps up to homotopy. Applying H gives an interleaving.





Persistence Stability for Metric Thickenings

We've compared X to a finite sample F. We find

$$H(\mathrm{VR}^m(X;\_))$$

is interleaved with

$$H(\mathrm{VR}^m(F;\_))$$

is isomorphic to

$$H(\mathrm{VR}(F;\_))$$

is interleaved with

$$H(\mathrm{VR}(X;\_))$$

Persistence Stability for Metric Thickenings

- So $H(\mathrm{VR}^m(X; \_))$ is interleaved with $H(\mathrm{VR}(X; \_))$, with the interleaving depending on the finite sample.
- If X is totally bounded, the sample can be made arbitrarily fine, so the persistence modules are ε-interleaved for any $\varepsilon > 0$.

Results

Theorem

If X is a totally bounded metric space, then $H(\mathrm{VR}^m(X;\_))$ and $H(\mathrm{VR}(X;\_))$ have identical persistence diagrams.

This implies

Theorem

If X and Y are totally bounded metric spaces, then

$$\mathrm{d_b}\Big(\mathrm{dgm}\big(H(\mathrm{VR}^m(X;\_))\big), \mathrm{dgm}\big(H(\mathrm{VR}^m(Y;\_))\big)\Big) \leq 2d_{GH}(X, Y)$$

Similar results hold for both intrinsic and ambient Čech complexes.

Interpretation, Implications, and Future Work

- Vietoris–Rips complexes and metric thickenings carry the same persistent homology information.
- Metric thickenings provide an alternate approach for infinite spaces.
- If the persistent homology of either $\mathrm{VR}(X;\_)$ or $\mathrm{VR}^m(X;\_)$ can be found, then the other is known.
- These results motivate further work on the homotopy types of metric thickenings.

References I

Michał Adamaszek, Henry Adams, and Florian Frick.
Metric reconstruction via optimal transport.
SIAM Journal on Applied Algebra and Geometry, 2(4):597–619, 2018.

Frédéric Chazal, Vin de Silva, and Steve Oudot.
Persistence stability for geometric complexes.
Geometriae Dedicata, 173(1):193–214, 2014.

Frederic Chazal, Vin de Silva, Marc Glisse, and Steve Oudot.
The structure and stability of persistence modules, 2013.

Michal Adamaszek and Henry Adams.
The Vietoris–Rips complexes of a circle.
Pacific Journal of Mathematics, 290:1–40, July 2017.

Nathaniel Saul and Chris Tralie.
Scikit-tda: Topological data analysis for python, 2019.

References II

Robert Ghrist.
Barcodes: The persistent topology of data.
Bull. Amer. Math. Soc. (N.S.), 45:61–75, 2008.

Francis C Motta, Rachel Neville, Patrick D Shipman, Daniel A Pearson, and R Mark Bradley.
Measures of order for nearly hexagonal lattices.
Physica D: Nonlinear Phenomena, 380:17–30, 2018.

Paul Bendich, James S Marron, Ezra Miller, Alex Pieloch, and Sean Skwerer.
Persistent homology analysis of brain artery trees.
The Annals of Applied Statistics, 10(1):198, 2016.