Persistence and Simplicial Metric Thickenings

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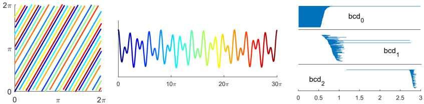
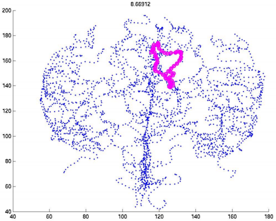
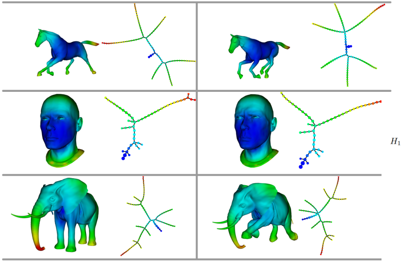
Applied topology/Topological data analysis



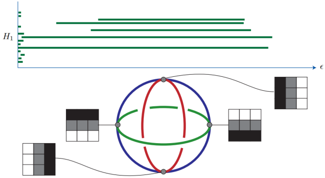
Topological time series analysis - Perea



Persistent homology analysis of brain artery trees - Bendich et al.



Topological methods for the analysis of high dimensional data sets and 3D object recognition - Singh et al.



Barcodes: The persistent topology of data - Ghrist
On the local behavior of spaces of natrual images - Carlsson et al.

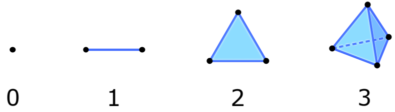
- Roughly 25 year history
- Topological techniques for applied problems
- Heavy emphasis on mathematical theory

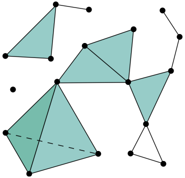
Setting

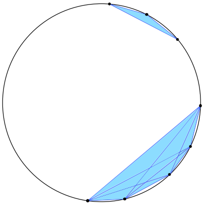
My work...

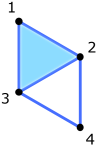
- revolves around the theory of one-parameter persistent homology
- builds on previous research on geometric constructions used in persistent homology
- solves some preexisting problems and provides tools for future research

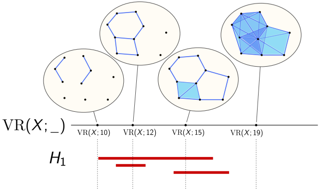
Simplicial complexes

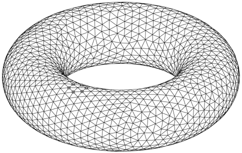








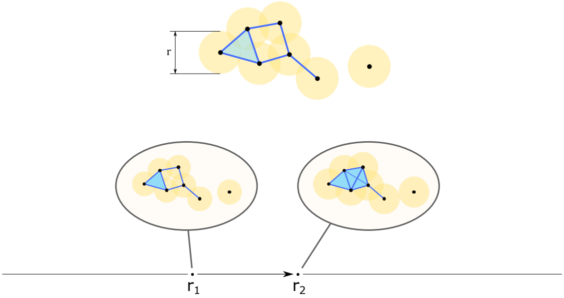




Vietoris–Rips

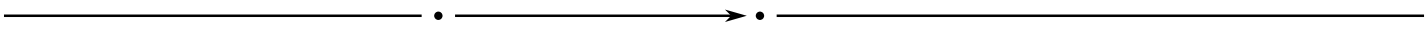
"Connect points that are close"

- $\mathrm{VR}_{\leq}(X;r) = \{\sigma \subseteq X \mid \sigma \text{ finite}, \operatorname{diam}\sigma \leq r\}$



Background: Filtrations and persistence

One-parameter filtration: “spaces evolving over time”



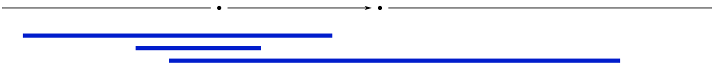
$$X_{s\leq t}\colon X_s \longrightarrow X_t$$

Apply H_n to get the *persistent homology module* $H_n(X)$

Barcodes

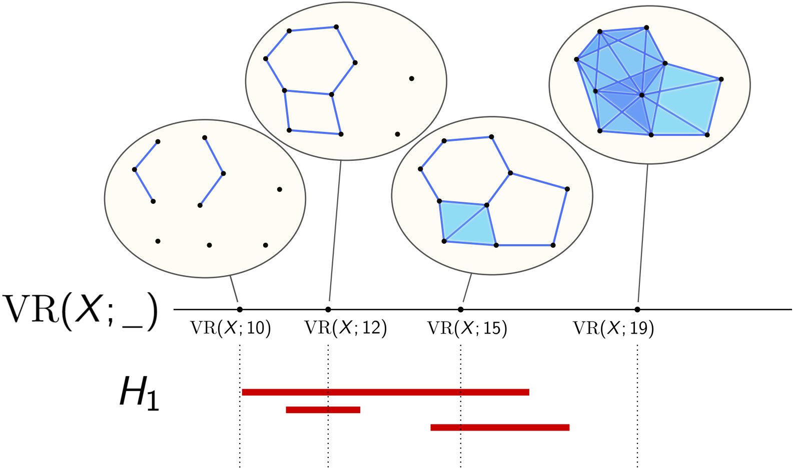
The "output" of persistent homology

The *barcode* of $H_n(X)$ records the lifetimes of homological features:



Barcodes exist under reasonable conditions and are "stable" to perturbations

Vietoris–Rips persistent homology



Topology of simplicial complexes

- Standard coherent / colimit topology
- Classical metric topology (not used here)
- Metric thickening topology

All agree for finite simplicial complexes but can differ for infinite

Simplicial metric thickenings

- *Metric reconstruction via optimal transport* Adamaszek, Adams, and Frick (2018)

If K is a simplicial complex with vertex set a metric space (X, d),

$$K^m = \left\{ \sum_{i=1}^{n} \lambda_i \delta_{x_i} \;\middle|\; \lambda_i \geq 0 \text{ for all } i, \sum_{i=1}^{n} \lambda_i = 1, [x_1, \ldots, x_n] \in K \right\},$$

equipped with the 1-Wasserstein metric

- Isometric embedding $X \hookrightarrow K^m$
- Vietoris–Rips metric thickenings: $\mathrm{VR}^m(X; r)$

Vietoris–Rips properties

Theorem (Hausmann / Adamaszek, Adams, and Frick)

For a closed Riemannian manifold M,
$\mathrm{VR}(M; r) \simeq \mathrm{VR}^m(M; r) \simeq M$ *for small enough r.*

Theorem (Equivalence of Vietoris–Rips persistent homology)

If X is a totally bounded metric space, then $H_n(\mathrm{VR}(X))$ and $H_n(\mathrm{VR}^m(X))$ have identical barcodes up to open vs. closed endpoints.

Theorem (Gillespie)

The natural map $\mathrm{VR}_<(X; r) \to \mathrm{VR}^m_<(X; r)$ is a weak homotopy equivalence.

Vietoris–Rips properties

"Close filtrations produce close barcodes"

Theorem (Stability of Vietoris–Rips persistent homology)

If X and Y are totally bounded metric spaces, then

$$d_B\Big(\mathrm{bar}\big(H_n(\mathrm{VR}(X))\big), \mathrm{bar}\big(H_n(\mathrm{VR}(Y))\big)\Big) \leq 2d_{GH}(X, Y)$$

and

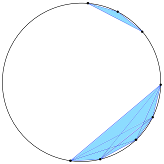
$$d_B\Big(\mathrm{bar}\big(H_n(\mathrm{VR}^m(X))\big), \mathrm{bar}\big(H_n(\mathrm{VR}^m(Y))\big)\Big) \leq 2d_{GH}(X, Y)$$

Applies even in the case of spaces with infinitely many points.

- Homotopy types of $\mathrm{VR}(S^1; r)$: Adamaszek and Adams, 2015.
- Matching homotopy types of the metric thickenings $\mathrm{VR}^m_{\leq}(S^1; r)$: *Vietoris-Rips Metric Thickenings of the Circle*, Journal of Applied and Computational Topology, 2023

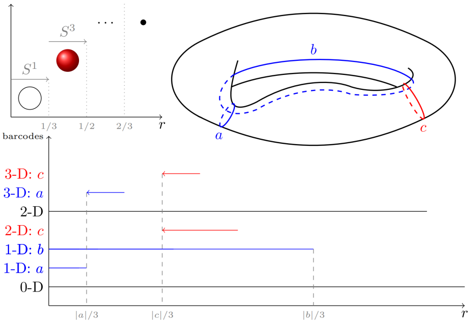
Theorem

If $r \in \left[\frac{2k\pi}{2k+1}, \frac{(2k+2)\pi}{2k+3}\right)$, *then* $\mathrm{VR}^m_{\leq}(S^1; r) \simeq S^{2k+1}$.



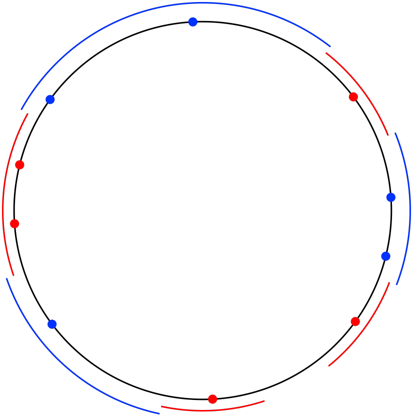
Implications for persistence

- Interpretation of persistent homology in practice
- New techniques in persistent homology



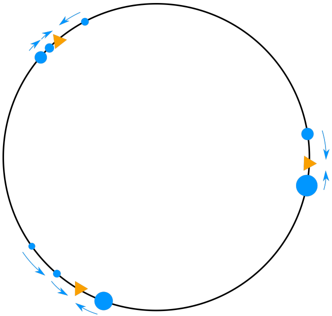
Footprints of Geodesics in Persistent Homology – Virk

What measures are possible?



Method outline

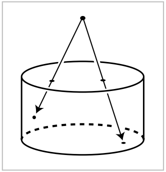
- Gather clusters
- Identify every measure with an odd polygonal measure while preserving homotopy type



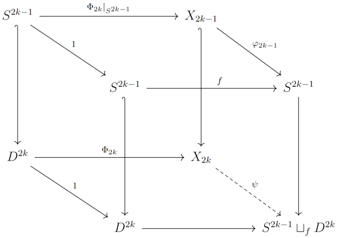
Technical properties

- Need properties of homotopies of simplicial metric thickenings
- Support homotopies
- Homotopy extension property

Extend classical ideas:

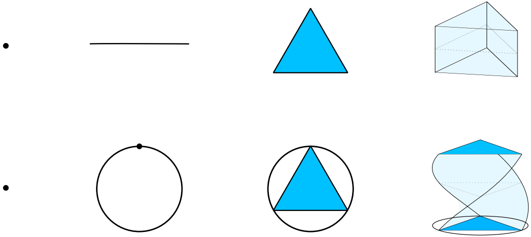


Algebraic Topology – Hatcher



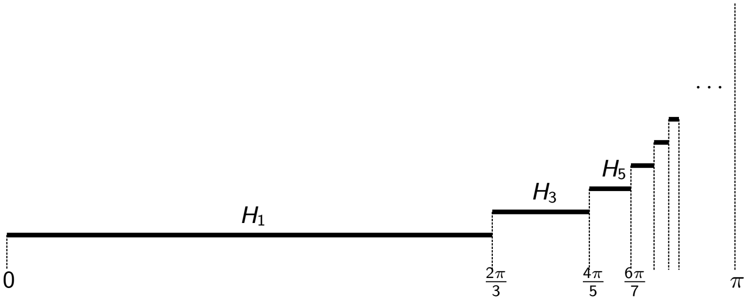
The homotopy types

- Result: a CW complex
- Induction to find homotopy types



D^0 S^1 D^2 S^3

Barcodes of $\mathrm{VR}^m_{\leq}(S^1)$



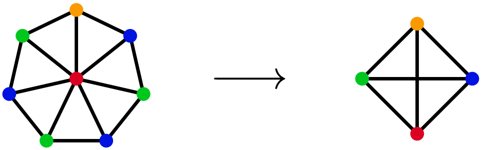
Anti-Vietoris–Rips

"Connect points that are far"

- $\mathrm{AVR}(X; r) = \{\sigma \subseteq X \mid \sigma \text{ finite}, \mathrm{spread}\,\sigma \geq r\}$
- Contravariant filtration
- $\mathrm{AVR}^m(X; r)$: metric thickening topology
- Connections/applications to graph coloring

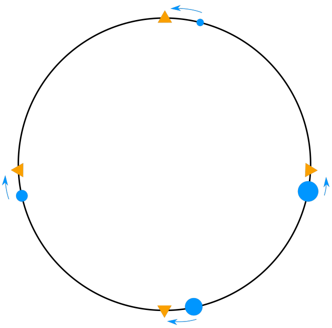
Graph coloring

- An n-coloring of G is the same as a homomorphism $G \to K_n$

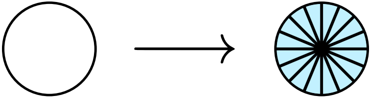


- Replace K_n with $\mathrm{AVR}(S^1; \frac{2\pi}{n})$
- Circular chromatic number: $\lceil \chi_C \rceil = \chi$

- Again, collapse to regular polygonal measures to get a cell complex
- Now we get n-gons for both even and odd n

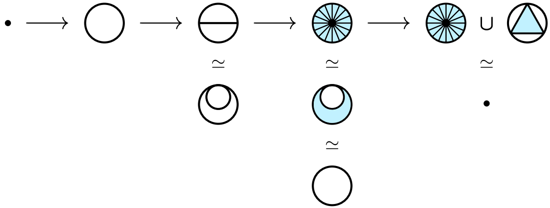


Gluing



- First step: gluing in “diameters” produces Möbius strip M
- Degree two map $S^1 \to M \simeq S^1$

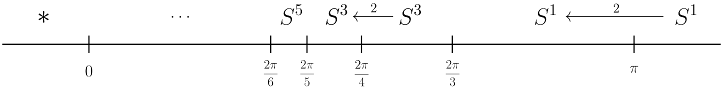
Techniques for higher dimensions



- Rely on degrees of attaching maps
- Third step uses homotopy group $\pi_n(S^n \vee S^n)$
- Fourth step introduces a manifold and applies Mayer–Vietoris

Homotopy types

Theorem
If $r \in \left(\frac{2\pi}{2k+1}, \frac{2\pi}{2k-1}\right]$, *then* $\mathrm{AVR}^m_{\geq}(S^1; r) \simeq S^{2k-1}$.



Degree two maps $\implies$ persistent homology depends on characteristic of field

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